Monotonicity formulae for smooth extremizers of integral functionals

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Abstract

A general monotonicity formula for smooth constrained local extremizers of first-order integral functionals subject to non-holonomic constraints is established. The result is then applied to recover some known monotonicity formulae and to discover some new monotonicity formulae of potential value.

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1 Introduction

Our aim in this paper is to investigate the validity of monotonicity formulae for extremizers of certain variational integrals. An example of such a result is given by what is now a classical monotonicity formula for harmonic maps. This formula was found for the first time by Price [13] for maps with values on the sphere. Monotonicity formulae for minimizing harmonic maps and for stationary harmonic maps between manifolds were established by Schoen and Uhlenbeck [16] and Bethuel [3], respectively. Evans [8] used a monotonicity formula to prove partial regularity of harmonic maps with values on the sphere, as did Bethuel [3] for harmonic maps into general manifolds. In the expository paper, Evans [9] showed, formally, how to obtain monotonicity formulae for some first-order integral functionals. Here we build on that work to obtain monotonicity formulae for functionals subject to non-holonomic constraints. In so doing, we follow the argument of Evans [9], which is based on the use of the strong form of the Euler–Lagrange equation of the functionals under consideration, and therefore consider only smooth integrands and smooth stationary points.

The paper is organized as follows. In Section 2, we will deal with constrained integral functionals of type

$$F[u] := \int_{\Omega} f(u, Du)\, dv, \quad u \in N, \tag{1}$$

where f is of class C^2 on $\mathbb{R}^m \times \mathbb{R}^{m\times n}$, dv is the volume element on $\mathbb{R}^n$, and N is a smooth submanifold of $\mathbb{R}^m$. Under suitable conditions on f and u, we establish a general monotonicity formula along constrained local extremizers of F. In Section 3, we present some illustrative examples, many of which are known, to which we are able to apply rigorously our result. Finally, in Section 4 we discuss further examples, emulating the formal spirit of the paper by Evans [9]. Since our main result deals only with smooth extremizers, it cannot in principle be applied to establish regularity. However, we believe that it should be useful for guessing which kind of monotonicity formula should hold for weak constrained local extremizers of integral functional of the type we consider.

2 General monotonicity formula

Let $h : \mathbb{R}^m \to \mathbb{R}$ be of class C^1 and satisfy $\partial_y h(y) \neq 0$ for all y in $\mathbb{R}^m$. Then N defined by

$$N := \{y \in \mathbb{R}^m : h(y) = 0\} \tag{2}$$

is a submanifold of $\mathbb{R}^m$ and given y in $\mathbb{R}^m$ sufficiently close to N there exists a unique smooth orthogonal projection of y, denoted by $\Pi(y)$, onto N. Let Ω be an open and bounded subset of $\mathbb{R}^n$, define a space X of functions by

$$X := \{u \in C^2(\Omega; \mathbb{R}^m) : u(x) \in N \text{ for any } x \in \Omega\}, \tag{3}$$

and consider an integral functional $F : X \to \mathbb{R}$ of the form

$$F[u] := \int_{\Omega} f(u, Du)\, dv, \tag{4}$$

where $f : \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is prescribed and of class C^2.

We say that an element u of X is a *constrained local extremizer* of F if

$$\left(\frac{d}{dt} F[\Pi(u + t\phi)]\right)_{|t=0} = 0 \quad \forall \phi \in C_c^{\infty}(\Omega; \mathbb{R}^m). \tag{5}$$

The next lemma provides the basis for obtaining a general monotonicity formula.

Lemma 2.1 *Let u be a constrained local extremizer of F and define $L : \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}^{n\times n}$ by*

$$L(y,G) := f(y,G)I - G^{\top}S(y,G), \tag{6}$$

where I denotes the identity on $\mathbb{R}^n$ and $S : \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ is defined by

$$S(y,G) := \partial_G f(y,G). \tag{7}$$

Then

$$\mathrm{div} L(u,Du) = 0 \qquad \textit{and} \qquad \mathrm{div}\,(L^{\top}(u,Du)v) = \mathrm{tr} L(u,Du) \tag{8}$$

on Ω, where $L^{\top} : \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}^{n\times n}$ is the transpose of L and $v : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$v(x) := x - \bar{x}, \tag{9}$$

with $\bar{x}$ being a fixed element of $\mathbb{R}^n \setminus (\Omega \cup \partial\Omega)$.

Proof As a consequence of (5), we find that there must exist a Lagrangian multiplier field $\lambda : \Omega \to \mathbb{R}$ such that

$$\partial_u f(u,Du) - \mathrm{div} S(u,Du) = \lambda \partial_u h(u) \tag{10}$$

on Ω. Since $Dh(u) = 0$ for all u in X, we see from (10) that

$$\begin{aligned}
\mathrm{div} L(u,Du) &= Df(u,Du) - D((Du)^{\top})S(u,Du) - (Du)^{\top}\mathrm{div} S(u,Du) \\
&= (Du)^{\top}(\partial_u f(u,Du) - \mathrm{div} S(u,Du)) \qquad (11)\\
&\qquad + D((Du)^{\top})(S(u,Du) - S(u,Du)) \\
&= \lambda (Du)^{\top}\partial_u h(u) \\
&= \lambda Dh(u) \\
&= 0 \qquad (12)
\end{aligned}$$

on Ω, which verifies that the first of (8) holds on Ω. We next note that, by the identity $\mathrm{div}\,(M^{\top}w) = w \cdot \mathrm{div} M + M \cdot Dw$, which applies for all $M : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $w : \mathbb{R}^n \to \mathbb{R}^n$ of class C^1, and the first of (8) that

$$\begin{aligned}
\mathrm{div}\,(L^{\top}(u,Du)v) - \mathrm{tr} L(u,Du) &= v \cdot \mathrm{div} L(u,Du) + L(u,Du)\cdot I - \mathrm{tr} L(u,Du) \\
&= 0 \qquad (13)
\end{aligned}$$

on Ω, which verifies that the second of (8) holds on Ω and completes the proof.

Remark The identity $(8)_2$ is a conservation law that can be derived by applying Noether's [12] theorem on invariant variational principles to the functional F of (4). The conserved quantity $L(u,Du) = f(u,Du)I - (Du)^{\top}S(u,Du)$ can be recognized as the counterpart, for the argument u of the functional F, of Eshelby's [7, §7] energy-momentum tensor. In a work concerned with maps between Riemannian manifolds, Baird & Eells [2] were perhaps the first workers to recognize how such conserved quantities can be exploited to derive monotonicity formulae. Subequent developments along these lines are reviewed by Dong & Lin [5]. Evans [9] exploits the existence of such conserved quantities to obtain monotonicity formulae for some particular unconstrained first-order integral functionals with integrands of the form $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ but does not construct a general argument akin to that obtained in the present work.

We next use Lemma 2.1 to establish a monotonicity formula for a suitable integral quantity involving constrained local extremizers of F granted that its integrand f satisfies certain conditions to be made explicit. Specifically, we have the following result:

Theorem 2.2 *Let $A : \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ of class C^0, $B : \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$, and q in $\mathbb{R}$ be chosen such that*

$$\mathrm{tr}\, L(y, G) = (n - q)A(y, G) + B(y, G) \quad \forall (y, G) \in \mathbb{R}^m \times \mathbb{R}^{m\times n}. \tag{14}$$

Suppose that $u \in X$ is a constrained local extremizer of F. Assume that the inequalities

$$A(u, Du) \geq f(u, Du) - (Du)e \cdot S(u, Du)e \tag{15a}$$

and

$$B(u, Du) \geq 0 \tag{15b}$$

apply on Ω, where, recalling the definition (9) of $v : \mathbb{R}^n \to \mathbb{R}^n$, $e : \mathbb{R}^n \to \mathbb{S}^{n-1}$ is defined by

$$e(x) = \frac{v(x)}{|v(x)|} = \frac{x - \bar{x}}{|x - \bar{x}|}. \tag{16}$$

Then, given $x_0 \in \Omega$ and $R > 0$ such that $B_R(x_0) \subset \Omega$, $E : (0, R) \to \mathbb{R}$ defined by

$$E(r) := \frac{1}{r^{n-q}} \int_{B_r(x_0)} A(u, Du)\, dv \tag{17}$$

is monotonically increasing.

Remark The quantity on the right-hand side of (15a) arises in the second corner condition of Weierstrass [17] and Erdmann [6] and, thus, is not a newly discovered object.

Proof We first note that, for any r in $(0, R)$,

$$E'(r) = -\frac{n-q}{r^{n-q+1}} \int_{B_r(x_0)} A(u, Du)\, dv + \frac{1}{r^{n-q}} \frac{d}{dr} \int_{B_r(x_0)} A(u, Du)\, dv. \tag{18}$$

To compute the derivative with respect to r in the integral in the second term on the right-hand side of (18), we observe that, by the coarea formula,

$$\int_{B_r(x_0)} A(u, Du)\, dv = \int_0^r \int_{\{|x-x_0|=t\}} A(u, Du)\, da\, dt, \tag{19}$$

where da denotes the surface area measure. Thus, since A is of class C^0, we have that

$$\frac{d}{dr} \int_{B_r(x_0)} A(u, Du)\, dx = \int_{\partial B_r(x_0)} A(u, Du)\, da. \tag{20}$$

By (14), Lemma 2.1, and the divergence theorem, we next see that

$$\begin{aligned}(n-q) \int_{B_r(x_0)} A(u, Du)\, dv - \int_{B_r(x_0)} B(u, Du)\, dv &= \int_{B_r(x_0)} \operatorname{tr} L(u, Du)\, dv \\ &= \int_{B_r(x_0)} \operatorname{div}(L^\top(u, Du) z)\, dv \\ &= r \int_{\partial B_r(x_0)} \nu \cdot L(u, Du)\nu\, da,\end{aligned} \tag{21}$$

where ν denotes an oriented unit normal on $\partial B_r(x_0)$. Using (20) and (21) in (18), recalling the definition (6) of L, and invoking the assumptions (15a) and (15b), we deduce that

$$\begin{aligned}E'(r) = \frac{1}{r^{n-q}} \int_{\partial B_r(x_0)} (A(u, Du) - f(u, Du) + S(u, Du)\nu \cdot (Du)\nu)\, da \\ + \frac{1}{r^{n-q+1}} \int_{B_r(x_0)} B(u, Du)\, dv \geq 0\end{aligned} \tag{22}$$

on $(0, R)$. Thus, as claimed, that E defined in (17) is monotonically increasing.

The conclusion of Theorem 2.2 follows from more restrictive, but nevertheless reasonable, hypotheses, as the next corollary shows.

Corollary 2.3 *Let $A : \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ of class C^0, $B : \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$, and q in $\mathbb{R}$ be such that*

$$nf(y, G) - G \cdot S(y, G) = (n-q)A(y, G) + B(y, G) \quad \forall (y, G) \in \mathbb{R}^m \times \mathbb{R}^{m\times n}. \tag{23}$$

Suppose that $u \in X$ is a constrained local extremizer of F. Assume that the inequalities

$$A(u, Du) \geq f(u, Du), \quad (Du)e \cdot S(u, Du)e \geq 0, \quad \text{and} \quad B(u, Du) \geq 0 \tag{24}$$

apply on Ω, where $e : \mathbb{R}^n \to \mathbb{S}^{n-1}$ is as defined in (16). Then, given $x_0 \in \Omega$ and $R > 0$ such that $B_R(x_0) \subset \Omega$, $E : (0, R) \to \mathbb{R}$ as defined in (17) is monotonically increasing.

3 Some examples in the unconstrained case

In this section we consider some examples for which it is possible to prove, applying standard regularity results for Dirichlet minimizers (see, for instance, Dacorogna [4, Thm. 4.11]), the existence of C^2-local extremizers.

3.1 Homogeneous case

Assume that f belonging to $C^2(\mathbb{R}^{m\times n})$ is positively p-homogeneous for some $p > 1$, so that $f(tG) = t^p f(G)$ for each $t > 0$ and for each $G \in \mathbb{R}^{m\times n}$. Then, by Euler's Theorem on homogeneous functions, $G \cdot S(G) = pf(G)$. To ensure that the hypothesis (23) of Corollary 2.3 holds, it is then natural to select A and B such that

$$A := f \qquad \text{and} \qquad B := 0, \tag{25}$$

in which case $(24)_1$ and $(24)_3$ are trivially satisfied. If $(24)_2$ holds as well, we thus find that $E : (0, R) \to \mathbb{R}$ defined by

$$E(r) = \frac{1}{r^{n-q}} \int_{B_r(x_0)} f(Du)\, dv \tag{26}$$

increases monotonically for any C^2-extremizer u of F.

A simple but illustrative example of a positively q-homogeneous integrand $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is given by

$$f(G) := \frac{\alpha}{p}|G|^p, \quad \alpha > 0. \tag{27}$$

In this case, $S(G) = \alpha|G|^{p-2}G$ for all G in $\mathbb{R}^{m\times n}$. Thus, $(24)_2$ reduces to

$$\alpha|G|^{p-2}|Ge|^2 \geq 0 \quad \forall (e, G) \in \mathbb{S}^{n-1} \times \mathbb{R}^{m\times n}, \tag{28}$$

and, recalling that $p > 1$, holds trivially.

3.2 Ginzburg–Landau functional

Let $W : \mathbb{R}^m \to [0, +\infty)$ be a smooth function, let $\alpha > 0$, and suppose that f has the form

$$f(y, G) := W(y) + \frac{\alpha}{2}|G|^2, \quad \alpha > 0. \tag{29}$$

Notice that $(24)_2$ holds as with the previously discussed example $f(G) = \alpha|G|^q/q$. Moreover, we see that

$$nf(y, G) - G \cdot S(y, G) = nW(y) + \frac{n\alpha}{2}|G|^2 - |G|^2 = nW(y) + \frac{(n-2)\alpha}{2}|G|^2. \tag{30}$$

We can therefore consider some particular choices for A and q. First, let $q = 2$. Then:

- A natural choice for A might be $A := f$, in which case $B = 2W \geq 0$ and Corollary 2.3 yields a monotonicity formula for $E : (0, R) \to \mathbb{R}$ defined by

$$E(r) = \frac{1}{r^{n-2}} \int_{B_r(x_0)} \Big(W(u) + \frac{\alpha}{2}|Du|^2 \Big)\, dv. \tag{31}$$

  Although Alikakos [1] previously established this result for $n = m \geq 2$, our argument shows that it holds for all choices of m and n satisfying $n \geq 1$ and $m \geq 1$.

- If, alternatively, $n \geq 2$ and we take $A : \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ to be of the form

$$A(y, G) := \frac{n}{n-2} W(y) + \frac{\alpha}{2}|G|^2, \tag{32}$$

  then $A \geq f$ and $B = 0$, whereby Corollary 2.3 yields a monotonicity formula for $E : (0, R) \to \mathbb{R}$ defined by

$$E(r) = \frac{1}{r^{n-2}} \int_{B_r(x_0)} \Big(nW(u) + \frac{\alpha(n-2)}{2}|Du|^2 \Big)\, dv. \tag{33}$$

  The result was previously established by Sourdis [14], who referred to it as the *weak monotonicity formula*.

3.3 Ginzburg–Landau functional in the presence of Modica's gradient bound

In this paragraph we still consider f of the form

$$f(y, G) := W(y) + \frac{\alpha}{2}|G|^2, \quad \alpha > 0, \tag{34}$$

with the provision that u satisfies the gradient bound

$$\frac{\alpha}{2}|Du|^2 \leq W(u) \tag{35}$$

on Ω. Modica [10] established this bound for $m = 1$ under the assumption that u is a bounded solution, on the whole $\mathbb{R}^n$, of the Euler–Lagrange equation

$$\alpha \Delta u = W'(u) \tag{36}$$

corresponding to the choice (34) of the integrand f. Although (35) holds for $m = 1$, Smyrnelis [15] provides counterexamples showing that it can fail for $m > 1$. For this reason in the present subsection we confine our attention to the case $m = 1$.

- For the choices $A := f$ and $q = 1$, we see, from (14) and (35), that

$$B(u, Du) = W(u) - \frac{\alpha}{2}|Du|^2 \geq 0 \tag{37}$$

on Ω. Referring to Corollary 2.3, we thus obtain a monotonicity formula for

$$E(r) = \frac{1}{r^{n-1}} \int_{B_r(x_0)} \Big(W(u) + \frac{\alpha}{2}|Du|^2 \Big)\, dv. \tag{38}$$

This result was previously established by Modica [10].

- If we choose A to be of the form

$$A(y, G) := \frac{n}{n-1} W(y) + \frac{(n-2)\alpha}{2(n-1)}|G|^2 \quad (y, G) \in \mathbb{R}^m \times \mathbb{R}^{m\times n} \tag{39}$$

while continuing to take $q = 1$, then we see from (35) that

$$A(u, Du) \geq f(u, Du) \qquad \text{and} \qquad B(u, Du) = 0 \tag{40}$$

on Ω. Thus, by Corollary 2.3, we obtain a monotonicity formula for $E : (0, R) \to \mathbb{R}$ defined by

$$E(r) = \frac{1}{r^{n-1}} \int_{B_r(x_0)} \Big(nW(y) + \frac{(n-2)\alpha}{2}|Du|^2 \Big)\, dv. \tag{41}$$

This result was previously established by Sourdis [14] for $n \geq 2$, who called it the *strong monotonicity formula*. Notice that, as a consequence of (35), the monotonicity formula for (41) holds also for $n = 1$.

- If $n \geq 2$, $q = n-1$, and $A := 2W$, then the first of (40) holds on Ω as in the previous example but

$$B(u, Du) = (n-2)\Big(W(u) + \frac{\alpha}{2}|Du|^2 \Big) \geq 0, \tag{42}$$

whereby Corollary 2.3 yields a monotonicity formula for $E : (0, R) \to \mathbb{R}$ defined by

$$E(r) = \frac{1}{r} \int_{B_r(x_0)} W(u)\, dv. \tag{43}$$

This generalizes a result obtained by Smyrnelis [15] for $n = 2$ to all $n \geq 2$.

3.4 Anisotropic Ginzburg–Landau functional

We next investigate an anisotropic variant of the previous example in which $W : \mathbb{R}^m \to [0, +\infty)$ is of class C^2 but $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is defined such that

$$f(y, G) := W(y) + \frac{1}{2}\alpha(\hat{G})|G|^2, \quad \hat{G} = \frac{G}{|G|}. \tag{44}$$

In this case,

$$S(y, G) = \alpha(\hat{G})G + \frac{1}{2}|G|\partial_{\hat{G}}\alpha(\hat{G}) \quad \forall (y, G) \in \mathbb{R} \times \mathbb{R}^n, \tag{45}$$

where $\partial_{\hat{G}}\alpha(\hat{G})$ is such that $G \cdot \partial_{\hat{G}}\alpha(\hat{G}) = 0$ for all G in $\mathbb{R}^{m\times n}$. Choosing $A := f$ and $q = 1$ and stipulating that the generalization

$$\frac{1}{2}\alpha\Big(\frac{Du}{|Du|}\Big)|Du|^2 \le W(u) \tag{46}$$

of the gradient bound (35) holds on Ω, we then see that $B(u, Du) \ge 0$ on Ω and, as a consequence of Corollary 2.3, arrive as a monotonicity formula for $E : (0, R) \to \mathbb{R}$ defined by

$$E(r) = \frac{1}{r^{n-2}} \int_{B_r(x_0)} \Big(W(u) + \frac{1}{2}\alpha\Big(\frac{Du}{|Du|}\Big)|Du|^2\Big)\, dv. \tag{47}$$

3.5 Area-type functionals

As a penultimate application, we consider area-type functionals F of the form

$$F(u) := \int_\Omega (c + |Du|^t)^s\, dv, \tag{48}$$

with c, t, and s being positive elements of $\mathbb{R}$. The integrand $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ of this functional is of class C^2 if either $t = 2$ or $t \ge 4$. To apply standard regularity results, we must also assume that t and s satisfy $ts > 1$, which ensures that f grows superlinearly with its argument (ruling out the special case $c = 1$, $t = 2$, and $s = 1/2$ corresponding to the area functional for a graph). For this choice, $S : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ is given by

$$S(G) = ts(c + |G|^t)^{s-1}|G|^{t-2}G \tag{49}$$

and we see that

$$\begin{aligned} nf(G) - G\cdot S(G) &= n(c + |G|^t)^s - ts(c + |G|^t)^{s-1}|G|^t \\ &= (n - ts)(c + |G|^t)^s + ts(c + |G|^t)^s\left(1 - \frac{|G|^t}{c + |G|^t}\right). \end{aligned} \tag{50}$$

We may now choose A and B such that

$$A(G) := f(G) \quad \text{and} \quad B(G) := ts(c+|G|^t)^s)\Big(1 - \frac{|G|^t}{c + |G|^t}\Big) \qquad \forall G \in \mathbb{R}^{m\times n} \tag{51}$$

and set $q := ts$. Since $c > 0$, we then infer that $B \ge 0$. We therefore confirm that $(24)_2$ holds and, applying Corollary 2.3, arrive at a monotonicity formula for $E : (0, R) \to \mathbb{R}$ defined by

$$E(r) = \frac{1}{r^{n-ts}} \int_{B_r(x_0)} (c + |Du|^t)^s\, dv. \tag{52}$$

4 Some formal examples

In this final section, we consider further examples only formally, without considering the requirements needed to ensure that all regularity assumptions required of the integrand f and argument u of the functional F are satisfied.

4.1 Constrained harmonic maps

Emulating the essential features of the discussion of p-homogeneous integrands in Section 3.1, we may apply Theorem 2.2 to the choice $f(G) := |G|^2$ of $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ with an effective constraint manifold N and on that basis obtain a monotonicity formula for harmonic maps with values belonging to N.

4.2 1-Laplacian

The (constrained) 1-Laplacian, that is

$$F(u) := \int_\Omega |Du|\, dx, \quad u \in N, \tag{53}$$

can be treated as in Section 3.1 and we obtain a monotonicity formula for $E : (0, R) \to \mathbb{R}$ defined by

$$E(r) = \frac{1}{r^{n-1}} \int_{B_r(x_0)} |Du|\, dv. \tag{54}$$

Such a formula has been considered by Evans [9], used it to formally deduce that, for a level set Γ of u,

$$\frac{\mathcal{H}^{n-1}(B_r(x_0)\cap\Gamma)}{r^{n-1}} \leq \frac{\mathcal{H}^{n-1}(B_R(x_0)\cap\Gamma)}{R^{n-1}}, \tag{55}$$

where x_0 belongs to Γ. This is a standard monotonicity formula for minimal hypersurfaces.

4.3 A singular quasi-convex functional

In conclusion, we show that Theorem 2.2 formally applies also to a generalization of an interesting example investigated by Evans [9] in the case $m = n$. Specifically, we consider the functional

$$F[u] = \int_\Omega f(Du)\, dv, \tag{56}$$

with integrand $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ defined according to

$$f(G) := \begin{cases} \dfrac{\mu}{2}|G|^2 + \dfrac{\kappa}{\det(G^\top G)^{\gamma/2}}, & \text{if } \det(G^\top G) > 0, \\ +\infty, & \text{if } \det(G^\top G) \leq 0, \end{cases} \tag{57}$$

where μ, κ, and γ are given positive elements of $\mathbb{R}$. A distinguishing feature of this example is that (15a) holds for a suitable choice of A but not $(24)_1$ and $(24)_2$ separately, meaning that we must rely on Theorem 2.2 instead of Corollary 2.3. First, we observe that, for any G in $\mathbb{R}^{m\times n}$ with $\det(G^\top G) > 0$ and for any $t > 0$, we have that

$$S(G) = \mu G - \frac{\kappa\gamma G(G^\top G)^{-1}}{\det(G^\top G)^{\gamma/2}}. \tag{58}$$

Thus, granted that $\det(G^\top G) > 0$, we see that

$$nf(G) - G\cdot S(G) = \frac{(n-2)\mu}{2}|G|^2 + \frac{n\kappa(\gamma+1)}{\det(G^\top G)^{\gamma/2}}. \tag{59}$$

Next, we take A to be given by

$$A(G) := \frac{\mu}{2}|G|^2 + \frac{\kappa(\gamma+1)}{\det(G^\top G)^{\gamma/2}} \quad \forall G \in \mathbb{R}^{m\times n} \tag{60}$$

and suppose that $q = 2$. We then find that

$$B(G) = \frac{2\kappa(\gamma+1)}{\det(G^\top G)^{\gamma/2}} \geq 0 \tag{61}$$

and, moreover, that

$$A(G) - f(G) + Ge\cdot S(G)e = \mu|Ge|^2 \geq 0. \tag{62}$$

Thus, we obtain a monotonicity formula for

$$E(r) = \frac{1}{r^{n-2}} \int_{B_r(x_0)} \Big(\frac{\mu}{2}|Du|^2 + \frac{\kappa(\gamma+1)}{\det((Du)^\top Du)^{\gamma/2}}\Big)\, dv. \tag{63}$$

This generalizes a result obtained by Evans [9] in the special case $m = n$.

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