Spatial distributions of the fields in guided normal modes of two coupled parallel optical nanofibers

Author
Fam Le Kien, Lewis Ruks, Sile Nic Chormaic, Thomas Busch

Journal or publication title
New Journal of Physics

Volume
23

Number
4

Page range
043006

Year
2021-04-02

Publisher
IOP Publishing Ltd on behalf of the Institute of Physics and Deutsche Physikalische Gesellschaft

Rights
(C) 2021 The Author(s).

Author's flag

URL
http://id.nii.ac.jp/1394/00001920/

doi: info:doi/10.1088/1367-2630/abea44
Spatial distributions of the fields in guided normal modes of two coupled parallel optical nanofibers

To cite this article: Fam Le Kien et al 2021 New J. Phys. 23 043006
Spatial distributions of the fields in guided normal modes of two coupled parallel optical nanofibers

Fam Le Kien, Lewis Ruks, Síle Nic Chormaic and Thomas Busch

Okinawa Institute of Science and Technology Graduate University, Onna, Okinawa 904-0495, Japan

E-mail: kienle.pham@oist.jp

Keywords: guided normal modes, coupled nanofibers, field distributions

Abstract

We study the cross-sectional profiles and spatial distributions of the fields in guided normal modes of two coupled parallel optical nanofibers. We show that the distributions of the components of the field in a guided normal mode of two identical nanofibers are either symmetric or antisymmetric with respect to the radial principal axis and the tangential principal axis in the cross-sectional plane of the fibers. The symmetry of the magnetic field components with respect to the principal axes is opposite to that of the electric field components. We show that, in the case of even $E_z$-cosine modes, the electric intensity distribution is dominant in the area between the fibers, with a saddle point at the two-fiber center. Meanwhile, in the case of odd $E_z$-sine modes, the electric intensity distribution at the two-fiber center attains a local minimum of exactly zero. We find that the differences between the results of the coupled mode theory and the exact mode theory are large when the separation distance between the fibers is small and either the fiber radius is small or the wavelength of light is large. We show that a slight difference between the radii of the nanofibers leads to strong asymmetry of the intensity distributions of the guided normal modes.

1. Introduction

Coupled waveguides form the central working component in numerous optical devices such as multicore fibers, optical directional couplers, polarization splitters, ring resonators, and interferometers [1–3]. Most of the previous work on coupling between parallel fibers was devoted to conventional fibers where the refractive indices of the core and the cladding differ only slightly from each other and the fiber radius is large compared to the wavelength of light [1–3]. It is desirable to study the properties of guided light fields in coupled subwavelength-diameter optical fibers due to their increasing relevance in current research efforts [4].

Optical nanofibers are tapered fibers that have a subwavelength diameter and significantly differing core and cladding refractive indices [4]. Such ultrathin fibers allow for a guided light field, which is tightly confined radially, to propagate along the fiber for a long distance (with several millimeters being typical) and to interact efficiently with nearby quantum or classical emitters, absorbers, and scatterers [5–7]. Optical nanofibers have been investigated for use in a variety of applications in nonlinear optics, atomic physics, quantum optics, and nanophotonics [4–7]. Nanofibers have been used for atom trapping [8–11], efficient channeling of emission from atoms into fiber-guided modes [12–14], efficient absorption of guided light by atoms [15, 16], generation of Rydberg states of atoms [17], and excitation of quadrupole transitions of atoms [18, 19]. Additionally, slot nanofibers, where the center of the nanofiber has been removed to create two parallel waveguide channels, have been proposed as atom traps [20]. Trapping atoms between two parallel nanofibers and interfacing them with the guided fields of the two different nanofibers may open up more possibilities for applications in nonlinear optics, quantum optics, and quantum information.

Recently, miniaturized optical devices comprising of twisted or knotted nanofibers have been produced [21]. Coupling between two nanofibers in such systems has been studied by using the linear coupling theory...
New J. Phys. 23 (2021) 043006 FLK i e n

[21, 22], which is an approximate theory valid under the condition of weak coupling [1–3]. It has been shown that butt coupling and self coupling [1–3] could be quite substantial for nanofibers due to the significant mode spread and overlap [22]. In the case of strong coupling between two fibers, a rigorous treatment for the normal modes is required.

The exact guided normal modes of two coupled dielectric rods can be calculated by the expansion of circular harmonics [23]. This method has been extended to the case of multicore fibers [24–26]. A vector theory that uses the circular harmonics expansion method and the finite-element method has been developed for two-core fibers with core index profiles that are radially inhomogeneous [27]. The propagation constant and the flux density of the field in a guided normal mode have been calculated [23, 27, 28]. It has been shown that the coupled mode theory (CMT) performs well when the separation between the fibers is large [23, 27, 28], and gives reasonable results even for touching fibers when the fiber radii are sufficiently large [28]. The polarization patterns [27] and the mode cutoffs [29] have been investigated. In optomechanics, forces arising from internal illumination by light traveling in coupled waveguides have been studied [30], and light-guiding arrays of mechanically compliant glass nanospikes have recently been fabricated [31].

In this work, we investigate the spatial distributions of the fields in guided normal modes of two coupled parallel optical nanofibers. We find that the distributions of the components of the fields in guided normal modes of two coupled identical nanofibers are either symmetric or antisymmetric with respect to the principal axes of the cross-sectional plane of the fibers. We reveal that the intensity distributions of the fields in guided normal modes of two identical fibers attain a local critical point at the two-fiber center that may be used for atom trapping and guiding. Additionally, we show that the discrepancy between the results of the CMT and the exact theory is large when the fiber separation distance is small and either the fiber radius is small or the light wavelength is large.

The key advance of the present work in relation to our previous work [22] is that here we perform an exact treatment for the spatial distributions of the fields in guided normal modes of two nanofibers, while in [22] we used the CMT and calculated the coupling coefficients.

The paper is organized as follows. In section 2 we describe the model of two coupled parallel nanofibers and present the basic equations for guided normal modes. Section 3 contains the numerical calculations of the spatial distributions of the fields in the guided normal modes. Finally, we conclude in section 4.

2. Two coupled parallel optical nanofibers

We study two vacuum-clad, optical nanofibers that are parallel to each other in the direction of the fiber axis z (see figure 1). The fibers are labeled by the indices \( j = 1, 2 \). Each nanofiber \( j \) can be treated as a dielectric cylinder with a radius \( a_j \) and a refractive index \( n_j > 1 \), surrounded by an infinite background of vacuum or air with a refractive index \( n_0 = 1 \). The nanofiber diameters are a few hundreds of nanometers. An individual nanofiber \( j \) can support either a single or multiple modes and this depends on the fiber size parameter \( V_j = k a_j \sqrt{n_j^2 - n_0^2} \). Here, \( k = \omega / c \) is the wave number of light with optical frequency \( \omega \) in free space. We neglect the van der Waals interaction between the fibers assuming that they are fixed.

We note that, although the configurations of twisted and knotted nanofibers have been experimentally realized [21], the technique used in that work does not allow one to obtain the configuration of two parallel nanofibers with high quality. Since nanofibers are usually fabricated by tapering of single-mode fibers [4], the large conic transition regions make it very challenging to obtain the configuration of two coupled parallel nanofibers with high quality. Despite this fact, we believe that the model can, in principle, be experimentally realized with a reasonable quality in the future. Alternatively, pulled nanofibers do not have a comparable conical transition region and offer a configuration that could be more easily experimentally realizable [32].

We introduce the global Cartesian coordinate system \( \{ x, y, z \} \). Here, the \( z \) axis is parallel to the \( z_1 \) and \( z_2 \) axes of the fibers, the \( x \) axis is perpendicular to the \( z \) axis and connects the centers \( O_1 \) and \( O_2 \) of the fibers, and the \( y \) axis is perpendicular to the \( x \) and \( z \) axes (see figure 1). The plane \( xy \) is the transverse (cross-sectional) plane of the fibers. The \( x \) and \( y \) axes are called the radial and tangential axes, respectively, of the two-fiber system (see figure 1(b)). The positions of the fiber centers \( O_1 \) and \( O_2 \) on the \( x \) axis are \( O_1 = -(a_1 + d_1) \) and \( O_2 = a_2 + d_2 \), where \( d_1 + d_2 = d \) is the fiber separation distance. Without loss of generality, we choose \( d_1 = d_2 = d/2 \). For each individual fiber \( j \), we use the local fiber-based system \( \{ r_j, \varphi_j \} \) of polar coordinates.

The normal modes of the coupled fibers are termed array modes. We study the array modes of a light field with an optical frequency \( \omega \) which propagates in the \( +z \) direction with a propagation constant \( \beta \). The
The electric and magnetic components of the field can be written as 

$$
\mathbf{E} = \mathbf{E}_e - i(\omega t - \beta z) + c.c.
$$

and

$$
\mathbf{H} = \mathbf{H}_e - i(\omega t - \beta z) + c.c.,
$$

respectively, where \( \mathbf{E} \) and \( \mathbf{H} \) are the slowly varying complex envelopes.

We are interested in the guided (bound) modes whose fields decay in the transverse direction. The exact theory for the guided normal modes of two parallel dielectric cylinders has been formulated in [23]. The flux density and the beat wavelength for the energy beating between the guided normal modes have been calculated for the cylinders with \( a_j \gg \lambda/2 \) and \( n_j/n_0 \approx 1 \). We follow the theory of [23] and use it to treat the spatial distributions of the fields in the guided normal modes of the coupled nanofibers.

According to the theory of [23], the longitudinal components \( \mathbf{E}_z \) and \( \mathbf{H}_z \) of the electric and magnetic parts, respectively, of the field in a guided normal mode are given, inside fiber \( j = 1, 2 \), as

$$
\mathbf{E}_z = \sum_{n=0}^{\infty} \left[ A_{nj} J_n(h_{jr_j}) \cos n\varphi_j + E_{nj} J_n(h_{jr_j}) \sin n\varphi_j \right],
$$

$$
\mathbf{H}_z = \sum_{n=0}^{\infty} \left[ B_{nj} J_n(h_{jr_j}) \sin n\varphi_j + F_{nj} J_n(h_{jr_j}) \cos n\varphi_j \right],
$$

and, outside the fibers, as

$$
\mathbf{E}_z = \sum_{j=1}^{2} \sum_{n=0}^{\infty} \left[ C_{nj} K_n(q_{jr_j}) \cos n\varphi_j + G_{nj} K_n(q_{jr_j}) \sin n\varphi_j \right],
$$

$$
\mathbf{H}_z = \sum_{j=1}^{2} \sum_{n=0}^{\infty} \left[ D_{nj} K_n(q_{jr_j}) \sin n\varphi_j + H_{nj} K_n(q_{jr_j}) \cos n\varphi_j \right].
$$

Here, we have introduced the fiber parameters

$$
h_j = \sqrt{k^2n_j^2 - \beta^2}, \quad q = \sqrt{\beta^2 - k^2n_0^2},
$$

which determine the scales of the spatial variations of the field both inside and outside the fibers. In equations (1) and (2), \( J_n \) represents the Bessel functions of the first kind and \( K_n \) represents the modified Bessel functions of the second kind. The sets \( \{ A_{nj}, B_{nj}, C_{nj}, D_{nj} \} \) and \( \{ E_{nj}, F_{nj}, G_{nj}, H_{nj} \} \) contain the mode expansion coefficients for the \( \mathbf{E}_z \)-cosine (\( x \)-polarized) and \( \mathbf{E}_z \)-sine (\( y \)-polarized) modes, respectively. These coefficients characterize the contributions of the partial waves, associated with the circular harmonics, to

![Figure 1. Two coupled parallel optical nanofibers (a) and the geometry of the system (b).](image-url)
the mode. We emphasize that equations (2) stand for guided (bound) modes but not for radiation (unbound) modes. Therefore, equations (2) do not contain the Hankel functions and the modified Bessel functions of the first kind.

It follows from the Maxwell equations and the translational symmetry of the system in the $z$ direction that the transverse components $E_{x,y}$ and $H_{x,y}$ of the electric and magnetic parts of the field can be expressed in terms of the longitudinal components $E_z$ and $H_z$ as [1–3]

$$
E_x = \frac{i\beta}{k^2 n_{ref}^2 - \beta^2} \left( \frac{\partial}{\partial x} E_z + \frac{\omega n_{ref}}{\beta} \frac{\partial}{\partial y} H_z \right),
$$
$$
E_y = \frac{i\beta}{k^2 n_{ref}^2 - \beta^2} \left( \frac{\partial}{\partial y} E_z - \frac{\omega n_{ref}}{\beta} \frac{\partial}{\partial x} H_z \right),
$$
$$
H_x = \frac{i\beta}{k^2 n_{ref}^2 - \beta^2} \left( \frac{\partial}{\partial x} H_z - \frac{\omega n_{ref}}{\beta} \frac{\partial}{\partial y} E_z \right),
$$
$$
H_y = \frac{i\beta}{k^2 n_{ref}^2 - \beta^2} \left( \frac{\partial}{\partial y} H_z + \frac{\omega n_{ref}}{\beta} \frac{\partial}{\partial x} E_z \right).
$$

(4)

Here, $n_{ref}$ is the spatial distribution of the refractive index in the presence of the two-fiber system, that is, $n_{ref} = n_j$ inside fiber $j = 1, 2$ and $n_{ref} = n_0$ outside the two fibers.

The field in the mode must satisfy the boundary conditions at the surfaces of both fibers. These conditions say that the tangential components of the electric and magnetic fields are continuous at the boundary. By using the boundary conditions, the algebraic equations for the mode expansion coefficients \{A$_{n_j}$, B$_{n_j}$, C$_{n_j}$, D$_{n_j}$\} and \{E$_{n_j}$, F$_{n_j}$, G$_{n_j}$, H$_{n_j}$\} have been derived [23]. These equations are summarized in appendix A.

For the $E_z$-cosine modes, the expansion coefficients $E_{n_j}$, $F_{n_j}$, $G_{n_j}$, and $H_{n_j}$ vanish. For these modes, the coefficients $A_{n_j}$ and $B_{n_j}$ for the field inside the fibers are given by equations (A.4), while the coefficients $C_{n_j}$ and $D_{n_j}$ for the field outside the fibers are nonzero solutions of equations (A.5).

For the $E_z$-sine modes, the expansion coefficients $A_{n_j}$, $B_{n_j}$, $C_{n_j}$, and $D_{n_j}$ vanish. For these modes, the coefficients $E_{n_j}$ and $F_{n_j}$ for the field inside the fibers are given by equations (A.10), while the coefficients $G_{n_j}$ and $H_{n_j}$ for the field outside the fibers are nonzero solutions of equations (A.11).

The dispersion equation for the $E_z$-cosine or $E_z$-sine modes is $\Delta = 0$, where $\Delta$ is the determinant of the system of linear equations (A.5) for $C_{n_j}$ and $D_{n_j}$ or (A.11) for $G_{n_j}$ and $H_{n_j}$. The solution to the equation $\Delta = 0$ determines the propagation constant $\beta$, which allows us to calculate the other fiber parameters $h_j$ and $q$ [see equations (3)].

Note that the coefficients associated with $C_{n_j}$ and $D_{n_j}$ in equations (A.5) and with $G_{n_j}$ and $H_{n_j}$ in equations (A.11) are real-valued coefficients. Therefore, when we omit a common global phase, we can make \{A$_{n_j}$, B$_{n_j}$, C$_{n_j}$, D$_{n_j}$\} and, similarly, \{E$_{n_j}$, F$_{n_j}$, G$_{n_j}$, H$_{n_j}$\} to be real-valued coefficients. Then, the longitudinal field components $E_z$ and $H_z$, given by equations (1) and (2), are real-valued, while the transverse components $(E_x, E_y)$ and $(H_x, H_y)$, given by equations (4), are imaginary-valued. Thus, we have

$$
E_x^* = E_x, \quad H_z^* = H_z,
$$
$$
E_y^* = -E_y, \quad H_x^* = -H_x,
$$
$$
E_z^* = E_z, \quad H_y^* = H_y.
$$

(5)

Equations (5) indicate that the longitudinal components $E_z$ and $H_z$ of the field in a guided normal mode are $\pi/2$ out of phase with respect to the transverse components $E_x$, $E_y$, $H_x$, and $H_y$. This relative phase is a typical feature of guided [1–3] and other transversely confined light fields [33].

Let us consider the particular case where the two fibers are identical, that is, the two fibers have the same radius $a_1 = a_2 = a$ and the same core refractive index $n_1 = n_2 = n_f$. In this case, for the $E_z$-cosine modes, we find

$$
A_{n_2} = (-1)^n \mu A_{n_1}, \quad B_{n_2} = (-1)^n \nu B_{n_1},
$$
$$
C_{n_2} = (-1)^n \nu C_{n_1}, \quad D_{n_2} = (-1)^n \mu D_{n_1}.
$$

(6)

and, for the $E_z$-sine modes, we get

$$
E_{n_2} = (-1)^n \nu E_{n_1}, \quad F_{n_2} = (-1)^n \nu F_{n_1},
$$
$$
G_{n_2} = (-1)^n \nu G_{n_1}, \quad H_{n_2} = (-1)^n \nu H_{n_1}.
$$

(7)
The vanishing of the electric field about the \( z \) axis. It follows from these relations that, for the odd \( E_z \)-sine modes, the electric field \( E \) at the two-fiber center \((x, y) = (0, 0)\) is zero, that is, \( E(0, 0) = 0 \).

The vanishing of the electric field \( E \) of the odd \( E_z \)-sine mode at the two-fiber center can be used to produce a local minimum of a blue-detuned optical dipole potential to trap ground-state atoms [34–36] or a local minimum of a ponderomotive optical Rydberg-electron potential to trap Rydberg atoms [37, 38].
Similarly, we can show that, in the case of the odd $E_z$-cosine mode, the magnetic field $\mathbf{H}$ at the two-fiber center is zero, that is, $\mathbf{H}(0,0) = 0$.

3. Numerical calculations

In this section, we numerically calculate the propagation constants and spatial distributions of the fields in guided normal modes of two parallel vacuum-clad silica nanofibers. The refractive index of the vacuum cladding is $n_0 = 1$. To calculate the refractive index $n_1 = n_2 = n_3$ of the silica cores of the nanofibers, the four-term Sellmeier formula for fused silica is used [39, 40]. In particular, for light with the wavelength $\lambda = 800$ nm, we have $n_1 = 1.4533$. In our numerical calculations, the infinite number of circular harmonics is truncated at a finite number $N_{\text{max}}$ in the range from 9 to 19. The value of $N_{\text{max}}$ is chosen such that the propagation constant converges and the boundary conditions are satisfied with a reasonable accuracy [23].

According to the previous section, in the case of identical fibers, there are four kinds of normal modes, denoted as even $E_z$-cosine, odd $E_z$-cosine, even $E_z$-sine, and odd $E_z$-sine modes, or as $I_{0}^{(\cos)}$, $I_{2}^{(\cos)}$, $I_{1}^{(\sin)}$, and $I_{2}^{(\sin)}$ modes, respectively [23]. The even $E_z$-cosine, odd $E_z$-cosine, even $E_z$-sine, and odd $E_z$-sine modes can also be labeled by the letters OO, EE, OE, and EO, respectively. These letters indicate the symmetry (E) and antisymmetry (O) of the field component $E_z$ about the $x$ (first letter) and $y$ (second letter) axes [27]. We are interested in the case where the fiber radii are small enough that no more than one normal mode of each of the four kinds can be supported.

3.1. Propagation constants of guided normal modes of two identical nanofibers

Let us assume that the two nanofibers have the same fiber radius, that is, $a_1 = a_2 = a$. We plot in figures 2–4 the propagation constant $\beta$, normalized to the free-space wave number $k$, as functions of the fiber radius $a$, the light wavelength $\lambda$, and the fiber separation distance $d$. We note that the ratio $n_{\text{eff}} = \beta/k$ between $\beta$ and $k$ is called the effective refractive index. It is clear from figures 2–4 that $n_{\text{eff}}$ depends on the parameters of the fibers, the wavelength of light, and the mode type. We observe from the figures that there are four guided normal modes, identified as even $E_z$-cosine, odd $E_z$-cosine, even $E_z$-sine, and odd $E_z$-sine modes [23].

Figures 2–4 show that there are two pairs of adjacent curves. The upper pair corresponds to the even modes and the lower pair to the odd modes. This indicates that the propagation constant of an even mode (see the upper pair of curves) is larger than that of the corresponding odd mode (see the lower pair of curves). The differences between the propagation constants for the even $E_z$-cosine and odd $E_z$-sine modes (see the lower pair of curves) are smaller than those for the even $E_z$-cosine and even $E_z$-sine modes (see the upper pair of curves). We see from figure 4 that the differences between the propagation constants for the four guided array modes reduce with increasing fiber separation distance $d$. We observe from figures 2 and 3 that the odd $E_z$-cosine and odd $E_z$-sine modes have cutoffs, but the even $E_z$-cosine and even $E_z$-sine modes have no cutoff [23, 29]. The reason is the following:

For a single propagation direction $+z$, each single-mode nanofiber can support a superposition of two fundamental modes $HE_{41}$ that are quasilinearly polarized along the $x$ and $y$ axes and are called as $E_z$-cosine and $E_z$-sine modes, respectively. We expect that two coupled parallel single-mode nanofibers can support up to four guided normal modes. We introduce the notation $\mathbf{e}_j^{(p)}$ for the profile function of the single-fiber mode with the quasilinear polarization $p = x, y$ of nanofiber $j = 1, 2$. According to the CMT [3], there is no coupling between the principal $x$ and $y$ polarizations. For a suitable choice of the global phases of the mode functions, the approximate profile functions of the guided normal modes can be given as $\mathbf{e}_\pm^{(p)} \approx \mathbf{e}_1^{(p)} \pm \mathbf{e}_2^{(p)}$, where the sign + and − corresponds to the even and odd modes, respectively. Note that $\mathbf{e}_z^{(p)}(r) = \mathbf{e}_x^{(p)}(r + O_{1,2})$. When either the fiber radius is small enough or the light wavelength is large enough, we can use the approximation $\mathbf{e}_\mp^{(p)}(r) \approx \mathbf{e}_1^{(p)}(r)$. In these limits, the even array modes with the profile functions $\mathbf{e}_z^{(p)} \approx \mathbf{e}_1^{(p)} + \mathbf{e}_2^{(p)}$ approach the modes of single nanofibers, while the odd array modes with the profile functions $\mathbf{e}_z^{(p)} \approx \mathbf{e}_1^{(p)} - \mathbf{e}_2^{(p)}$ are widely spread out in the outside of the nanofibers. Consequently, the propagation constant of an even mode is larger than that of the corresponding odd mode (see

<table>
<thead>
<tr>
<th>Mode type</th>
<th>$\mathbf{e}_x$</th>
<th>$\mathbf{e}_y$</th>
<th>$\mathbf{e}_z$</th>
<th>$\mathbf{H}_x$</th>
<th>$\mathbf{H}_y$</th>
<th>$\mathbf{H}_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_z$-cosine</td>
<td>+</td>
<td>–</td>
<td>+</td>
<td>–</td>
<td>+</td>
<td>–</td>
</tr>
<tr>
<td>$E_z$-sine</td>
<td>–</td>
<td>+</td>
<td>–</td>
<td>+</td>
<td>–</td>
<td>+</td>
</tr>
</tbody>
</table>
Figure 2. Propagation constant $\beta$, normalized to the free-space wave number $k$, as a function of the fiber radius $a$ for two identical nanofibers. The wavelength of light is $\lambda = 800$ nm and the separation distance between the two fibers is $d = 0$ (a) and $200$ nm (b). The refractive index of the fibers is $n_f = 1.4533$ and that of the surrounding medium is $n_0 = 1$.

Figure 3. Propagation constant $\beta$, normalized to the free-space wave number $k$, as a function of the wavelength $\lambda$ of light for two identical nanofibers. The fiber radius is $a = 200$ nm and the separation distance between the two fibers is $d = 0$ (a) and $200$ nm (b). The refractive index $n_f$ of the nanofibers is calculated from the four-term Sellmeier formula for fused silica [39, 40] and that of the surrounding medium is $n_0 = 1$.

Comparison between figures 2(a) and (b) and between figures 3(a) and (b) shows that the cutoff values of the fiber radius $a$ and the light wavelength $\lambda$ for the odd $E_z$-cosine and $E_z$-sine modes depend on the separation distance $d$ between the two fibers. A smaller $d$ leads to a larger cutoff value of the fiber radius $a$ and to a smaller cutoff value of the light wavelength $\lambda$. We see from figure 4(a) that, if the fiber radius $a$ is large enough or, equivalently, the light wavelength is small enough, there is no cutoff of the guided normal modes. However, figures 4(b) and (c) show that, if the fiber radius $a$ is small enough or, equivalently, the light wavelength is large enough, a cutoff of an odd guided normal mode may appear at a nonzero fiber separation distance $d$. Comparing the solid and dashed curves of figures 2–4 shows that the difference between the propagation constants of the $E_z$-cosine and $E_z$-sine modes reduces with increasing fiber separation distance $d$. This feature is due to the fact that the difference between the propagation constants...
of the $x$- and $y$-polarized array modes is determined by the coupling between the nanofibers, which depends on the mode overlap and consequently reduces with increasing $d$.

Nanofibers are effective media for nonlinear optics due to the tight confinement of the fields in their guided modes [5–7, 41, 42]. The control of dispersion properties of tapered fibers has led to many interesting applications such as supercontinuum generation [41] and photon triplet generation [42]. An advantage of the system of two coupled nanofibers is that the dispersion properties of the array modes depend on not only the fiber radius, the wavelength of light, and the material refractive index but also the separation distance between the fibers. Consequently, one can control and manipulate nonlinear processes by varying the fiber separation distance in addition to the other parameters.

3.2. Spatial profiles of the fields in the guided normal modes of two identical nanofibers

In this subsection, we study the spatial distributions of the fields in the guided normal modes of two identical nanofibers. We display the cross-sectional profiles of the electric intensity distributions $|E|^2$ for different guided array modes. We also show the dependencies of the components $E_x$, $E_y$, and $E_z$ and the intensity $|E|^2$ of the electric field on the $x$ and $y$ coordinates (since $E_z$ and $E_x$ are imaginary-valued and $E_z$ is real-valued, we plot $\text{Im}(E_x) = -iE_x$, $\text{Im}(E_y) = -iE_y$, and $\text{Re}(E_z) = E_z$).

3.2.1. Even $E_z$-cosine mode

In figure 5, we plot the cross-sectional profile of the electric intensity distribution $|E|^2$ of the field in the even $E_z$-cosine mode of two identical parallel nanofibers. The figure shows that the electric field intensity is dominant in the area between the fibers. This feature can be used to attract atoms [34–36] using gas in single red-detuned array-mode light field. We note that the intensity distribution of the even $E_z$-cosine mode is similar to that of the perpendicular-polarization symmetric mode of a slot nanofiber, where high intensities can be realized in the vacuum region between the slot walls [20].

We show in figure 6 the dependencies of the components $E_x$, $E_y$, and $E_z$ and the intensity $|E|^2$ of the electric field in the even $E_z$-cosine mode on the $x$ and $y$ coordinates. Comparing the scales of the vertical axes in figure 6 shows that all the three components $E_x$, $E_y$, and $E_z$ of the field are significant, while $E_x$ (see figures 6(a) and (e)) is dominant. These features are in agreement with the fact that, in the case of single fibers, the $E_z$-cosine modes are quasilinearly polarized along the $x$ axis [1–3].

Figures 5 and 6 show that a significant portion of the field is in the outside of the nanofibers. The figures also show that abrupt changes of the fields occur at the surfaces of the fibers. Such dramatic changes arise...
Figure 5. Cross-sectional profile of the electric intensity distribution $|\mathbf{E}|^2$ of the field in the even $\mathbf{E}_z$-cosine mode of two identical parallel nanofibers. The fiber radius is $a = 200$ nm and the separation distance between the two fibers is $d = 50$ nm (a) and 200 nm (b). The power of light is the same in the two cases. Other parameters are as in figure 2.

Figure 6. Dependencies of the components $\mathbf{E}_x$, $\mathbf{E}_y$, and $\mathbf{E}_z$ and the intensity $|\mathbf{E}|^2$ of the electric field in the even $\mathbf{E}_z$-cosine mode of two identical fibers on the $x$ and $y$ coordinates. The fiber radius is $a = 200$ nm and the separation distance between the two fibers is $d = 50$ nm. The parameters used are the same as for figure 5(a).

from the boundary conditions and the sharp contrast between the refractive index $n_f$ of the silica nanofibers and the refractive index $n_0 = 1$ of the vacuum medium outside the nanofibers.

The solid curve of figure 6(d) shows that the peaks of the intensity distribution occur at the facing points $(x, y) = (\pm d/2, 0)$. Meanwhile, the center $(x, y) = (0, 0)$ of the two-fiber system is a saddle point.
Figure 7. Cross-sectional profile of the electric intensity distribution $|\mathbf{E}|^2$ of the field in the odd $\mathcal{E}_z$-cosine mode of two identical parallel nanofibers. The parameters used are the same as for figure 5.

Figure 8. Dependencies of the components $\mathcal{E}_x$, $\mathcal{E}_y$, and $\mathcal{E}_z$ and the intensity $|\mathbf{E}|^2$ of the electric field in the odd $\mathcal{E}_z$-cosine mode of two identical fibers on the $x$ and $y$ coordinates. The fiber radius is $a = 200$ nm and the separation distance between the two fibers is $d = 50$ nm. The parameters used are the same as for figure 7(a).

(see figures 6(d) and (h)). Despite these facts, the electric field intensity in the area between the two fibers is substantially higher than that in the surrounding area. This behavior of the electric field intensity distribution can, as already mentioned above, be used to attract atoms using red-detuned light to produce an attractive optical dipole potential.
3.2.2. Odd $E_z$-cosine mode

We depict in figure 7 the cross-sectional profile of the electric intensity distribution $|\mathbf{E}|^2$ of the field in the odd $E_z$-cosine mode of two identical parallel nanofibers. We observe that the electric field intensity is dominant in the outer vicinities of the left-side surface of the left-side fiber and the right-side surface of the right-side fiber.

We display in figure 8 the dependencies of the components $E_x$, $E_y$, and $E_z$ and the intensity $|\mathbf{E}|^2$ of the electric field in the odd $E_z$-cosine mode on the $x$ and $y$ coordinates. We see from the scales of the vertical axes in figure 8 that all the three components $E_x$, $E_y$, and $E_z$ of the field are significant.

3.2.3. Even $E_z$-sine mode

We show in figure 9 the cross-sectional profile of the electric intensity distribution $|\mathbf{E}|^2$ of the field in the even $E_z$-sine mode of two identical parallel nanofibers. The figure shows that the electric field intensity is dominant in the outer vicinities of the top and bottom parts of the surfaces of the fibers, and is significant in the area between the fiber surfaces.

In figure 10, the dependencies of the components $E_x$, $E_y$, and $E_z$ and the intensity $|\mathbf{E}|^2$ of the electric field in the even $E_z$-sine mode on the $x$ and $y$ coordinates are plotted. We see from the scales of the vertical axes in figure 10 that all the three components $E_x$, $E_y$, and $E_z$ of the field are significant, while $E_y$ (see figures 10(b) and (f)) is dominant. These features are in agreement with the fact that, in the case of single fibers, the $E_z$-sine modes are quasilinearly polarized along the $y$ axis [1–3].

3.2.4. Odd $E_z$-sine mode

The cross-sectional profile of the electric intensity distribution $|\mathbf{E}|^2$ of the field in the odd $E_z$-sine mode of two identical parallel nanofibers is displayed in figure 11. We observe that the electric field intensity is dominant in the vicinities of the top and bottom parts of the surfaces of the fibers, and is significant in the outer vicinities of the left-side surface of the left-side fiber and the right-side surface of the right-side fiber.
Figure 10. Dependencies of the components $E_x$, $E_y$, and $E_z$ and the intensity $|E|^2$ of the electric field in the even $E_z$-sine mode of two identical fibers on the $x$ and $y$ coordinates. The fiber radius is $a = 200$ nm and the separation distance between the two fibers is $d = 50$ nm. The parameters used are the same as for figure 9(a).

Figure 11. Cross-sectional profile of the electric intensity distribution $|E|^2$ of the field in the odd $E_z$-sine mode of two identical parallel nanofibers. The parameters used are the same as for figure 5.

We plot in figure 12 the dependencies of the components $E_x$, $E_y$, and $E_z$ and the intensity $|E|^2$ of the electric field in the odd $E_z$-sine mode on the $x$ and $y$ coordinates. We see from the scales of the vertical axes in figure 12 that all the three components $E_x$, $E_y$, and $E_z$ of the field are significant.

Figure 11 and the solid curves of figures 12(d) and (h) confirm the prediction that the electric field of the odd $E_z$-sine mode at the two-fiber center $(x, y) = (0, 0)$ is zero. This feature of the odd $E_z$-sine mode can
be used to trap ground-state atoms with a blue-detuned optical dipole potential [34–36], or to trap Rydberg atoms with a ponderomotive optical Rydberg-electron potential [37, 38]. We emphasize that the existence of a local minimum of exact zero at the two-fiber center is a specific property of the intensity of the electric field in the odd $E_z$-sine mode, and occurs for any fiber separation distance $d$. To reduce to effects of the fiber surfaces on the atoms at the two-fiber center, we can increase the separation between the fibers.

A common feature of figures 5, 7, 9 and 11 is that the field intensity $|E|^2$ is symmetric with respect to the principal axes $x$ and $y$. This symmetry is an obvious consequence of the geometry of the system of two identical parallel nanofibers.

We note that the symmetry properties of the field components $E_x$, $E_y$, and $E_z$, shown in figures 6, 8, 10 and 12, are in agreement with equations (8)–(11) and tables 1 and 2. In particular, the symmetry properties of the curves for the field components in the left panel of figure 6 (for the even $E_z$-cosine mode) with respect to the coordinate $x$ are the same as those of the corresponding curves in the left panel of figure 12 (for the odd $E_z$-sine mode) and are opposite to those of the corresponding curves in the left panels of figure 8 (for the odd $E_z$-cosine mode) and figure 10 (for the even $E_z$-sine mode). Meanwhile, the symmetry properties of the curves for the field components in the right panel of figure 6 (for the even $E_z$-cosine mode) with respect to the coordinate $y$ are the same as those of the corresponding curves in the right panel of figure 8 (for the odd $E_z$-cosine mode) and are opposite to those of the corresponding curves in the right panels of figure 10 (for the even $E_z$-sine mode) and figure 12 (for the odd $E_z$-sine mode).

It is worth emphasizing that the electric intensity of the even $E_z$-cosine mode is dominant in the area between the fibers while the electric intensity of the odd $E_z$-sine mode attains a local minimum of exactly zero at the two-fiber center. These features may be used to attract atoms with a red-detuned light field in the even $E_z$-cosine mode or a blue-detuned light field in the odd $E_z$-sine mode. We note that the effects of the van der Waals potential on the trapping of atoms must be considered. In addition, it is challenging to excite a single array mode of the two coupled nanofibers. These issues deserve separate studies.

Comparison between figures 5 and 7 and between figures 9 and 11 shows that, in the region between the two nanofibers, the interference between the fields of the individual nanofibers is constructive for even modes (see figures 5 and 9) and destructive for odd modes (see figures 7 and 11). Thus, the even and odd modes behave like the bright and dark states of a three-level atom interacting with two driving light fields [43]. This observation is consistent with the results of the CMT [1–3].

3.3. Comparison between the exact mode theory and the coupled mode theory
From the CMT [1–3], the coupling length for the fiber modes with the principal polarization $p = x, y$ of two parallel fibers is $L_p = \pi / 2 \eta_p$, where $\eta_p = (\kappa_p - c_p \chi_p) / (1 - c_p^2)$ is the power transfer coefficient. Here, $\kappa_p, \chi_p$, and $c_p$ are the coefficients of directional coupling, self coupling, and butt coupling, respectively. The calculations of the power transfer coefficient $\eta_p$ for two parallel nanofibers have been reported in [22].
Figure 13. Power transfer coefficients $\eta_x$ (a) and $\eta_y$ (b) between two identical fibers calculated from the exact theory for the guided array modes (solid lines) and the coupled mode theory (dashed lines) as functions of the fiber radius $a$ for the fiber separation distances $d = 0$ and 200 nm. The light wavelength is $\lambda = 800$ nm. Other parameters are as in figure 2. The vertical dashed lines indicate the positions of the cutoffs of the odd array modes.

Figure 14. Power transfer coefficients $\eta_x$ (a) and $\eta_y$ (b) between two identical fibers calculated from the exact theory for the guided array modes (solid lines) and the coupled mode theory (dashed lines) as functions of the light wavelength $\lambda$ for the fiber separation distances $d = 0$ and 200 nm. The fiber radius is $a = 200$ nm. Other parameters are as in figure 3. The vertical dashed lines indicate the positions of the cutoffs of the odd array modes.

In the framework of the coupled mode theory [1–3], the transfer of power between the coupled fibers is a result of the beating between two normal modes. The power transfer coefficient is $\eta_p = (\beta_a^{(p)} - \beta_b^{(p)})/2$, where $\beta_a^{(p)}$ and $\beta_b^{(p)}$ are the propagation constants of the normal modes $a$ and $b$ with the principal polarization $p = x$ or $y$ (the $E_z$-cosine or $E_z$-sine type). For two identical nanofibers, the normal modes $a$ and $b$ are the even ($\nu = -1$) and odd ($\nu = 1$) modes, respectively. Hence, the power transfer coefficient $\eta_p$ can be calculated from the exact theory for the guided array modes [23].

We plot in figures 13–15 the power transfer coefficients $\eta_x$ and $\eta_y$ between two identical fibers calculated from the exact theory for the guided array modes (solid curves) and the coupled mode theory (dashed curves) as functions of the fiber radius $a$, the light wavelength $\lambda$, and the fiber separation distance $d$. In these figures, the vertical dashed lines indicate the positions of the cutoffs of the odd array modes that occur in the framework of the exact theory when $d$ is small and either $a$ is small or $\lambda$ is large in the ranges plotted. The coupled mode theory is not able to predict the mode cutoffs. Below a cutoff, the overlap between the modes of the different individual fibers is so significant that only even array modes exist. In this regime, the concept of power transfer between the fibers becomes meaningless and, hence, the coupled mode theory is not valid.

Thus, when the fiber separation distance is too small and either the fiber radius is too small or the wavelength of light is too large, the two nanofibers cannot be considered as independent or weakly coupled
Figure 15. Power transfer coefficients $\eta_x$ (a) and $\eta_y$ (b) between two identical fibers calculated from the exact theory for the guided array modes (solid lines) and the coupled mode theory (dashed lines) as functions of the fiber separation distance $d$ for the fiber radii $a = 150, 200, \text{ and } 250 \text{ nm}$. The light wavelength is $\lambda = 800 \text{ nm}$. Other parameters are as in figure 2. The vertical dashed lines indicate the positions of the cutoffs of the odd array modes.

Figure 16. Left panel: cross-sectional profiles of the electric intensity distributions $|E|^2$ of the fields in the $E_z$-cosine modes $I^0_{(\cos)}$ and $I^\pi_{(\cos)}$ of two nonidentical parallel nanofibers. Right panel: dependencies of the intensities $|E|^2$ of the fields in the modes on the $x$ and $y$ coordinates. The fiber radii are $a_1 = 220 \text{ nm}$ and $a_2 = 200 \text{ nm}$. The fiber separation distance is $d = 100 \text{ nm}$. Other parameters are as in figure 2.

waveguides. In this case, only two even-type principal array modes can be supported (like for a single nanofiber). This condition also marks the limits of validity for the coupled mode theory, which assumes weak coupling.

We observe from figures 13–15 that the results of the coupled mode theory agree well with those of the exact theory in the far-above-cutoff regions where the fiber radius is large, the wavelength of light is small, or the fiber separation distance is large. Near to a cutoff, the differences between the results of the exact and coupled mode theories are significant but not dramatic. For touching nanofibers ($d = 0$), the differences between the results of the two theories become large when the fiber radius $a$ is small (see figure 13) or the light wavelength $\lambda$ is large (see figure 14). Comparison between figures 13(a) and (b) and between...
Figure 17. Left panel: cross-sectional profiles of the electric intensity distributions $|\mathbf{E}|^2$ of the fields in the $\mathcal{E}_x$-sine modes $I_x^{(0)}$ and $I_x^{(\pi)}$ of two nonidentical parallel nanofibers. Right panel: dependencies of the intensities $|\mathbf{E}|^2$ of the fields in the modes on the $x$ and $y$ coordinates. The parameters used are the same as for figure 16.

figures 14(a) and (b) shows that, for touching nanofibers, the differences between the results of the two theories for the $x$-polarized modes are larger than those for the $y$-polarized modes.

3.4. Nonidentical nanofibers

It is desirable to know the effects of a difference between the radii of the nanofibers on the properties of the guided normal modes and the symmetry of the field intensity distributions. When the two nanofibers are not identical, the symmetry relations (8) and (9) for the $x$ coordinate are not valid and, hence, the guided array modes cannot be identified as even and odd modes anymore. We follow [23] and label the first two modes of the $\mathcal{E}_x$-sine type as $I_x^{(0)}$ and $I_x^{(\pi)}$. Similarly, we label the first two modes of the $\mathcal{E}_x$-cosine type as $I_x^{(0)}$ and $I_x^{(\pi)}$. Like the case of identical nanofibers, modes $I_x^{(0)}$ and $I_x^{(\pi)}$ have no cutoff, while modes $I_x^{(0)}$ and $I_x^{(\pi)}$ may have cutoffs. Note that the symmetry relations (10) and (11) for the $y$ coordinate remain valid for nonidentical nanofibers.

In the left panel of figure 16 the cross-sectional profiles of the electric intensity distributions $|\mathbf{E}|^2$ of the fields in the $\mathcal{E}_x$-cosine modes of two nonidentical parallel nanofibers are plotted. We show in the right panel of figure 16 the dependencies of $|\mathbf{E}|^2$ on the $x$ and $y$ coordinates. The corresponding results for the $\mathcal{E}_x$-sine modes are shown in figure 17. We observe from the figures that the intensity distribution $|\mathbf{E}|^2$ is asymmetric with respect to the coordinate $x$ and symmetric with respect to the coordinate $y$. These properties are obvious consequences of the geometry of the system of two nonidentical parallel nanofibers.

Figures 16(a), (c) and (d) show that the electric field intensity of the $I_x^{(\cos)}$ mode is dominant in the area between the nanofibers and, hence, atoms can be attracted to this area using a single red-detuned light field. We observe from figures 17(b) and (e) that the electric field intensity of the $I_x^{(\sin)}$ mode has a local minimum of exact zero at a point on the $x$ axis in the region between the nanofibers and, hence, atoms can be attracted to this area using a single blue-detuned light field.

Figures 16 and 17 show that, in the cases of $I_x^{(\cos,\sin)}$ modes, the intensity of the field in the area of the bigger nanofiber (nanofiber 1) is higher than that in the area of the smaller nanofiber (nanofiber 2). Meanwhile, in the cases of $I_x^{(\cos,\sin)}$ modes, the intensity of the field in the area of the smaller nanofiber (nanofiber 2) is higher than that in the area of the bigger nanofiber (nanofiber 1). We see that a slight difference between the radii of the nanofibers ($a_1 = 220$ nm and $a_2 = 200$ nm) leads to strong asymmetry of the intensity distributions of the guided normal modes. This feature is a consequence of the fact that the dependencies of the coupling between the nanofibers on the fiber radii are nonlinear.
4. Summary

We have studied the cross-sectional profiles and spatial distributions of the fields in guided normal modes of two coupled parallel optical nanofibers. We have shown that the distributions of the components of the field in a guided normal mode of two identical nanofibers are either symmetric or antisymmetric with respect to the radial principal axis $x$ and the tangential principal axis $y$ in the cross-sectional plane of the fibers. The symmetry of the magnetic field components with respect to the principal axes is opposite to that of the electric field components. We have found that, in the case of even guided modes, symmetric about the radial principal axis and symmetric about the tangential principal axis, the intensity distribution is dominant in the area between the fibers, with a saddle point at the two-fiber center. This feature may be used to trap atoms with a single red-detuned guided normal-mode light field. Meanwhile, in the case of odd guided modes, symmetric about the radial principal axis $x$ but asymmetric about the tangential principal axis $y$, the intensity distribution is dominant in the area between the fibers, with a saddle point at the two-fiber center. This feature may be used to attract atoms with a single blue-detuned guided normal-mode light field.

In this appendix, we summarize the key results of [23].

Appendix A. Equations for the expansion coefficients of the array modes

In this appendix, we summarize the key results of [23].

According to [23], the longitudinal components $\mathcal{E}_z$ and $\mathcal{H}_z$ of the electric and magnetic parts, respectively, of the field in a guided normal mode are given by equations (4) and (2). Since we consider only guided (bound) modes, equations (2) do not contain the Hankel functions and the modified Bessel functions of the first kind.

The Cartesian-coordinate transverse components $\mathcal{E}_\alpha$ and $\mathcal{H}_\alpha$ of the field are given by equations (4). In an arbitrary polar coordinate system $(\rho, \varphi)$ of the transverse plane, the radial components $\mathcal{E}_r$ and $\mathcal{H}_r$ and the azimuthal components $\mathcal{E}_\varphi$ and $\mathcal{H}_\varphi$ of the field are given as [1–3]

$$
\mathcal{E}_r = \frac{i\beta}{k_n^2 n_{ref}^2 - \beta^2} \left( \frac{\partial}{\partial r} \mathcal{E}_z + \frac{\omega \mu_0}{\beta} \frac{\partial}{\partial \varphi} \mathcal{H}_z \right),
$$

$$
\mathcal{E}_\varphi = \frac{i\beta}{k_n^2 n_{ref}^2 - \beta^2} \left( \frac{\partial}{\partial \varphi} \mathcal{E}_z - \frac{\omega \mu_0}{\beta} \frac{\partial}{\partial r} \mathcal{H}_z \right),
$$

$$
\mathcal{H}_r = \frac{i\beta}{k_n^2 n_{ref}^2 - \beta^2} \left( \frac{\partial}{\partial r} \mathcal{H}_z - \frac{\omega \epsilon_0 n_{ref}^2}{\beta} \frac{\partial}{\partial \varphi} \mathcal{E}_z \right),
$$

$$
\mathcal{H}_\varphi = \frac{i\beta}{k_n^2 n_{ref}^2 - \beta^2} \left( \frac{\partial}{\partial \varphi} \mathcal{H}_z + \frac{\omega \epsilon_0 n_{ref}^2}{\beta} \frac{\partial}{\partial r} \mathcal{E}_z \right).
$$

(A.1)

Like equations (4), equations (A.1) are consequences of the Maxwell equations and the translational symmetry of the system in the $z$ direction. It is clear that the radial components $\mathcal{E}_r$ and $\mathcal{H}_r$ and the azimuthal components $\mathcal{E}_\varphi$ and $\mathcal{H}_\varphi$ of the field in the local fiber-based polar coordinate system $\{r, \varphi\}$ can be calculated from the longitudinal components $\mathcal{E}_z$ and $\mathcal{H}_z$ using equations (A.1) for $\{r, \varphi\} = \{r_j, \varphi_j\}$. Equations (A.1) allow us to work straightforwardly with the boundary conditions for the individual circular nanofibers.

Acknowledgments

This work was supported by the Okinawa Institute of Science and Technology Graduate University.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).
The boundary conditions for the axial components $E_z$ and $H_z$ and the local-fiber-coordinate azimuthal components $E_{\phi_j}$ and $H_{\phi_j}$ of the electric and magnetic fields at the surface $S_j$ of fiber $j$ read

$$
E_z|_{S_j^-} = E_z|_{S_j^+}, \quad H_z|_{S_j^-} = H_z|_{S_j^+},
$$
$$
E_{\phi_j}|_{S_j^-} = E_{\phi_j}|_{S_j^+}, \quad H_{\phi_j}|_{S_j^-} = H_{\phi_j}|_{S_j^+},
$$
(A.2)

where the symbols $S_j^-$ and $S_j^+$ indicate the inner and outer sides of the surface of fiber $j$.

To apply the boundary conditions at the surface of nanofiber $j$, we express $E_z$ and $H_z$ outside the fibers as functions of $\eta$ and $\varphi_j$ using the Graf’s and Gegenbauer’s addition theorem [44]. According to this theorem, we have

$$
K_n(qr_1) \left( \frac{\cos \theta_1}{\sin \theta_1} \right) = \sum_{m=-\infty}^{+\infty} K_{n+m}(qW)I_m(qr_1) \left( \frac{\cos m\theta_1}{\sin m\theta_1} \right),
$$
$$
K_n(qr_2) \left( \frac{\cos \theta_2}{\sin \theta_2} \right) = \sum_{m=-\infty}^{+\infty} K_{n+m}(qW)I_m(qr_2) \left( \frac{\cos m\theta_2}{\sin m\theta_2} \right),
$$
(A.3)

where $r_1 = PO_1$, $r_2 = PO_2$, $W = O_1O_2$, $\theta_1 = \angle PO_1O_2 = \varphi_1$, and $\theta_2 = \angle PO_2O_1 = \pi - \varphi_2$ (see figure 1).

For the $E_z$-cosine modes, the expansion coefficients $E_{nij}$, $F_{nij}$, $G_{nij}$, and $H_{nij}$ vanish. We apply the boundary conditions (A.2) to these modes. Then, we find that the coefficients $A_{nij}$ and $B_{nij}$ for the field inside the fibers are given by the equations

$$
J_n(u_j)A_{nij} = K_n(w_j)C_{nij} + I_n(w_j) \sum_{m=0}^{\infty} f_{jmn}C_{nij},
$$
$$
(A.4)
$$
$$
J_n(u_j)B_{nij} = K_n(w_j)D_{nij} + I_n(w_j) \sum_{m=0}^{\infty} g_{jmn}D_{nij},
$$

while the coefficients $C_{nij}$ and $D_{nij}$ for the field outside the fibers are nonzero solutions of the equations [23]

$$
n \left( \frac{1}{u_j^2} + \frac{1}{w_j^2} \right) \left[ K_n(w_j)C_{nij} + I_n(w_j) \sum_{m=0}^{\infty} f_{jmn}C_{nij} \right] + \frac{\omega \mu_0}{\beta} \left[ J_n(u_j) \sum_{m=0}^{\infty} g_{jmn}D_{nij} \right] = 0,
$$
$$
n \left( \frac{1}{u_j^2} + \frac{1}{w_j^2} \right) \left[ K_n(u_j)D_{nij} + I_n(u_j) \sum_{m=0}^{\infty} f_{jmn}C_{nij} \right] + \frac{\omega \epsilon_0}{\beta} \left[ J_n(u_j) \sum_{m=0}^{\infty} g_{jmn}D_{nij} \right] = 0.
$$
(A.5)

Here, we have introduced the notation $j = 2$ or 1 for $j = 1$ or 2, respectively. We have also introduced the parameters

$$
u_j = h_1a_j, \quad w_j = q_0a_j,
$$
(A.6)

and the coefficients

$$
f_{1mn} = (-1)^m f_{nm}, \quad g_{1mn} = (-1)^m g_{nm},
$$
(A.7)

and

$$
f_{2mn} = (-1)^m f_{nm}, \quad g_{2mn} = (-1)^m g_{nm},
$$
(A.8)

where

$$
f_{nm} = K_{m+n}(qW) + K_{m-n}(qW) \text{ for } n > 0,$$
$$
f_{0,m} = K_m(qW),
$$
g_{0,m} = -K_{m+n}(qW) + K_{m-n}(qW),
$$
(A.9)

with $W = d + a_1 + a_2$ being the distance between the fiber centers.

For the $E_z$-sine modes, the expansion coefficients $A_{nij}$, $B_{nij}$, $C_{nij}$, and $D_{nij}$ vanish. We apply the boundary conditions (A.2) to these modes. Then, we find that the coefficients $E_{nij}$ and $F_{nij}$ for the field inside the fibers...
are given by the equations

\[ J_n(u_j)E_{nj} = K_n(w_j)G_{nj} + I_n(w_j) \sum_{m=0}^{\infty} g_{nm} G_{nj} \]  
(A.10)

\[ J_n(u_j)F_{nj} = K_n(w_j)H_{nj} + I_n(w_j) \sum_{m=0}^{\infty} f_{nm} H_{nj}, \]

while the coefficients \( G_{nj} \) and \( H_{nj} \) for the field outside the fibers are nonzero solutions of the equations [23]

\[ n \left( \frac{1}{u_j^2} + \frac{1}{u_j^2} \right) \left[ K_n(w_j)G_{nj} + I_n(w_j) \sum_{m=0}^{\infty} g_{nm} G_{nj} \right] = \frac{\omega \mu_0}{\beta} \left[ \frac{J_n'(u_j)}{u_j I_n(u_j)} + \frac{K_n'(w_j)}{w_j K_n(w_j)} \right] K_n(w_j)H_{nj} \]

\[ - \frac{\omega \mu_0}{\beta} \left[ \frac{J_n'(u_j)}{u_j I_n(u_j)} + \frac{I_n'(w_j)}{w_j I_n(w_j)} \right] I_n(w_j) \sum_{m=0}^{\infty} f_{nm} H_{nj} = 0, \]

\[ n \left( \frac{1}{u_j^2} + \frac{1}{u_j^2} \right) \left[ K_n(w_j)H_{nj} + I_n(w_j) \sum_{m=0}^{\infty} f_{nm} H_{nj} \right] = \frac{\omega \epsilon_0}{\beta} \left[ \frac{n_i^2 J_n'(u_j)}{u_j I_n(u_j)} + \frac{n_i^2 K_n'(w_j)}{w_j K_n(w_j)} \right] K_n(w_j)G_{nj} \]

\[ - \frac{\omega \epsilon_0}{\beta} \left[ \frac{n_i^2 J_n'(u_j)}{u_j I_n(u_j)} + \frac{n_i^2 I_n'(w_j)}{w_j I_n(w_j)} \right] I_n(w_j) \sum_{m=0}^{\infty} g_{nm} G_{nj} = 0. \]  
(A.11)

We emphasize that equations (A.5) and (A.11) contain the mode expansion coefficients with the indices \( j \) and \( j \), where \( j = 2 \) or 1 for \( j = 1 \) or 2, respectively. They are in agreement with the results of [23], presented in the form of \( 4 \times 4 \) matrix minors.

We consider the particular case where the two fibers are identical, that is, the two fibers have the same radius \( a_1 = a_2 = a \) and the same core refractive index \( n_1 = n_2 = n_\rlap{c} \). In this case, we have \( h_1 = h_2 = u_1 = u_2 \), and \( v_1 = v_2 \). Then, for the \( \varepsilon_x \)-cosine modes, we find

\[ A_{m2} = (-1)^m \nu A_{m1}, \quad B_{m2} = (-1)^m \nu B_{m1}, \]

\[ C_{m2} = (-1)^m \nu C_{m1}, \quad D_{m2} = (-1)^m \nu D_{m1}, \]  
(A.12)

and, for the \( \varepsilon_x \)-sine modes, we get

\[ E_{m2} = (-1)^m \nu E_{m1}, \quad F_{m2} = (-1)^m \nu F_{m1}, \]

\[ G_{m2} = (-1)^m \nu G_{m1}, \quad H_{m2} = (-1)^m \nu H_{m1}, \]  
(A.13)

where \( \nu = -1 \text{ or } +1 \) corresponds to the even or odd mode [23]. Hence, equations (A.5) for the \( \varepsilon_x \)-cosine modes take the form [23]

\[ n \left( \frac{1}{u_j^2} + \frac{1}{u_j^2} \right) \left[ K_n(w)C_{m1} + I_n(w) \sum_{m=0}^{\infty} \nu g_{nm} C_{m1} \right] + \frac{\omega \mu_0}{\beta} \left[ \frac{J_n'(u)}{u I_n(u)} + \frac{K_n'(w)}{w K_n(w)} \right] K_n(w)D_{m1} \]

\[ + \frac{\omega \mu_0}{\beta} \left[ \frac{J_n'(u)}{u I_n(u)} + \frac{I_n'(w)}{w I_n(w)} \right] I_n(w) \sum_{m=0}^{\infty} \nu g_{nm} D_{m1} = 0, \]

\[ n \left( \frac{1}{u_j^2} + \frac{1}{u_j^2} \right) \left[ K_n(w)D_{m1} + I_n(w) \sum_{m=0}^{\infty} \nu g_{nm} D_{m1} \right] + \frac{\omega \epsilon_0}{\beta} \left[ \frac{n_i^2 J_n'(u)}{u I_n(u)} + \frac{n_i^2 K_n'(w)}{w K_n(w)} \right] K_n(w)C_{m1} \]

\[ + \frac{\omega \epsilon_0}{\beta} \left[ \frac{n_i^2 J_n'(u)}{u I_n(u)} + \frac{n_i^2 I_n'(w)}{w I_n(w)} \right] I_n(w) \sum_{m=0}^{\infty} \nu g_{nm} C_{m1} = 0, \]  
(A.14)

and equations (A.11) for the \( \varepsilon_x \)-sine modes become [23]

\[ n \left( \frac{1}{u_j^2} + \frac{1}{u_j^2} \right) \left[ K_n(w)G_{m1} + I_n(w) \sum_{m=0}^{\infty} \nu g_{nm} G_{m1} \right] - \frac{\omega \mu_0}{\beta} \left[ \frac{J_n'(u)}{u I_n(u)} + \frac{K_n'(w)}{w K_n(w)} \right] K_n(w)H_{m1} \]

\[ - \frac{\omega \mu_0}{\beta} \left[ \frac{J_n'(u)}{u I_n(u)} + \frac{I_n'(w)}{w I_n(w)} \right] I_n(w) \sum_{m=0}^{\infty} \nu g_{nm} H_{m1} = 0, \]
\[
\begin{aligned}
n \left( \frac{1}{w^2} + \frac{1}{u^2} \right) & \left[ K_n(w)H_{n1} + I_n(w) \sum_{m=0}^{\infty} \nu_{m,n} H_{m1} \right] - \omega_0 \frac{\eta_n^2}{\beta} \left[ \frac{\eta_n^2 K_n^2(w)}{u J_n(u)} + \frac{\eta_n^2 K_n^2(w)}{w K_n(w)} \right] K_n(w) G_{n1} \\
- \omega_0 \frac{\eta_n^2}{\beta} \left[ \frac{\eta_n^2 K_n^2(w)}{u J_n(u)} + \frac{\eta_n^2 K_n^2(w)}{w K_n(w)} \right] & I_n(w) \sum_{m=0}^{\infty} \nu_{m,n} G_{m1} = 0.
\end{aligned}
\] (A.15)

**ORCID iDs**

Fam Le Kien [https://orcid.org/0000-0002-3463-6053](https://orcid.org/0000-0002-3463-6053)
Síle Nic Chormaic [https://orcid.org/0000-0003-4276-2014](https://orcid.org/0000-0003-4276-2014)

**References**

[34] Chu S 1998 Rev. Mod. Phys. 70 685