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Geodesic string condensation from symmetric tensor gauge theory: A unifying framework of holographic toy models

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We propose that there is a universal picture for different constructions of holographic toy models, and its continuous limit can be described by a gravitylike field theory, namely, the rank-2 U(1) gauge theory. First, we show that two different toy models for holography—the perfect tensor networks and the hyperbolic fracton models—are both equivalent to a picture of evenly distributed bit threads on geodesics in the anti–de Sitter space. We name this picture "geodesic string condensation." It is actually a natural leading-order approximation to the holographic entanglement structure. Then, we reason that the rank-2 U(1) gauge theory with linearized diffeomorphism as its gauge symmetry, also known as a case of Lifshitz gravity, is the bulk field theory that gives rise to this picture. The Gauss's laws and spatial curvature require the geodesic gauge field lines, instead of the local loops (magnetic fields), to be the fundamental dynamical variables, which lead to geodesic string condensation. These results provide an intuitive way to understand the entanglement structure of gravity in anti–de Sitter/conformal field theory.

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Introduction. Modern physics has entered an exciting era of exploring profound connections between quantum many-body systems, quantum gravity, and quantum information. Many of these interdisciplinary conversations revolve around the holographic principle [1,2] and anti–de Sitter/conformal field theory (AdS/CFT) correspondence [3,4]. As a conjectured duality between quantum gravity in $(d+1)$-dimensional asymptotically AdS space-time and a d-dimensional CFT on its boundary, the AdS/CFT duality is a profound insight of quantum gravity, and also acts as a powerful tool for condensed matter problems [5].

In 2006, Ryu and Takayanagi conjectured that the entanglement entropy of a boundary CFT segment is measured by the area of the corresponding extremal covering surface in the AdS geometry [6,7]. This conjecture reveals the intimate relation between entanglement and geometry in quantum gravity. Following the insight of Swingle [8], various tensor-network holographic toy models were built [8–14], and they uncover the quantum-informational correcting feature of holographic entanglement.

On the condensed matter hemisphere, the fracton states of matter were studied intensively in recent years [15–23]. The gapless versions of fracton states, namely, rank-2 U(1) (R2-U1) gauge theories [21,22,24,25], were found to be certain limits of Lifshitz gravity [26,27]. The charge excitations dubbed "fractons" also show gravitational attractions [28]. Inspired by these discoveries, a toy fracton model in AdS space was studied and shown to satisfy holographic properties in a similar fashion as the holographic tensor networks [29,30].

Some important questions remain unanswered despite this progress. First, what is the connection between the different holographic toy models? Is there a universal picture behind them? Furthermore, how can we derive its continuous limit from a bulk field theory, and how is the bulk theory related to gravity? The perfect tensor networks are clever constructions directly based on holographic entanglement properties, but their correspondence to any concrete bulk field theory is still unknown. The hyperbolic fracton model so far is a classical spin model in the bulk, and still far from a field theory that shows satisfactory resemblance to gravity.

In this work, we advance our understanding of holographic toy models by addressing these questions. First, we point out that there is a universal picture behind different constructions of holographic toy models: a homogeneous and isotropic distribution of bit threads (up to some lattice discretization). We then show that the traceful, vector charged R2-U1, a theory with the spatial part of linearized diffeomorphsim as its gauge symmetry, gives rise to this continuous bit-thread picture. We reason that in the presence of spatial curvature, the gauge symmetry forbids the local magnetic field (local loops of gauge field), and only allows gauge field lines along a geodesics to be the fundamental dynamical variables. Hence the entanglement structure is determined by the "geodesic string condensation," a name given in analogy to "string net condensation" [31]. It is exactly the continuous bit-thread picture. As such, we establish the connection between the holographic toy models and a concrete gravitylike bulk field theory, and shows how entanglement structure emerges from linearized diffeomorphism.

Bit-thread type holographic toy models as a universal picture. We first make the observation that different holographic toy models have the same underlying universal picture. The toy models are based on two different constructions:

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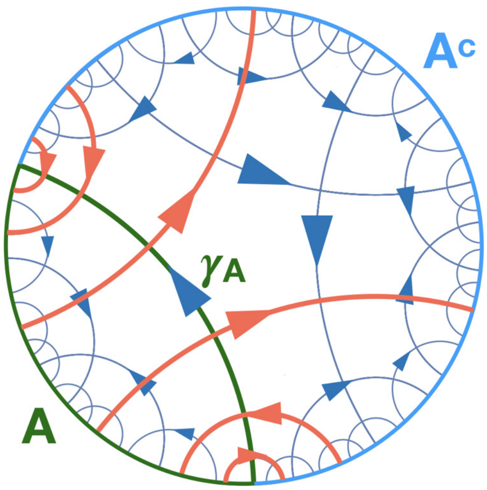


FIG. 1. Universal picture of holographic toy models: bit threads distributed evenly on the hyperbolic lattice. In the continuous case it is bit threads distributed homogeneously and isotropically in AdS space. The bit threads connecting boundary subregion A and its complement A^c are highlighted in orange. Their number is proportional to the length of covering geodesic γ_A, which yields the Ryu-Takayanagi formula [Eq. (1)].

The perfect tensor networks [8–14], and more recently the hyperbolic fracton models [29,30]. They are equivalent to bit threads distributed on a tessellation of the hyperbolic disk.

A bit thread is a line with entangled qubits (or more generally, we can consider any quantum/classical degrees of freedom) at its two ends [32–34]. A flow of the bit threads in the AdS space, when saturating the bound of minimal covering surface of a boundary subregion, gives the correct entanglement entropy described by the Ryu-Takayanagi formula (RT formula]) [6,7]

$$S_A = \frac{\text{area}(\gamma_A)}{4G_N}. \tag{1}$$

The hyperbolic fracton model was shown to be equivalent to bit threads in our earlier work [30]. It is dual to the eight-vertex model defined on the edges of the pentagon tessellation (Fig. 1). At low temperature, the eight-vertex model becomes a web of independent one-dimensional spin chains with ferromagnetic couplings. Each chain is then a classical bit thread with its two ends correlated.

The perfect tensor networks were also discovered to be equivalent to bit threads of Majorana modes in recent works by Jahn *et al.* [35,36]. They show that the perfect tensor network can be described by Majorana modes via Jordan-Wigner transformation. In the Mjaorana fermion language, the tensor-network state becomes a collection of Majorana dimers. Each dimer locates at the two ends of a geodesic on the hyperbolic lattice, which is the bit thread.

Following these observations, a universal picture of the holographic toy model emerges: By arranging the bit threads in the AdS space homogeneously and isotropically in the continuous limit, the simplest toy model of holography can be constructed. The hyperbolic fracton models and perfect tensor networks are discretized lattice versions of this picture.

This universal picture captures some of the crucial properties of holographic entanglement structure, but also misses some finer ones. It captures the RT formula for entanglement entropy of any *connected* boundary subregion. Instead of adjusting the bit-thread flow to saturate the target covering surface like in the original proposal [32], the bit threads in this picture are in a fixed configuration, but the RT formula is satisfied due to their even distribution in the bulk. The bulk information is defined in the dual model in both the hyperbolic fracton model and the perfect tensor networks. Its reconstruction obeys the Rindler reconstruction rule [37,38], again when the boundary subregion is connected.

However, in these models, the entanglement spectrum of a boundary subregion is always flat, thus the nth-Rényi entanglement entropy

$$S_n(\rho_A) = \frac{1}{1-n}\log \mathrm{Tr}\rho_A^n \tag{2}$$

has no n dependence, while in AdS/CFT the n dependence is nontrivial [39]. Also, these models deviate from the RT formula when the boundary subregion has multiple disconnected components. This deviation is due to the bit threads connecting different components of the boundary subregion, which is discussed in Ref. [30].

In Table I, we have summarized the comparison between genuine AdS/CFT, the bit-thread type toy models, and for completeness also the holographic random tensor networks proposed by Yang *et al.* [11–13]. The random tensor network satisfies the RT formula for the arbitrary boundary subregion, and does not belong to the universal picture proposed here.

Hence we can conclude that this evenly distributed bit-thread picture can be viewed as a "leading order" approximation to the entanglement structure of AdS/CFT. Built upon a collection of two-body entangled qubits/bits only, it fails to capture the finer entanglement structure of genuine gravitational AdS/CFT. Nonetheless, it is a helpful starting point for us to visualize the holographic entanglement structure, and gain a more intuitive understanding of it.

These observations also lead to the next question this work addresses: what is the bulk field theory that gives rise to the bit-thread type of holographic toy models? One can make some informed guesses: These toy models capture the RT formula and Rindler reconstruction at leading order, but fail at "higher order." Thus, a reasonable speculation is that such bulk theory cannot be the full-fledged general relativity, but it has to share certain essential features of gravity or is a special limit of it.

We will show that this is indeed the case. The bulk theory that describes the bit-thread type toy models is the traceful, vector charged rank-2 U(1) gauge theory [21,22,24,25,27,40] in the condensed matter physics literature, which is also known to be a special case of the Lifshitz gravity [26,41] in the high energy theory literature. As a special case of general relativity, its gauge symmetry is the spatial part of the linearized diffeomorphism. As we will elaborate, the consequence of such gauge symmetry is that the electric field lines can only travel along the geodesics in AdS space, instead of forming local loops like in conventional gauge theory. Hence, the entanglement structure is the continuous bit-thread distribution.

TABLE I. Comparison of the holographic entanglement properties between genuine AdS/CFT, bit-thread type holographic toy models, and random tensor networks. The bit-thread type holographic toy models, as a leading order approximation to holographic entanglement entropy, capture the Ryu-Takayanagi formula [Eq. (1)] for connected boundary subregion, but not other finer details.

	AdS/CFT	Bit-thread type toy models	Random tensor networks
RT formula for connected boundary subregion	Yes	Yes	Yes
RT formula for disconnected boundary subregion	Yes	No	Yes
n dependence of Rényi entropy	Yes	No	No
Nonflat entanglement spectrum	Yes	No	No

Rank-2 U(1) theory and its flat-space dynamics. Let us first quickly review the traceful, vector-charged version of R2-U1 theory [21,22,24,25]. Here we work in the two-dimensional space, but the physics extends to higher dimensions.

The R2-U1 theory features the gauge symmetry

$$A^{ij} \to A^{ij} + \partial^i \lambda^j + \partial^j \lambda^i, \tag{3}$$

where the gauge field A^{ij} is the rank-2 symmetric tensor. Taking A^{ij} as the perturbation of the metric $A^{ij} = h^{ij} - \delta^{ij}$ in general relativity, the transformation is the spatial part of the linearized diffeomorphism.

The gauge symmetry corresponds to the Gauss' law of the electric field at low energy [21,22]. In this case, the electric field is a rank-2 symmetric tensor

$$E^{ij} = E^{ji}. \tag{4}$$

The Gauss' law imposed on the electric field are

$$\partial_i E^{ij} = 0. \tag{5}$$

We take both Eq. (4) and Eq. (5) to be the Gauss' law at the low-energy sector of the theory.

The electric charge is a vector, defined as

$$\rho^i = \partial_j E^{ij}. \tag{6}$$

Besides the total vector charge conservation, the symmetric condition imposes an additional conservation law

$$\begin{aligned} &\int dv\, \epsilon^k{}_{ij} x^i \partial_l E^{jl} = -\int dv\, \epsilon^k{}_{ij} E^{ij} = 0, \\ &\text{i.e., } \int dv\, \boldsymbol{\rho} \times \mathbf{x} = 0. \end{aligned} \tag{7}$$

It restricts the movement of a vector charge $\boldsymbol{\rho}$. The charge can only move in the direction of $\boldsymbol{\rho}$ but not perpendicularly. It has crucial consequences in the entanglement structure, as we shall see.

Finally, in the *flat* space, the magnetic field is the simplest gauge symmetry-invariant term,

$$B = \epsilon^{ai}\epsilon^{bj}\nabla_a\nabla_b A_{ij}. \tag{8}$$

And the Hamiltonian is

$$\mathcal{H}_{\text{R2-U1-flat}} = U E_{ij}E^{ij} + tB^2. \tag{9}$$

The dynamics B, however, do not survive in the presence of spatial curvature, as we will explain later.

The Hamiltonian of Eq. (9) is also a case of Lifshitz gravity [27]. Treating A^{ij} as the perturbation of the metric, the magnetic field squared term B^2 is equivalent to R^2, R being the Ricci scalar. Here, the conventional linear term R and cosmological constant Λ in general relativity, as well as the self-interacting, nonlinear terms are forbidden due to the time-reversal, lattice translation, and spatial reflection symmetries. This was carefully analyzed in Ref. [27]. So, the theory of Eq. (9) can be viewed as a special version of linearized gravity.

Entanglement structure from gauge symmetry: Conventional U(1) as an example. Before examining the entanglement structure of the R2-U1 in AdS space, let us first review the topological entanglement entropy in the conventional U(1) gauge theory from the condensed matter point of view [31,42–44]. It is the string-net condensation picture proposed by Levin and Wen [31]. This helps to understand the logical chain of how gauge symmetry determines the entanglement structure.

The gauge symmetry for conventional U(1) theory

$$A^i \to A^i + \partial^i \lambda \tag{10}$$

as our starting point determines the Gauss' law to be electric charge conservation

$$\int dv\, \partial_i E^i = 0. \tag{11}$$

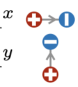
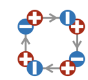
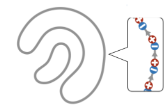
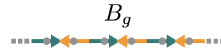
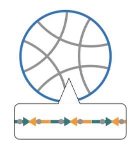
At low energy, the operations of gauge field A^x, A^y are to construct microscopic electric fields, or dipoles (cf. Table II). The gauge field operators respect the charge conservation globally, but not locally. Mathematically, that is to say the gauge field themselves are not gauge invariant.

To construct the gauge-invariant operator, that is, to build an object that respects the Gauss' law in any infinitesimal region, the dipole operators have to be connected head-to-tail together to form a loop. The minimal loop is the magnetic field $B = \epsilon^{ij}\nabla_i A_j$ (cf. Table II), which is now gauge invariant.

The term B^2 makes the system tunnel between different electric field configurations by creating loops of electric field line. As a consequence, the vacuum of the system is the fluctuation of the electric-field-line loops, or a superposition of all loop configurations [31]. This enables the calculation of topological entanglement entropy.

Here we can identify the crucial chain of logic: the gauge symmetry chosen determines the Gauss' law; the gauge operators are those objects (dipoles) obeying Gauss' law globally but not locally; they can be used to construct the magnetic field that respects Gauss' law in any local region (minimal loops); the magnetic field determines the configuration of electric field lines at low energy (all loop configurations), which then determine the entanglement structure of the system. In Table II, the above logical chain is shown on the second row.

TABLE II. From gauge symmetry to the entanglement structure. This table demonstrates the logical chain leading from the gauge symmetry to the configurations of electric field lines as the dynamical variables. The second row is for conventional U(1), where the electric field lines can be arbitrary loops. The third row is for rank-2 U(1) in AdS space, where the electric field lines are on the geodesics extending to infinity.

Gauge symmetry & Gauss' law	gauge operator **A**	gauge invariant operator magnetic field **B**	electric field line
$A^i \to A^i + \partial^i \lambda$ $\partial_i E^i = 0$			
$A^{ij} \to A^{ij} + \nabla^i \lambda^j + \nabla^j \lambda^i$ $\nabla^i E_i^j = 0 \quad E^{ij} = E^{ji}$ curved space			

Entanglement structure of R2-U1 in AdS space: Geodesic string condensation. Now let us examine the entanglement structure of R2-U1 in the two-dimensional AdS space following the same mechanism. We will see that instead of string-net condensation, the picture will be "*geodesic string condensation*." That is, the strings of electric fields travel along geodesics only, and their superposition as the vacuum determines the entanglement structure. The facts that the gauge symmetry is linearized diffeomorphism, and the space is curved, play crucial roles in determining the entanglement structure.

Like in the previous section, the gauge operators play the role of creating vector charge multipoles. They are listed in Table II. The diagonal terms A^{xx}, A^{yy}, or in general $A^{ij}s_i s_j$ for direction $\hat{\boldsymbol{s}}$ is to move a vector charge along the direction it points. The off-diagonal term A^{xy} creates a vector charge multipole. All these operators have vanishing $\int \boldsymbol{\rho}$ and $\int \boldsymbol{\rho} \times \boldsymbol{x}$. In flat space, the dynamics, or the magnetic fields B [Eq. (8)], is again a composite of the gauge operators in which the Gauss' laws are satisfied locally. It is illustrated in Table II.

However, the magnetic fields are very different in the curved space compared to the case in the flat space. This has been carefully studied by Slagle *et al.* in Ref. [40]. When a vector charge is parallel transported around a finite region back to its starting point, it will in general be different from the original vector due to the spatial curvature. Thus such parallel transport over a closed loop is energy-costly.

Consequently, the local dynamics of B [Eq. (8)] is forbidden. The pictorial intuition is that the dynamics of B as illustrated by Fig. 2 always happen over a finite-sized plaquette in the system. In flat lattice, such combination of A^{ij} operators does not violate the Gauss' laws [Eqs. (4) and (5)] in any microscopic region. But in curved space it is not true anymore.

A more convincing evidence is that B [Eq. (8)] is not gauge invariant in the presence of curvature. Promoting $\boldsymbol{\nabla}$ to the covariant derivative, we have

$$B \to B - R g^{aj} \nabla_a \lambda_j, \tag{12}$$

under gauge transformation, where R is the Ricci scalar [40]. In fact, higher order local terms up to $\nabla_a \nabla_b \nabla_c \nabla_d \lambda^e$ were systematically examined but no gauge-invariant B term was found in Ref. [40].

So what are the dynamics allowed in the AdS space? We note that the difficulty is rooted in parallel transporting a vector charge around a loop. To avoid this, we have to consider parallel transporting the charge on a geodesic extending from one infinity to the other instead of forming a loop.

In the lattice model, it has the following picture: A vector charge, for example, $\boldsymbol{\rho} = (\rho^x, 0)$, can be moved along the x direction by acting A^{xx} operators on the path. To make sure that any local region respects Gauss' laws, however, such line operation has to extend to infinity in both directions.

In the field theory, for a given geodesic g with unit vector $\hat{\boldsymbol{s}}$ along it, this operation is the dynamics

$$B_g = \int_g ds\, A^{ij} \hat{s}_i \hat{s}_j. \tag{13}$$

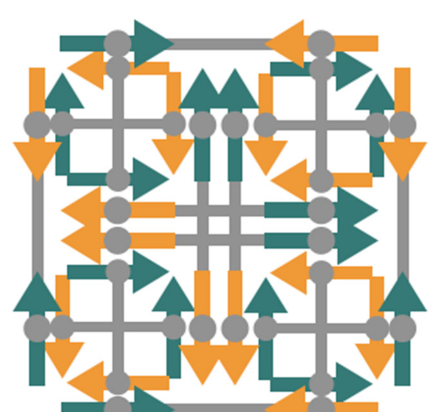


FIG. 2. The operator B of rank-2 U(1) theory in the flat space [Eq. (8)]. It is a composite of multiple A^{xx}, A^{yy}, and A^{xy} operators. It acts on a finite-area plaquette in the system, and does not survive the spatial curvature.

The fact that locally no Gauss' laws are broken is reflected by its invariance under gauge transformation

$$B_g \to B_g + \int_g ds(\nabla^i\lambda^j + \nabla^i\lambda^j)\hat{s}_i\hat{s}_j = B_g + 2(\boldsymbol{\lambda}\cdot\hat{\mathbf{s}})\Big|_{-\inf}^{\inf}, \tag{14}$$

where the second term vanishes assuming vanishing gauge transformation at infinity.

We can thus write down the theory as

$$H_{\text{R2-U1-AdS}} = \int dv\, U E^{ij}E_{ij} + \int_{g\in\text{all geodesics}} dg t_g B_g^2. \tag{15}$$

Such nonlocal dynamics are normally unfavored in many disciplines of physics. However, they are the ones stable in the presence of spatial curvature. Let us bear with them, and examine the corresponding entanglement structure.

With such dynamics on geodesics, a drastic change happens to the electric field lines. In AdS space, instead of forming loops, they travel along geodesics from one boundary point to another. The vacuum is then a superposition of all possible geodesic electric-field-line configurations. We name this the *geodesic string condensation*. As a result, the entanglement structure for each geodesic string is that the two boundary points are entangled by the corresponding geodesic dynamics. As the B_g distribute in AdS space homogeneously and isotropically, we have exactly the continuous bit-thread picture we speculated at the beginning of this work. Upon lattice discretization, and also assigning $\boldsymbol{E}$ discrete/continuous values, one can obtain toy models of the same universal picture but different in details, including the perfect tensor networks and the hyperbolic fracton models.

Discussion. In this work we obtained a very pictorial, intuitive understanding of the leading order entanglement structure of holography, and the mechanism that generates it. The leading order entanglement structure is a web of evenly distributed bit threads, and we noted that several holographic toy models belong to this picture. We reason that, taking the linearized diffeomorphism as the gauge symmetry, the corresponding symmetric tensor gauge theory gives rise to this picture by geodesic string condensation. Retrospectively, it is sensible that a theory mimicking gravity at first order yields the entanglement structure of gravity also at first order.

Many questions follow. One exciting question to ask is that, can we understand the finer entanglement structure (some are listed in Table I) in a similar way? For example, the random tensor networks proposed by Yang *et al.* [11–13] satisfy the RT formula for the arbitrary boundary subregion. What modification of the geodesic string condensation picture is needed to capture such properties? Another question is how to introduce a nonflat entanglement spectrum to match the nth Rényi entropy. These projects will be very useful for us to gain improved intuition of the entanglement structure in holography.

The argument presented here is based on a chain of reasoning at a qualitative level. It would deepen our understanding to rederive these results through more explicit calculations of the entanglement entropy for the R2-U1, or its different variations, on flat and AdS space. It is also intriguing to know if the physics demonstrated in the work is connected to other aspects of gravity, including its ground state degeneracy [45], and its relation to topological order [46].

We hope our work will provide new, useful insight in understanding both fracton states of matter and quantum gravity.

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