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Higher-spin holography in de Sitter space: horizon modes, black holes, and the boundary partition function

by

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Declaration of Original and Sole Authorship

I, Adrian David, declare that this thesis entitled *Higher-spin holography in de Sitter space: horizon modes, black holes, and the boundary partition function* and the data presented in it are original and my own work.

I confirm that:

- No part of this work has previously been submitted for a degree at this or any other university.
- References to the work of others have been clearly acknowledged. Quotations from the work of others have been clearly indicated, and attributed to them.
- In cases where others have contributed to part of this work, such contribution has been clearly acknowledged and distinguished from my own work.
- Parts this work has been previously published in Physical Review **D**, 045005 as "Spinor-helicity variables for cosmological horizons in de Sitter space"; Journal of High Energy Physics **2020**, 127 as "Higher-spin symmetry vs. boundary locality, and a rehabilitation of dS/CFT"; and as preprint arXiv:2009.02893 [hep-th] "Bulk interactions and boundary dual of higher-spin-charged particles".

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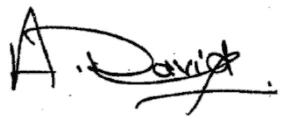
Signature: 

Abstract

Higher-spin holography in de Sitter space: horizon modes, black holes, and the boundary partition function

Higher-spin holographic realizations of quantum gravity on de Sitter spacetimes is a promising model for addressing quantum gravity in an universe with positive cosmological constant. In this body of work we focus on issues related to (i) cosmological horizon modes, (ii) higher-spin black hole worldlines, and (iii) the boundary partition function. (i) We introduce a spinor-helicity formalism to encode the data of massless fields of arbitrary spin on a cosmological horizon. The evolution of free fields between past and future horizons reduces to a simple Fourier transform in terms of these variables. We show how this arises, by decomposing the problem into a pair of horizon-to-twistor problems. (ii) We decompose the boundary partition function Z in terms of spherical modes in the spinor-helicity basis. Even though the correlators agree, we observe a persistent discrepancy between the higher-spin-algebraic calculation of Z and the result of a direct CFT calculation. This suggests a failure of locality in higher-spin theory, even on the boundary. (iii) We show that the linearized version of the Didenko–Vasiliev black hole solves the Fronsdal field equations with a particle-like source. These fields are precisely the linearized bulk higher-spin fields corresponding to a bilocal source on the boundary. We show that the boundary correlator of two bilocal operators agrees with the bulk action describing the corresponding particles interacting in the bulk.

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Introduction

Lord Kelvin (disputedly) claimed in 1900 that *"there is nothing new to be discovered in physics now; all that remains is more and more precise measurement."* In spite of this, the 20th century has seen the development of two great conceptual schemes of modern theoretical physics: on one hand quantum mechanics and quantum field theory, on the other hand, general relativity. The latest quest has been to unify these two grand yet stubbornly contradicting theories into a single framework.

General Relativity, through Einstein's equations, describes a world model consisting of a spacetime manifold with a metric, whose curvature is constrained by the stress-energy-momentum of energy and matter distributions; physical quantities are classical, as in they have definite real values. Our fundamental theories of matter and energy, however, are all quantum theories. In general, physical quantities are described by quantum states which give probability distributions over a range of values, and canonically conjugate properties have inversely related specificities as expressed by Heisenberg's Uncertainty Principle.

In the naïve attempt to follow the model of, say, quantum electrodynamics and quantize the gravitational field similar to the way in which the electromagnetic field was quantized, various serious difficulties arise. In a sense, these are technical difficulties: the gravitational interaction is non-renormalizable and thus the perturbative methods that have been successfully employed in ordinary quantum field theories prove ineffective. At the conceptual level, these difficulties arise from the nature of the gravitational interaction, as a property of spacetime itself, rather than as a field propagating on top of a static spacetime background. Given the uncertainty principle and the probabilistic nature of quantum theory, the geometry of spacetime itself is fluctuating, but ordinary quantum theory presupposes a well-defined classical background against which to define these fluctuations.

One can argue that the two theories differ on an ontological level [1]: general relativity discards the fixed kinematical structure of spacetime so that localization is rendered relational, however, quantum field theory requires a fixed flat background in its construction, which is used to derive standard features of the theory. Moreover, quantum field theory involves quantum fluctuations at arbitrarily short distances in the vicinity of a point, while general relativity involves the use of smooth geometry.

In a strict sense, it could be said that the problem of *quantum gravity* has been solved: string theory, through the AdS/CFT correspondence, offers an account of the problem if questions are restricted to the spatial infinity of a universe with negative cosmological constant.

The idea behind string theory is to replace the point particles of ordinary quantum field theory with one-dimensional extended objects called strings. In the early development of the theory, it was recognized that the construction of a consistent quantum theory of strings requires a higher number of spatial dimensions; in fact, supersymmetric string theories must be formulated in 9+1 dimensions. Strings can be open or closed, and have a characteristic tension and hence vibrational spectrum, with the various modes of vibration corresponding to various particles, one of which is the spin-2 graviton. One of the advantages of such theories is that they are perturbatively renormalizable.

The AdS/CFT correspondence [2–4] was introduced as the statement that observable properties of a particular string theory defined on anti-de Sitter (AdS) spacetime are equivalent to those of a conformal field theory (CFT) on the conformal boundary of AdS. This is a concrete example of the more general *holographic principle* [5–7], which states that a gravitational theory over a *bulk* spacetime region can be completely described by a theory defined on the lower-dimensional *boundary* of the bulk region.

We are interested in formulating a theory of quantum gravity over spacetimes with a positive cosmological constant where an observer only has access to partial information. One of the reasons why such questions are interesting is the astronomical observations [8, 9] which indicate that the cosmological constant in our universe is in fact positive.

De Sitter (dS) space is a natural toy model to describe quantum gravity over finite regions as it is the maximally symmetric spacetime of positive cosmological constant containing horizons; however, descriptions of de Sitter are currently unavailable or difficult to address within the context of string theory. The de Sitter vacua that have been found in string theory turn out to be metastable [10–12], (that is, they exhibit a non-zero probability of decay to a different vacuum state of lower energy), and moreover are difficult to work with. One reason for this situation is the lack of supersymmetry. Due to its thermal nature, de Sitter space cannot be supersymmetric, namely, the thermal distribution function at finite temperature breaks supersymmetry. More formally, de Sitter space is inconsistent with supersymmetry in the sense there is no supergroup that includes the dS isometries and has unitary representations [13]. It has further been argued that even metastable de Sitter vacua might belong to the string swampland, *i.e.* unrealizable in string theory [14].

Descriptions of Sitter space may be connected to those on AdS via analytic continuation, which suggests that one may be able to import the holographic language from the negative cosmological constant case; thus, the achievements of the AdS/CFT correspondence have lead to the idea of dS/CFT. This duality has been conjectured before [15–17], but until recently [18] there has been no non-trivial proposition on how to realize it. The system that will be discussed in this body of work has as its bulk the Vasiliev bosonic higher-spin theory [19–21], while on CFT side we have a $Sp(2N)$ vector model. Importantly, due to the different causal structures of the two spaces, the holographic dictionary between bulk and boundary needs to change, and with it the physical interpretation of mathematical quantities. It is worth noting that while in AdS the radial direction emerges through holography, in dS the time direction would emerge holographically.

One approach to dS / CFT is to consider the Lorentzian bulk physics of global de Sitter space, where the CFT partition function is used to define a Hartle–Hawking wavefunction [17, 18, 22]. This approach is suitable to describe temporary de Sitter phases, as is the case for cosmological inflation, where the would-be future boundary of dS_4 is rendered observable.

In contrast, we consider truly asymptotically de Sitter spacetime, where the future boundary is unobservable, and one must focus on the causal patch defined by a pair of cosmological horizons. In particular, we will explore some of the questions encountered in the course of constructing suitable descriptions of higher-spin fields in de Sitter, their corresponding boundary data on the cosmological horizons, and non-local dictionaries with the unobservable boundary. Although the final aim of the described approach is to construct a full holographic description of quantum gravity within the causal patch of a de Sitter observer, it will be convenient to consider some of the above questions over spacetimes of Euclidean signatures, namely Euclidean anti-de Sitter; these descriptions can be analytically continued to those over de Sitter space.

In this body of work we focus on issues related to (i) cosmological horizon modes, (ii) higher-spin black holes from boundary bilocals, and (iii) the boundary CFT partition function.

(i) We introduce [23] a spinor-helicity formalism to encode the data of massless fields of arbitrary spin on a cosmological horizon in de Sitter space. The evolution of free fields between past and future horizons (what might be called the free S-matrix in an observers causal patch) reduces to a simple Fourier transform in terms of these variables. We show how this arises via twistor theory, by decomposing the horizon-to-horizon problem into a pair of (more symmetric) horizon-to-twistor problems.

(ii) We investigate [24] the decomposition of the boundary CFT partition function in terms of spherical modes in the spinor-helicity basis. Further, even though the n-point correlators agree, we observe a discrepancy between the higher-spin-algebraic calculation of the partition function and the result of a direct calculation in the boundary CFT [25]; this disagreement persists even when considering the Legendre transform of the local action and accounting for contact pieces. This paradox suggests a failure of locality in higher-spin theory, even on the boundary. A way forward from here is to introduce spin-locality as a replacement for spacetime locality, echoing recent developments in the bulk theory.

(iii) We show that the linearized version of the Didenko–Vasiliev black hole [26] solves the Fronsdal field equations with a particle-like source. Furthermore, these fields are precisely the linearized bulk higher-spin fields corresponding to a bilocal source on the boundary. We will also show that the boundary correlator of such two bilocal operators agrees with the bulk action describing the two corresponding particles interacting in the bulk.

Chapter 1

Background

1.1 Space-time and twistor geometry

1.1.1 De Sitter space

We model de Sitter space dS_d as the hyperboloid of spacelike directions in flat $d+1$-dimensional Minkowski space $\mathbb{R}^{1,d}$

$$\mathrm{dS}_d = \left\{ x^\mu \in \mathbb{R}^{1,d} \,\middle|\, x_\mu x^\mu = l^2 \right\} \tag{1.1}$$

where l is a parameter of unit length called the de Sitter radius. We use indices $(\mu, \nu, \ldots)$ for vectors in $\mathbb{R}^{1,d}$, which are raised and lowered the $\eta_{\mu\nu}$ metric of signature $(-, +, \ldots, +)$. The isometries of de Sitter space are given by the Lorentz group $O(1, d)$. Hence, the metric has $d(d+1)/2$ independent Killing vector fields, thus it is maximally symmetric and of constant curvature. Its curvature is given by the Riemann tensor

$$R_{\rho\sigma\mu\nu} = \frac{1}{l^2}\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\right) .$$

Note that the Ricci tensor is proportional to the metric

$$R_{\mu\nu} = \frac{d-1}{l^2} g_{\mu\nu} ,$$

making de Sitter an Einstein manifold. It further follows that de Sitter space is a vacuum solution to the Einstein equations with *positive* cosmological constant

$$\Lambda = \frac{(d-2)(d-1)}{2l^2}$$

and scalar curvature

$$R = \frac{d(d-1)}{l^2} = \frac{2d}{d-2}\Lambda .$$

Hereafter we specialize to de Sitter space with radius $l = 1$.

Besides the embedded $\mathbb{R}^{1,d}$ description one can introduce a number of coordinate systems on dS_d that provide different insights into the structure of this space. Of

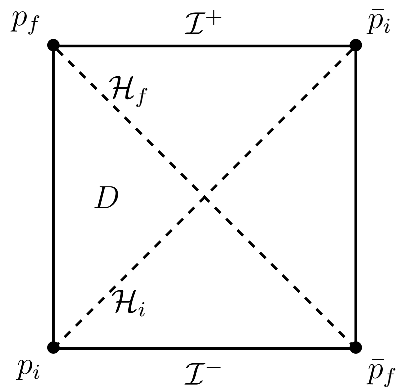


Figure 1.1: A Penrose diagram of dS_4. Past and future infinity are denoted by $\mathcal{I}^-$ and $\mathcal{I}^+$ respectively. The boundary points $p_i \in \mathcal{I}^-$ and $p_f \in \mathcal{I}^+$ define light cones $\mathcal{H}_i$ and $\mathcal{H}_f$, respectively, which divide the spacetime into four distinct quadrants; in particular D is the causal patch. The antipodes of p_i and p_f are denoted by $\bar{p}_i$ and $\bar{p}_f$, respectively.

immediate interest are the conformal coordinates (T, θ_i), which relate to the embedded flat coordinates as

$$X_0 = \sinh \tau\,, \quad \cosh \tau = \frac{1}{\cos T}\,,$$

so that $T \in (\pi/2, \pi/2)$ and the θ_i's parametrize spherical coordinates on S^{d-1}, with usual metric $d\Omega^2_{d-1}$. In these coordinates the metric becomes

$$ds^2 = \frac{1}{\cos^2 T}\left(-dT^2 + d\Omega^2_{d-1}\right). \tag{1.2}$$

This enables us to better understand the causal structure of de Sitter, since null geodesics with respect to the conformal metric (1.2) will also be null with respect to the *conformally* related metric

$$d\tilde{s} = \cos^2 T ds^2 = -dT^2 + d\Omega^2_{d-1}\,.$$

The Penrose diagram (Fig. 1.1) contains all the information regarding the causal structure of dS_d, with each point representing a S^{d-2} sphere, except for the points that lie on the left and right edges. The boundaries of dS_d can be identified with the asymptotes of the hyperboloid (1.1). These are the spaces of future-pointing and past-pointing null directions in $\mathbb{R}^{1,d}$ respectively, which are identified with future infinity $\mathcal{I}^+$ and past infinity $\mathcal{I}^-$. They are the surfaces where all null and timelike geodesics originate and terminate, and have the geometry of a spacelike conformal $d-1$-dimensional sphere, *i.e.* the $O(1,d)$ symmetry group reduces to the conformal symmetry of the boundary. We represent the boundary as the set of null vectors ℓ^μ, up to the identification $\ell^\mu \cong \lambda \ell^\mu$.

One for the peculiarities of de Sitter space is that no observer can access the entire spacetime, which can be problematic when trying to construct a theory of quantum gravity. An observer in de Sitter can be identified with a pair of non-antipodally related boundary points $p_i \in \mathcal{I}^-$ and $p_f \in \mathcal{I}^+$ which we think of as the past and future endpoints of an observer's worldline, respectively. These points cast ligthcones $\mathcal{H}_i$ and $\mathcal{H}_f$, respectively, into the bulk, which are the past and future horizons of the observer, and they divide de Sitter space into four disjoint quadrants. Note that the immortal observer can only receive signals from the half-spacetime in the past of p_f, hence a description of the entire space goes beyond what can be physically measured. This is qualitatively different from Minkowski space, where a timelike observer will eventually have access to the entire history of the universe in her light cone. Similarly, the observer can only send signals in the half-spacetime lying to the future of p_i. Thus the largest possible observable region is the *causal patch* D that lies in the common interior of $\mathcal{H}_i$ and $\mathcal{H}_f$, also known as a static patch.

Further, note that every point in de Sitter has an antipodal point. In the embedding spacetime picture (1.1) the antipodal map is given by $x^\mu \to -x^\mu$, which is invariant under the $O(1,d)$ isometry group. Antipodal points in the bulk are spacelike separated, however, when extended to the boundary, antipodal points $\pm\ell^\mu \in \mathcal{I}^\pm$ are, in fact, connected by null-geodesics[1]. This causal connection relating points on $\mathcal{I}^-$ to those on $\mathcal{I}^+$ breaks the two copies of the conformal group down to a single copy. This naively suggests that, when trying to construct holographic theories, the CFT dual lives on a single euclidean sphere.

It is thus natural to (topologically) identify antipodal points in de Sitter space, idea that goes back to Schrödinger [28]. This identification results in the quotient space $\mathrm{dS}_d/\mathbb{Z}_2$ referred to as the *elliptic*[2] *de Sitter space*. Like dS_d, this is maximally symmetric, with isometry group $O(1,d)/\mathbb{Z}_2 = SO(1,d)$, where $\mathbb{Z}_2$ is generated by the antipodal map. As hinted at in the previous paragraph, the asymptotic boundary becomes single S^{d-1} sphere $\mathcal{I}$ with conformal geometry, *i.e.* the identification of the $\mathcal{I}^-$ and $\mathcal{I}^+$ boundaries. Alternatively, one could identify antipodal points solely on the boundaries $\mathcal{I}^\pm$ of dS, leaving the bulk intact; this results in the *compactified de Sitter space.*

In addition to the real four-dimensional spacetime dS_4, we will also consider *complex de Sitter space* $\mathrm{dS}_{4,\mathbb{C}}$, the set of points in flat five-dimensional complex space $x^\mu \in \mathbb{C}^5$ satisfying $x_\mu x^\mu = 1$. As before we identify antipodal points, which gives us $\mathrm{dS}_{4,\mathbb{C}}/\mathbb{Z}_2$. Now identifying the complex infinities leads to the complexified infinity $\mathcal{I}_\mathbb{C}$, which we can view as the set of complex directions in $\mathbb{C}^5$. Of particular interests are the imaginary future and past slices of $\mathrm{dS}_{4,\mathbb{C}}$

$$\mathbb{H}^\pm = \left\{ x^\mu \in \mathrm{dS}_{4,\mathbb{C}} \,|\, \mathrm{Re}\, x^0 = 0,\ \mathrm{Im}\, x^0 \gtrless 0 \right\} , \tag{1.3}$$

which are isomorphic to hyperbolic space H_4.

[1] Considering null rays as degenerate spatial geodesics that bounce off null infinity, then all such light rays leaving a point converge, or refocus, onto the antipodal point. This, in fact, can be seen in the Greens functions of dS quantum fields, at the level of singularity structures. [15, 27]

[2] The term "elliptic" refers to the fact that the antipodally identified points are spacelike separated, *i.e.* related by elliptic generators, as opposed to being timelike (read hyperbolically) or null (read parabolically) separated.

Vectors at a point $x^\mu \in \mathrm{dS}_4$ are defined as elements v^μ of the tangent space at that point, *i.e.* $v_\mu x^\mu = 0$. Further, the metric at this point can be identified with the tangent space projector

$$q_{\mu\nu}(x) = \eta_{\mu\nu} - x_\mu x_\nu \,. \tag{1.4}$$

The covariant derivative is defined as the flat $\mathbb{R}^{1,4}$ derivate projected onto the dS_4 hyperboloid, namely

$$\nabla_\mu v_\nu = q_\mu^\rho(x) q_\nu^\sigma(x) \partial_\rho v_\sigma \,.$$

In addition to the ambient picture, it will also be useful to consider the Poincaré coordinates

$$x^\mu(z, \mathbf{r}) = -\frac{1}{z}\left(\frac{r^2 - z^2 + 1}{2}, \frac{r^2 - z^2 - 1}{r}, \mathbf{r}\right) \tag{1.5}$$

with $\mathbf{r}$ a flat 3d coordinate and metric

$$ds^2 = \frac{-dz^2 + d\mathbf{r}^2}{z^2} \,.$$

Note that in Poincaré coordinates the antipodal map reads as the operation $z \to -z$; this was invoked in some of the recent discussions of higher-spin holography [29].

One fixes a conformal frame on the boundary by choosing a section of the lightcone in the embedding $\mathbb{R}^{1,4}$. The simplest, flat sections have $\mathbb{R}^3$ geometry and can be obtained by choosing a particular point n^μ on the conformal boundary; this will play the role of the "point at infinity". The flat section is then found by intersecting the $\mathbb{R}^{1,4}$ lightcone and the null hyperplane defined by $\ell \cdot n = -\frac{1}{2}$. One particular convenient choice of flat frame is given by $n^\mu = \left(\frac{1}{2}, \frac{1}{2}, \mathbf{0}\right)$:

$$\ell^\mu(\mathbf{r}) = \left(\frac{r^2 + 1}{2}, \frac{r^2 - 1}{2}, \mathbf{r}\right) . \tag{1.6}$$

This can be viewed as the bulk-to-boundary or extreme boost limit $x^\mu \to \frac{\ell^\mu}{z}$ as $z \to 0$ of the Poincaré coordinates (1.5).

1.1.2 Twistor geometry

The twistor theory of Penrose [30, 31] was originally devised as a framework for quantum General Relativity. One effectively changes the fundamental principle from locality to causality, making away with spacetime by replacing points with twistors, the "maximally lightlike" extended objects in spacetime. Recently, twistor theory has been usefully employed for scattering amplitude calculations in supergravity [32] and maximally supersymmetric Yang–Mills theories [33, 34]. Moreover, these objects are well suited for describing massless fields and, as it will be described in Section 1.3, they are essential in the formulation of higher-spin theories.

We introduce the twistor space $\mathbb{T}$ of dS_4 as the space of 4-component Dirac spinors of the isometry group $SO(1,4)$. We will label twistor indices as $(a, b, \ldots)$. The twistor space $\mathbb{T}$ is endowed with a symplectic metric I_{ab} (also known as the *infinity twistor*) which raises and lowers indices as $U_a = I_{ab} U^b$, $U^a = U_b I^{ba}$. In particular I_{ab} and I^{ab} are lowered/raised-index analogues, *i.e.* $I_{ac} I^{bc} = \delta_a^b$.

We can map between tensor and twistor indices via the gamma matrices $(\gamma_\mu)^a{}_b$; these are the Clifford algebra generators in the embedding space $\mathbb{R}^{1,4}$, *i.e.* $\gamma^{(\mu}\gamma^{\nu)} = -\eta^{\mu\nu}$. One particular realisation of the gamma matrices $(\gamma_\mu)^a{}_b$ and twistor metric I_{ab} that will be useful throughout the rest of this body of work reads as, in matrix block notation,

$$(\gamma_0)^a{}_b = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (\gamma_4)^a{}_b = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad (\gamma_k)^a{}_b = \begin{bmatrix} -i\left(\sigma^k\right)^\beta{}_\alpha & 0 \\ 0 & i\left(\sigma^k\right)^\alpha{}_\beta \end{bmatrix}, \tag{1.7}$$

where, for $k = 1, 2, 3$, σ^k are the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

In particular

$$I_{ab} = \begin{bmatrix} 0 & -\delta^\alpha_\beta \\ \delta^\beta_\alpha & 0 \end{bmatrix}. \tag{1.8}$$

One can see that the matrices γ^μ_{ab} are traceless, and, by lowering indices, that they are antisymmetric in their ab twistor indices; moreover, they convert between traceless bitwistors and 4+1d vectors as $\xi^{ab} = \gamma^{ab}_\mu \xi^\mu$, $\xi^\mu = -\frac{1}{4}\gamma^\mu_{ab}\xi^{ab}$.

Further, we define the antisymmetric product $\gamma^{\mu\nu}_{ab} = \gamma^{[\mu}_{ac}\gamma^{\nu]c}{}_b$. These matrices are traceless, symmetric in their twistor indices, and can be used to convert between bivectors and symmetric twistor matrices $f^{ab} = \frac{1}{2}\gamma^{ab}_{\mu\nu}f^{\mu\nu}$, $f^{\mu\nu} = \frac{1}{4}\gamma^{\mu\nu}_{ab}f^{ab}$.

Note that the set of matrices $\{\mathbb{1}_4, \gamma^\mu, \gamma^{\mu\nu}\}$ spans the space of 4×4 matrices; in particular, the six matrices $\{I_{ab}, \gamma^\mu_{ab}\}$ span the antisymmetric subspace, whereas the ten matrices $\gamma^{\mu\nu}_{ab}$ span the symmetric one.

It is useful to also note that $\gamma^{[ab}_\mu \gamma^{cd]}_\nu = \frac{1}{3}\eta_{\mu\nu}\epsilon^{abcd}$, where ϵ^{abcd} is the totally antisymmetric symbol with inverse $\epsilon_{abcd} = 3I_{[ab}I_{cd]}$ normalized so that $\epsilon_{abcd}\epsilon^{abcd} = 4!$. The form I_{ab} has unit determinant with respect to ϵ^{abcd} and so we use ϵ^{abcd} to introduce a measure on $\mathbb{T}$

$$d^4U = \frac{\epsilon_{abcd}}{4!(2\pi)^2} dU^a dU^b dU^c dU^d. \tag{1.9}$$

For calculational simplicity we will occasionally employ the index free notation of [35] for products in $\mathbb{R}^{1,4}$ and twistor space: index-free products should be read as bottom-to-top index contractions. Namely, for twistors U^a, V^a and vectors ℓ^μ, x^μ we write

$$\begin{aligned} &\ell \cdot x \equiv \ell_\mu x^\mu \; ; \quad (xU)^a \equiv (x^\mu\gamma_\mu)^a{}_b U^b \; ; \\ &UV \equiv U_a V^a = -I_{ab}U^a V^b \; ; \\ &U\ell x U \equiv U_a \ell^a{}_b x^b{}_c U^c = -\ell_\mu x_\nu \gamma^{\mu\nu}_{ab} U^a U^b. \end{aligned} \tag{1.10}$$

In the following calculations, integral over complex twistor space are required. Thus one needs to worry about choices of contour. However, certain integrals, such as the delta function on twistors and the Gaussian described below, can be defined analogously to real-line integrals. Formally, we write

$$\int d^4U\, \delta(U) f(U) = f(0)\,, \tag{1.11}$$

where we define the twistor delta function as

$$\delta(U) = \int d^4V\ e^{iVU}\,. \tag{1.12}$$

We also note the Gaussian integral reads

$$\int d^4U\, e^{\frac{1}{2}UAU} = \frac{\pm 1}{\sqrt{\det A}}\,, \tag{1.13}$$

where $\det A = \frac{1}{8}\left(\operatorname{tr} A^2\right)^2 - \frac{1}{4}\operatorname{tr} A^4$ and A_{ab} is a symmetric twistor matrix .

The square root in (1.13) introduces a sign ambiguity which depends on the analytical continuation from the real contour. Note that this sign ambiguity is also reflected in the delta function (1.12): its integral definition can be regularised by introducing a Gaussian into the integrand. However, as previously noted, the result of the Gaussian integral has a sign ambiguity, therefore, while the composition with $\delta(U)$ (1.11) is well defined, the object itself is only defined as a limit of functions up to a sign. We will encounter such sign ambiguities throughout the following discussions that involve the twistor formalism.

1.1.3 Spinor decomposition

At a fixed point in de Sitter space x^μ the twistor space $\mathbb{T}$, which we introduced as the Dirac spinors of $SO(1,4)$, is in fact the Dirac spinor representation of the Lorentz group $SO(1,3)$ at that point. Further, this representation decomposes into right- and left-handed Weyl spinors, with corresponding projectors

$$P_\pm{}^a{}_b(x) = \frac{1}{2}\left(\delta^a_b \pm ix^\mu \gamma_\mu{}^a{}_b\right) \tag{1.14}$$

or equivalently

$$P^{ab}_\pm = \frac{1}{2}\left(I^{ab} \pm ix^\mu \gamma^{ab}_\mu\right)\,.$$

The matrices $P_\pm{}^a{}_b$ sum to unity and they act as projectors onto implicitly defined subspaces $P_\pm(x)$, as they satisfy

$$P_\pm{}^a{}_c P_\pm{}^c{}_b = P_\pm{}^a{}_b\,, \quad P_\pm{}^a{}_c P_\mp{}^c{}_b = 0\,,$$

where the latter equation implies that $P_\pm(x)$ are indeed orthogonal. In fact, these function as x-dependent versions of the chiral projectors in $\mathbb{R}^{1,3}$. Also note that the antipodal map $x^\mu \to -x^\mu$ interchanges P_+ and P_-. Given a twistor U^a, we denote its right-handed and left-handed Weyl spinor components at x as $u^a_\pm = P_\pm{}^a{}_b(x)U^b$, where we use $(a, b, \ldots)$ indices for both the Dirac spinors of $SO(1,4)$ and $SO(1,3)$. The projectors $P^\pm_{ab}$ can then be employed as spinor metrics on the right- and left-handed

Weyl spinor subspaces, respectively.

Moreover, the symplectic metric of a two-dimensional spinor space can also act as a measure, defined as

$$d^2u_\pm \equiv \frac{P^\pm_{ab}}{2(2\pi)} dU^a dU^b\,. \tag{1.15}$$

The twistor measure (1.9) can then be written as a product of the two chiral spinor measures, namely $d^4U = d^2u_- d^2u_+$. For a bulk point x^μ, the measure (1.15) can be rewritten as

$$d^2u_\pm = \frac{P^\pm_{ab}}{4\pi} du^a_\pm du^b_\pm\,. \tag{1.16}$$

Since twistors are flat they can be transported freely between points in dS_4; what changes at different points is the decomposition into left- and right-handed spinors. This property can be used when constructing the covariant derivative of a Weyl spinor in dS_4: first, take the flat $\mathbb{R}^{1,4}$ derivative of an embedding twistor and then project the result onto the relevant spinor subspace. For a right-handed spinor field $\psi^a_+(x)$, its derivative reads

$$\nabla_\mu \psi^a_+(x) = q_\mu{}^\nu(x) P_+{}^a{}_b \partial_\nu \psi^b_+(x)\,.$$

In particular, for a spacetime-independent twistor Y, its spinor components $y_\pm(x)$ have derivatives

$$\nabla_\mu y^a_\pm = \pm\frac{1}{2}(\gamma_\mu)^a{}_b y^b_\mp\,.$$

This is the Penrose twistor equation on spacetimes with cosmological constant [30].

On the boundary, at a point ℓ^μ, the twistor decomposition is qualitatively different, as the two subspaces degenerate into a single spinor subspace $P(\ell)$ via projector

$$P^{ab}(\ell) = \frac{1}{2}\ell^{ab}$$

which is totally null with respect to twistor metric I_{ab}. The metric and measure du^2 on $P(\ell)$ is defined analogously to the bulk definition (1.16)

$$\frac{du^a du^b}{2\pi} \equiv P^{ab}(\ell) d^2u\,.$$

Note that $P(\ell)$ is the space of cospinors at boundary point ℓ, whereas contravariant spinors live in the quotient space of twistors modulo elements of $P(\ell)$, *i.e.*

$$P^*(\ell) = \left\{ u^{*a} \cong u^{*a} + u^a \middle| u^a \in P(\ell) \right\}\,.$$

The cospinor space $P^*(\ell)$ is endowed with spinor metric $P^{ab}(\ell) = \frac{1}{2}\ell^{ab}$ and the integration measure (1.15) becomes

$$d^2u^* \equiv \frac{P_{ab}(\ell)}{4\pi} du^{*a} du^{*b}\,, \quad d^4U = -d^2u^* d^2u\,.$$

By fixing a second boundary point ℓ'^μ we can bypass the introduction of quotient spaces, as the elements of one spinor space will act as canonical representatives of equivalence classes in the other spinor space. Their measures are related as

$$d^2u = -\frac{2}{\ell \cdot \ell'} d^2u^* = -\frac{\ell'_{ab} du^a du^b}{4\pi(\ell \cdot \ell')} \,.$$

We also define a measure on the space as the inverse of the spinor metric

$$\frac{du^a du^b}{2\pi} = P^{ab}(\ell) d^2u \,.$$

For the purpose of calculational brevity, it will be useful to treat both bulk and boundary spinor decomposition spaces on the same footing. In fact, the spinor spaces $P_\pm(x), P(\ell)$ are spanned by relevant twistor matrices ${P_\pm}^a{}_b(x), P^a{}_b(\ell)$. These are special cases of a generic twistor matrix

$$P^a{}_b(\xi) = \frac{1}{2}\left(\sqrt{-\xi \cdot \xi}\; \delta^b_a + \xi^a{}_b\right)$$

with $\xi^\mu \in \mathbb{R}^{1,4}$ a null or timelike vector. In particular, for $\xi^\mu = \ell^\mu$ we recover the boundary spinor space $P(\ell)$ and for $\xi^\mu = \pm i x^\mu$ we have $P(\xi) = P_\pm(x)$, respectively. As before, we equip the spinor space $P(\xi)$ with metric and measure given as

$$\frac{du^a du^b}{2\pi} = P^{ab}(\xi) d^2u \,.$$

Similar to (1.12) we can use the above metric to introduce analogue of the twistor delta function integral over the twistor space $P(\xi)$ as

$$\delta_\xi(U) = \int_{P(\xi)} d^2v \, e^{ivU} \,. \tag{1.17}$$

In the particular cases of bulk spinors $\xi^\mu = \pm x^\mu$ we will denote the delta function as $\delta_\xi(U) = \delta^\pm_x(U)$. It will be useful to consider the integral of δ_ξ over a spinor space associated with a different spacetime point ξ'. This reads

$$\int_{P(\xi')} d^2u \, \delta_\xi(u) f(u) = \frac{2}{\sqrt{(\xi \cdot \xi)(\xi' \cdot \xi') - \xi \cdot \xi'}} f(0) \,. \tag{1.18}$$

In a similar fashion to (1.13), the Gaussian integral over $P(\xi)$, for symmetric twistor matrix A_{ab} is written as

$$\int_{P(\xi)} d^2u \, e^{\frac{1}{2}uAu} = \frac{\pm 1}{\sqrt{\det_\xi(A)}} \,,$$

where the determinant reduces to $\det_\xi(A) = -\frac{1}{2}\,\mathrm{tr}\,(P(\xi)A)^2$.

1.1.4 Euclidean de Sitter space

Throughout this body of work, and as is common in the literature, we will also consider spacetimes of Euclidean signature, namely Euclidean anti-de Sitter (EAdS_4) space.

Similarly, to our construction of dS_4 (1.1), we define EAdS_4 as the hyperboloid of unit future-point spacelike vectors embedded in $\mathbb{R}^{1,4}$

$$\mathrm{EAdS}_4 = \left\{ x^\mu \in \mathbb{R}^{1,4} \,\middle|\, x_\mu x^\mu = -1,\, x^0 > 0 \right\} . \tag{1.19}$$

Note that EAdS_4 is usually identified with the positive branch $\mathbb{H}_+$ of (1.3), and thus with hyperbolic space H_4. As before, a vector in EAdS_4 is a vector in the ambient space that is tangent to the hyperboloid (1.19), and its covariant derivative is defined as its flat $\mathbb{R}^{1,4}$ derivative projected back onto the hyperboloid. Note however, in contrast with (1.4), the EAdS_4 projector reads

$$q_{\mu\nu}(x) = \eta_{\mu\nu} + x_\mu x_\nu \, .$$

More generally, we will be able to analytically continue spacetime dependent quantities from dS_4 to EAdS_4 via $ix^\mu \to x^\mu$, as can be seen from the hyperboloid definitions (1.1, 1.19) and their respective projectors.

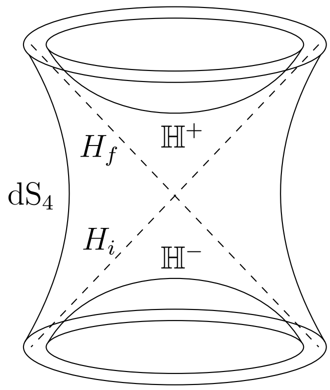


Figure 1.2: (EA)dS hyperboloids in embedding space $\mathbb{R}^{1,4}$ with two of the spacelike directions projected out. dS_4 is represented as the one-sheeted hyperboloid, whereas $\mathbb{H}^\pm$ are represented as the two-sheeted hyperboloid. EAdS_4 is usually identified with $\mathbb{H}^+$. The asymptotes of the hyperboloids are constituted by horizons H_f and H_i.

As for de Sitter, we identify the boundary of EAdS_4 with the asymptote of the hyperboloid (1.19). This has spherical topology S_3 and we represent its elements as the set of null vectors ℓ^μ, up to the identification $\ell^\mu \cong \lambda \ell^\mu$.

The isometry group of EAdS_4 is the same $SO(1,4)$ and thus we can define twistor space as we did for dS_4 in Section 1.1.2. Note however that the spinor projectors (1.14) now read

$$P_{\pm}{}^a{}_b(x) = \frac{1}{2}\left(\delta^a_b \pm x^\mu \gamma_\mu{}^a{}_b\right) .$$

Lastly, note that in Euclidean signature the antipodal map $x^\mu \to -x^\mu$ sends the hyperboloid (1.19) to its $x^0 < 0$ counterpart.

1.2 Gauge fields in dS_4

Let $\phi_{\mu_1\nu_1\mu_2\nu_2\ldots\mu_s\nu_s}(x)$ be spin-s gauge field-strength, that is, a rank-$2s$ tensor, antisymmetric within pairs of indices $\mu_i\nu_i$, but symmetric under pair exchange. Moreover, it is traceless and it vanishes upon antisymmetrizing any three indices.

At $s = 0$ we will consider a conformally massless scalar field as an honorary gauge field. The $s = 1$ case corresponds to the familiar Maxwell field strength $F_{\mu\nu}$, whereas for $s = 2$ we recognize the linearized Weyl curvature tensor $C_{\mu_1\nu_1\mu_2\nu_2}$. In general, for $s > 0$, the field-strength decomposes into a left-handed and right-handed piece, which are anti-self-dual and self-dual, respectively, in every $\mu_i\nu_i$ pair of indices.

The scalar field $\phi(x)$ satisfies the Klein–Gordon equation

$$\Box\phi - 2\phi = 0\,, \tag{1.20}$$

which is the field equation for a massless conformally coupled scalar in dS_4.

At $s = 1$ we recall the Maxwell equations

$$\nabla^\mu \phi_{\mu\nu} = 0\,; \quad \nabla_{[\rho}\phi_{\mu\nu]} = 0\,.$$

For spin $s \geq 2$ only the transverse equation is required

$$\nabla^{\mu_1}\phi_{\mu_1\nu_1\ldots\mu_s\nu_s} = 0\,. \tag{1.21}$$

To describe interacting fields we introduce gauge potentials [36] as totally symmetric rank-s tensors $h_{\mu_1\mu_2\ldots\mu_s}(x)$, double-traceless for $s \geq 4$, $g^{\mu_1\mu_2}g^{\mu_3\mu_4}h_{\mu_1\ldots\mu_s} = 0$.

For the scalar case we define the potential to coincide with the above field strength, $h(x) = \phi(x)$. At $s = 1$, as before, we retrieve the Maxwell potential A_μ, while for $s = 2$ we recognize the metric perturbation tensor $h_{\mu\nu}$.

For any spin s the potential $h_{\mu_1\mu_2\ldots\mu_s}(x)$ satisfies the Fronsdal field equation [37]

$$\begin{aligned}\big(\Box + (s^2 - 2s - 2)\big)\, h_{\mu_1\mu_2\ldots\mu_s} - s\nabla_{(\mu_1}\nabla^\nu h_{|\nu|\mu_2\ldots\mu_s)} + \\ + \frac{s(s-1)}{2}\nabla_{(\mu_1}\nabla_{\mu_2}h^\nu{}_{|\nu|\mu_3\ldots\mu_s)} + s(s-1)g_{(\mu_1\mu_2}h^\nu{}_{|\nu|\mu_3\ldots\mu_s)} = 0\,.\end{aligned} \tag{1.22}$$

Note that this admits a gauge symmetry of the form $\delta h_{\mu_1\ldots\mu_s} = \nabla_{(\mu_1}\theta_{\mu_2\ldots\mu_s)}$ for arbitrary totally symmetric and traceless $\theta_{\mu_2\ldots\mu_s}$. One can consistently define the field-strength $\phi_{\mu_1\nu_1\ldots\mu_s\nu_s}$ as the s-derivative gauge invariant quantity given by $\nabla_{\mu_1}\cdots\nabla_{\mu_s}h_{\nu_1\ldots\nu_s}$ antisymmetrized over all index pairs $\mu_i\nu_i$ and with removed traces. Note that, up to normalizations, this agrees with the standard terminology in the $s = 1, 2$ cases. Crucially, when the potential $h_{\mu_1\mu_2\ldots\mu_s}$ satisfies (1.22) the field-strength $\phi_{\mu_1\nu_1\ldots\mu_s\nu_s}$ so-defined satisfies the field equation (1.21).

We will also be interested in constructing boundary data for the field equations above. For the scalar case, the value of the field on, say, the final horizon H_f, constitutes good boundary data for the scalar field equation (1.20). For general spin, one constructs good boundary data out of two complex-conjugate scalar components for the left-handed and right-handed helicities, respectively. The standard construction has been given in [38–40], while [41] provides a general discussion in terms of field strengths.

1.3 Higher-spin gravity

Higher-spin gravity [19, 20] is a theory describing the interaction of an infinite number of massless fields with arbitrary spin; in its minimal version, there is an infinite tower of fields, one of each even integer spin. This infinite set of fields has a corresponding gauge symmetry, the infinite-dimensional higher-spin symmetry, which acts as an extension of the usual spacetime symmetry of General Relativity. The spin-2 graviton is joined by massless fields of spin $s > 2$, leading to higher-spin interactions among all of the fields, all at once, and at all orders in derivatives, which is very different from interactions in GR.

Higher-spin theory is formally defined through a set of non-pertubative equations of motion, which are invariant under the higher-spin algebra and diffeomorphisms. It can be formulated over curved spacetime with cosmological constant $\Lambda > 0$ and without requiring higher than four dimensions.

Similar to string theory, higher-spin theory has been brought into (A)dS/CFT dualities, as the bulk dual of vector models [18, 42]. Throughout this work we will not be directly talking about the non-linear bulk theory; in fact, assuming the holographic duality, one can bypass some of the difficulties of solving the bulk equations by leaving the non-linearity to the holographic dual. In particular, this can be done for the interaction vertices in the bulk theory, which have been reconstructed from the n-point functions of the boundary CFT [43]. In [44, 45] this was done for the quartic scalar vertex.

We will now proceed to described the linearized version of higher-spin theory.

1.3.1 Higher-spin algebra

In addition to spacetime coordinates x^μ, higher-spin theory is formulated over spacetime-independent twistor coordinates Y^a. We will be employing Penrose-style twistors, as introduced in Section 1.1.2; within higher-spin theory Penrose-style twistors and the Penrose transform were introduced in [46–48]. This is in contrast with standard higher-spin literature, where the "twistor" is used to refer to a pair of local spinors at a particular spacetime point, rather than a global geometric object in spacetime.

Twistor coordinates are acted upon by the non-commutative star product

$$Y^a \star Y^b = Y^a Y^b + iI^{ab} . \tag{1.23}$$

By associativity, this extends to a product on polynomial functions of Y

$$f(Y) \star g(Y) = f \exp\left(iI^{ab} \overleftarrow{\frac{\partial}{\partial Y^a}} \overrightarrow{\frac{\partial}{\partial Y^b}} \right) g$$

and also to products of non-polynomial functions, via an integral formula

$$f(Y) \star g(Y) = \int d^4U d^4V \; f(Y+U) g(Y+V) e^{-iUV} . \tag{1.24}$$

The higher-spin algebra is the infinite dimensional algebra of even functions $f(Y)$, *i.e.* of integer-spin, with associative product (1.24). It contains as a subalgebra the generators $M_{\mu\nu}$ of $SO(1,4)$, the isometry group of dS_4; in index-free notation they read as

$$M_{\mu\nu} = \frac{1}{8} Y \gamma_{\mu\nu} Y \,, \tag{1.25}$$

with commutator

$$[M^{\mu\nu}, M_{\rho\sigma}]_\star = 4\delta^{[\mu}_{[\rho} M^{\nu]}{}_{\sigma]} \,. \tag{1.26}$$

Since the infinitesimal rotations of $O(1,4)$ are generated by twistor products $Y_a Y_b$, finite rotations will be generated, through exponentiation, by Gaussian integrals (1.13). Recall however that such objects are defined only up to sign; it turns out that in the context of higher-spin symmetry, the sign ambiguity is crucial for consistent topology and cannot be fixed globally.

Further, one can define a trace operation $\mathrm{tr}_\star$ on twistor functions $f(Y)$ by evaluation at $Y = 0$, namely

$$\mathrm{tr}_\star f(Y) = f(0) \,. \tag{1.27}$$

Indeed, this respects the star product

$$\mathrm{tr}_\star(f \star g) = \int d^4U d^4V f(U) g(V) e^{iUV} = \mathrm{tr}_\star (g \star f) \tag{1.28}$$

by virtue of the fact that f and g are even functions of Y.

Taking the star product with the twistor delta function (1.12) performs a twistor Fourier transform

$$\begin{aligned} f(Y) \star \delta(Y) &= \int d^4U f(U) e^{iUY} \,; \\ \delta(Y) \star f(Y) &= \int d^4U f(U) e^{-iUY} \,. \end{aligned} \tag{1.29}$$

In particular

$$\delta(Y) \star \delta(Y) = 1 \,; \quad \delta(Y) \star f(Y) \star \delta(Y) = f(-Y) \,. \tag{1.30}$$

These properties establish $\delta(Y)$ as the Klein operator [49] of the higher-spin algebra (1.24), since it commutes with even twistor functions, and anti-commutes with odd ones. Thus, $\delta(Y)$ is invariant in the adjoint representation of the higher-spin symmetry group.

The star product (1.24), the star-trace (1.27), and the Klein operator (1.12) are in fact the only structures that preserve higher-spin symmetry. Note however that the action of $\delta(Y)$ is subtle, due to the contour ambiguities that arise in the integral construction (1.24) of higher-spin algebra with non-polynomial twistor functions. In particular, recall from Section 1.1.2 that even in the usual cases of delta functions (1.12) and Gaussians (1.13) sign ambiguities arise. Hence, one needs to be careful when interpreting the sign of $\delta(Y)$ or Fourier transforms (1.29).

Recall from Section 1.1.3 that choosing a point x in the bulk of dS_4 breaks the isometry group $SO(1,4)$ down to $SO(3)$. In the star product formalism, the action of the symmetry group on the left and right-handed subspaces is generated by bilinears of the form $y^a_+ y^b_+$ and $y^a_- y^b_-$, where we decomposed the twistor Y into Weyl spinors at

x as $Y^a = y_-^a + y_+^a$. Explicitly, the star product (1.23) decomposes as

$$y_\pm^a \star y_\pm^b = y_\pm^a y_\pm^b + iP_\pm^{ab}\,, \quad y_-^a \star y_+^b = y_+^a \star y_-^b = y_-^a y_+^b\,,$$

recalling that the projectors $P_\pm$ and the Weyl spinors $y_\pm$ depend on the point x.

The role of the twistor delta function is played by the spinor delta functions with respect to $y_\pm$ as defined in (1.17), namely they are Klein operators for the right- and left-handed higher-spin subalgebras. Note that these chiral delta functions $\delta_x^\pm(Y)$ depend on the twistor Y only through the spinor component $y_\pm$.

The boundary decomposition is a bit more subtle, but one can easily define a boundary spinor delta function $\delta_\ell(Y)$. In fact, both bulk and boundary delta functions are special cases of (1.17)

$$\delta_\xi(Y) = \int_{P(\xi)} d^2u\, e^{iuY}\,. \tag{1.31}$$

Similar to the Fourier-like identities (1.29) and (1.30) implemented through star products with $\delta(Y)$, the spinor deltas $\delta_\xi(Y)$ give us

$$\begin{aligned} f(Y)\star\delta_\xi(Y) &= \int_{P(\xi)} d^2u\, f(Y+u)e^{iuY}\,,\\ \delta_\xi(Y)\star f(Y) &= \int_{P(\xi)} d^2u\, f(Y+u)e^{-iuY}\,. \end{aligned} \tag{1.32}$$

In particular

$$\delta_\xi(Y)\star\delta(Y) = \delta(Y)\star\delta_\xi(Y) = \delta_{-\xi}(Y)\,.$$

At a bulk point x we have

$$\delta_x^\pm(Y)\star f(Y)\star\delta_x^\pm(Y) = f(\mp xY)\,,$$

where, in the index-free notation of (1.10), we read $xY \equiv (x^\mu\gamma_\mu)^a{}_b Y^b$; however, products of chiral delta function are x-independent

$$\begin{aligned} \delta_x^\pm(Y)\star\delta_x^\pm(Y) &= 1\,,\\ \delta_x^\pm(Y)\star\delta_x^\mp(Y) = \delta_x^-(Y)\delta_x^+(Y) &= \delta(Y)\,. \end{aligned}$$

To further investigate the x-dependence of $\delta_x^\pm(Y)$ we want to consider the x-derivative of the integral expression (1.31); such calculations are subtle, since the relevant domains of integration $P_\pm(x)$ are themselves functions of x. A useful technique is to consider a change of variable, *e.g.* $u_+ = P_+(x)u'_+$ where u'_+ is integrated over the spinor space $P_+(x')$ at an arbitrary fixed point x'. This method leads to

$$\nabla_\mu\delta_x^\pm = -\frac{i}{4}\,(Y\gamma_\mu xY)\star\delta_x^\pm = \frac{i}{4}\delta_x^\pm\star(Y\gamma_\mu xY)\,. \tag{1.33}$$

It will be useful to consider the star product of a pair of delta-functions at separated points. Using (1.18, 1.32) and related techniques of manipulating spinor spaces this reads

$$\delta_\xi(Y)\star\delta_{\xi'}(Y)=\frac{2}{\sqrt{(\xi\cdot\xi)(\xi'\cdot\xi')}-\xi\cdot\xi'}\exp\frac{1}{2}\frac{-iY\xi\xi' Y}{\sqrt{(\xi\cdot\xi)(\xi'\cdot\xi')}-\xi\cdot\xi'}\,. \tag{1.34}$$

Further star products with δ-functions will continue to result in Gaussian integrals. In particular, specializing to boundary points $\ell^\mu, \ell'^\mu, \ldots$ the delta-star-products simplify to

$$\delta_\ell(Y)\star\delta_{\ell'}(Y)=-\frac{2}{\ell\cdot\ell'}\exp\frac{iY\ell\ell' Y}{2\ell\cdot\ell'}\,, \tag{1.35}$$

$$\delta_\ell(Y)\star\delta_{\ell'}(Y)\star\delta_{\ell''}(Y)=\pm i\sqrt{-\frac{\ell\cdot\ell''}{2(\ell\cdot\ell')(\ell'\cdot\ell'')}}\,\delta_\ell(Y)\star\delta_{\ell''}(Y)\,. \tag{1.36}$$

Note that the three-point product (1.36) was reduced to the two-point one, and is imaginary; it also carries an ambiguous sign, as it has been performed as a Gaussian integral over a complex spinor space. Using (1.36) recursively, we can immediately derive an expression for the n-point function

$$\delta_{\ell_1}(Y)\star\cdots\star\delta_{\ell_n}(Y)=\frac{4(\pm i)^{n-2}}{\sqrt{\prod_{i=1}^n(-2\ell_i\cdot\ell_{i+1})}}\exp\frac{iY\ell_1\ell_n Y}{2\ell_1\cdot\ell_n}\,, \tag{1.37}$$

where, in the last term of the product, $\ell_{n+1}=\ell_1$.

1.3.2 Linearized higher-spin gravity and the Penrose transform

In its linearized limit, higher-spin theory describes a tower of free massless fields, one for every even spin. We describe a field of spin $s>0$ by the self-dual and anti-self-dual parts of the field strength, as discussed in Section 1.2, which now we encode in purely left- and right-handed totally symmetric spinors with $2s$ indices. Explicitly, we write the field content as

$$\text{spin }0: C^{(0,0)}\,,\quad \text{spin }1: C^{(2,0)}_{\alpha\beta}\,,\, C^{(0,2)}_{\dot\alpha\dot\beta}\,,\quad \text{spin }2: C^{(4,0)}_{\alpha\beta\gamma\delta}\,,\, C^{(0,4)}_{\dot\alpha\dot\beta\dot\gamma\dot\delta}\,,\quad \text{etc.} \tag{1.38}$$

where the bracketed upper index indicates the number of left- and right-handed spinor indices. The indices $(\alpha,\beta,\ldots)$ and $(\dot\alpha,\dot\beta,\ldots)$ are temporarily introduced to designate left- and right-handed spinor indices at a point x in the bulk; they correspond to the relevant twistor indices $(a,b,\ldots)$ with implied $P_\pm(x)$ chiral projections.

The above field strengths satisfy field equations as follows: the scalar $C^{(0,0)}(x)=h(x)$ obeys the wave equation of a conformally coupled massless field (1.20)

$$\nabla_\mu\nabla^\mu C^{(0,0)}=-2C^{(0,0)}, \tag{1.39}$$

whereas the $s>0$ field strengths satisfy the free massless equations (1.21)

$$\nabla^{\alpha_1}{}_{\dot\beta}\,C^{(2s,0)}_{\alpha_1\alpha_2\ldots\alpha_{2s}}=0\,,\quad \nabla_\beta{}^{\dot\alpha_1}\,C^{(0,2s)}_{\dot\alpha_1\dot\alpha_2\ldots\dot\alpha_{2s}}=0\,. \tag{1.40}$$

The *Penrose transform* [30, 31] is a closed form solution to the field equations (1.39, 1.40) in terms of an arbitrary even holomorphic twistor function $F(Y)$. Specifically, each separate helicity of the field content (1.38) is represented by a twistor function of homogeneity $-2 \pm 2s$. Thus, a general even function $F(Y)$ encodes one free massless field of every helicity, and hence the entire multiplet (1.38).

In the current formalism, the Penrose transform [35] reads

$$\begin{aligned} C^{(2s,0)}_{\alpha_1\alpha_2\ldots\alpha_{2s}} &= i\int_{P_+(x)} d^2u_+ \left.\frac{\partial^s F_+(u_- + u_+)}{\partial u_-^{\alpha_1}\ldots\partial u_-^{\alpha_{2s}}}\right|_{u_-=0}, \\ C_{(0,2s)}^{\dot\alpha_1\dot\alpha_2\ldots\dot\alpha_{2s}} &= i(-1)^s\int_{P_+(x)} d^2u_+\, u_+^{\dot\alpha_1}\ldots u_+^{\dot\alpha_{2s}} F_+(u_+)\,, \end{aligned} \tag{1.41}$$

where the numerical prefactors are added for later convenience and $F_+(Y)$ is an arbitrary twistor function; the subscript $\cdot_+$ refers to the domain of integration $P_+(x)$; equivalently, one can write the transform with respect to $P_-(x)$. The scalar field $C^{(0,0)}$ represents a special case, namely

$$C^{(0,0)} = i\int_{P_+(x)} d^2u_+\, F_+(u_+)\,. \tag{1.42}$$

Showing that the fields (1.41, 1.42) are solutions to the field equations (1.39, 1.40) is a fairly straightforward exercise upon changing variables so to shift the x-dependence from the domain of integration into the integrand (as described above for the derivation of (1.33)).

As previously discussed, the integral relations (1.41, 1.42) exhibit contour ambiguities. This is inherited from analogous ambiguity of the integral form (1.24) of the higher-spin algebra. The Penrose transform can be rigorously defined in terms of sheaf cohomologies [31, 50]. For the rest of the discussion in this work however we will keep in line with the current higher-spin literature and continue working in this naive formalism, keeping the ambiguity in mind.

A powerful tool at our disposal of the Vasiliev construction is the *unfolded* formalism of the field dynamics. First, we consider the full set of inequivalent on-shell derivatives of the fields (1.41) with spin $s \geq 0$

$$\begin{aligned} \left(C^{(2s+k,k)}\right)_{\alpha_1\ldots\alpha_{2s}\beta_1\ldots\beta_k}{}^{\dot\beta_1\ldots\dot\beta_k} &= \nabla_{(\beta_1}{}^{(\dot\beta_1}\ldots\nabla_{\beta_k}{}^{\dot\beta_k)}C^{(2s,0)}_{\alpha_1\alpha_2\ldots\alpha_{2s})}\,, \\ \left(C^{(k,2s+k)}\right)^{\beta_1\ldots\beta_k}{}_{\dot\beta_1\ldots\dot\beta_k\alpha_1\ldots\alpha_{2s}} &= \nabla^{(\beta_1}{}_{(\dot\beta_1}\ldots\nabla^{\beta_k}{}_{\dot\beta_k)}C^{(0,2s)}_{\dot\alpha_1\dot\alpha_2\ldots\dot\alpha_{2s})}\,. \end{aligned} \tag{1.43}$$

Thus, for every pair of integers $m+n \in 2\mathbb{Z}$, we have a field $C^{(m,n)}$, *i.e.* one for every integer-spin representation of the bulk rotation group. We can package the above into a unique scalar *master field* $C(x;Y)$ as follows

$$C(x;Y) = \sum_{m,n} \frac{1}{m!n!} C^{(m,n)}_{\alpha_1\ldots\alpha_m\dot\alpha_1\ldots\dot\alpha_n} y_-^{\alpha_1}\ldots y_-^{\alpha_m} y_+^{\dot\alpha_1}\ldots y_+^{\dot\alpha_n},$$

$$C^{(m,n)}_{\alpha_1\ldots\alpha_m\dot\alpha_1\ldots\dot\alpha_n} = (P_-)^{a_1}{}_{\alpha_1}\ldots(P_-)^{a_m}{}_{\alpha_m}(P_+)^{a_{m+1}}{}_{\dot\alpha_1}\ldots(P_+)^{a_{m+n}}{}_{\dot\alpha_n}\left.\frac{\partial^{m+n}C}{\partial Y^{a_1}\ldots\partial Y^{a_{m+n}}}\right|_{Y=0}, \tag{1.44}$$

where $y_\pm = P_\pm(x)Y$ are, as before, the chiral components of the twistor Y at the point x. The field equations (1.39, 1.40) and the definitions (1.43) can be encapsulated into the following unfolded equation

$$\nabla_\mu C = \frac{i}{4} C \star (Y\gamma_\mu x Y)\,.$$

This form makes it immediate to recognize that, by virtue of (1.33), the chiral delta function $\delta_x^\pm(Y)$ are solutions of the master field equation. Moreover, we can write a general solution as

$$C(x;Y) = \pm F_\pm(Y) \star i\delta_x^\pm(Y)\,. \tag{1.45}$$

In particular, using (1.32) and shifting the integration variable, we can read (1.45) as a Fourier transform in one of the right-handed spinor variable at x^μ, namely

$$C(x;Y) = i\int_{P_+(x)} d^2u_+\, F(u_+ + y_-)\, e^{iu_+y_+}\,. \tag{1.46}$$

The spacetime-independent functions $F_\pm(Y)$ are the Penrose transforms of the free massless solution encoded by $C(x;Y)$. In fact, they Fourier transform into each other, namely

$$F_+(Y) = -F_-(Y) \star \delta(Y)\,.$$

Recall that the chiral delta functions $\delta_x^\pm(Y)$ square to one; hence, we can relate the value master field at two points x and x' as

$$C(x';Y) = C(x;Y) \star \delta_x^+(Y) \star \delta_{x'}^+(Y)\,, \tag{1.47}$$

where the general two-point product is given by the Gaussian (1.34). The fact that we can deduce the master field at an arbitrary point x' from its value at a different point x is a feature of the unfolded formalism.

We are particularly interested in master fields with antipodal symmetry

$$C(-x;Y) = \pm C(x;Y)\,. \tag{1.48}$$

To see how this is realized in terms of spacetime-independent twistor functions, one can plug in the identity $\delta_x^-(Y) = \delta_{-x}^+(x)$ into the form of the general solution (1.45). It thus follows that antipodal symmetry (1.48) is equivalent to the following

$$F_-(Y) = \mp F_+(Y) \quad \Leftrightarrow \quad F_\pm(Y)\star\delta(Y) = \pm F_\pm(Y)\,.$$

After performing a Penrose transform on the above identity, the antipodal symmetry is re-expressed as a star-product symmetry at a point x, namely

$$C(x;Y)\star\delta(Y)=\pm C(x;Y)\,.$$

1.4 The holographic dual

As mentioned in the Introduction, the success of the AdS/CFT correspondence suggest that a similar approach could be taken to construct holography in de Sitter space via a dS/CFT correspondence. Some first insight into this construction was found in the context of dS_3 [15]. First, one notices that asymptotic symmetry of the dS_3 boundary is the conformal group $SL(2,\mathbb{C})$, as we mentioned in Section 1.1.1. Then, using the Brown–York prescription one can define the stress tensor associated with the boundary of a spacetime; its asymptotic behavior then allows one to associate a central charge to dS_3.

This was taken forward through a conjectured duality between type-A higher-spin gravity in dS_4 and a free vector model on the three-dimensional boundary at infinity [18]; this was constructed by flipping the sign of Λ in the corresponding higher-spin AdS/CFT model [42].

In this section we will describe the boundary theory corresponding to the linerized bulk higher-spin theory as expressed using the Penrose transform, in a fashion that keeps the effects of the interaction in a higher-spin organized form.

1.4.1 Boundary theory

In this section we will construct the partition function of the boundary free vector model in twistor language in a way that makes higher-spin conformal invariance manifest; this is achieved through the so-called "holographic dual" of the Penrose transform [35]. Recall that we will represent boundary points on the conformal three-sphere by null vectors ℓ^μ, up to the identification $\ell^\mu\cong\lambda\ell^\mu$.

The usual *local* action of N free massless scalars in the fundamental representation of an internal $U(N)$ symmetry [42] takes the form

$$S_{\mathrm{CFT}}=-\int d^3\ell\;\bar\phi_I\Box\phi^I, \tag{1.49}$$

where $I=1,\ldots,N$ is an internal index, ϕ^I and their complex conjugates $\bar\phi_I$ are dynamical fields of conformal weight $\Delta=\frac{1}{2}$, and with $\Box$ the conformal Laplacian operator. In the case of dS_4 the fields $\phi^I,\bar\phi_I$ have Fermi statistics, and restriction to even-spin reduces us to the $Sp(2N)$ model.

The single-trace primaries of this theory consist of an infinite tower of conserved currents $J^{(s)}$. In the three-dimensional flat section (1.6) with spatial indices $(i,j,k,\ldots)$ the spin-s currents read [51, 52]

$$J^{(s)}_{k_1\ldots k_s}=\frac{1}{i^s}\bar\phi_I\left(\sum_{m=0}^{s}(-1)^m\binom{2s}{2m}\overleftarrow{\partial}_{(k_1}\ldots\overleftarrow{\partial}_{k_m}\overrightarrow{\partial}_{k_{m+1}}\ldots\overrightarrow{\partial}_{k_s)}-\text{traces}\right)\phi^I\,, \tag{1.50}$$

where the "traces" terms contain contractions of one or more pairs of the k_i indices.

Note that this formulation includes an honorary scalar "current" $J^{(0)} = \bar{\phi}_I \phi^I$, the spin-1 current $J^{(1)}_i = \frac{1}{i}\bar{\phi}_I \overleftrightarrow{\partial}_i \phi^I$ as the ordinary $U(1)$ charge, and a spin-2 current proportional to the stress-energy tensor $J^{(2)}_{ij} = 8T_{ij}$.

The sources for the operators (1.50) are spin-s gauge potentials $A^{(s)}_{\mu_1\dots\mu_s}$; adding linear couplings to such external sources, the action (1.49) becomes

$$S_{\text{CFT}} = -\int d^3\ell \, \bar{\phi}_I \Box \phi^I - \int d^3\ell \, \sum_{s=0}^{\infty} A^{(s)}_{\mu_1\dots\mu_s}(\ell) J^{\mu_1\dots\mu_s}_{(s)}(\ell) \, , \tag{1.51}$$

where we can use the $\mathbb{R}^{1,4}$ indices $(\mu, \nu, \dots)$ for the currents since we are not using the explicit form (1.50) with its flat three-dimensional derivatives.

The *bilocal* formulation [35, 53, 54] is a convenient rewriting of the theory upon noticing that the local primaries $J^{(s)}_{k_1\dots k_s}$ are a cumbersome Taylor expansion of the two-point inner product $\phi^I(\ell)\bar{\phi}_I(\ell')$. Replacing $J^{(s)}$ with bilocal operators $\mathcal{O}(\ell,\ell') \equiv \phi^I(\ell)\bar{\phi}_I(\ell')$ which couple to bilocal sources $\Pi(\ell',\ell)$ allows for the CFT to be rewritten as

$$S_{\text{CFT}} = -\int d^3\ell \, \bar{\phi}_I \Box \phi^I - \int d^3\ell' d^3\ell \, \bar{\phi}_I(\ell') \Pi(\ell',\ell) \phi^I(\ell) \, . \tag{1.52}$$

In this formulation the partition function[3] of the theory can be easily obtained. First, rewrite the action in matrix-like notation

$$S_{\text{CFT}} \left[\Pi(\ell',\ell)\right] = -\bar{\phi}_I \left(\Box + \Pi\right) \phi^I \, ,$$

where $\phi(\ell)$ can be viewed as an infinite dimensional vector with dual $\bar{\phi}(\ell)$, and $\Box$ and Π as matrices/Hilbert space operators. The Gaussian path integral over ϕ and $\bar{\phi}$ gives

$$\begin{aligned} Z_{CFT}\left[\Pi(\ell',\ell)\right] = \int \mathcal{D}\phi \mathcal{D}\bar{\phi} \, e^{-S_{\text{CFT}}} = \left(\det\left(\Box + \Pi\right)\right)^N \sim \\ \sim \left(\det\left(1 + G\Pi\right)\right)^N = \exp\left(N \operatorname{tr} \ln\left(1 + G\Pi\right)\right) \, , \end{aligned} \tag{1.53}$$

where we define $\det M = \exp \operatorname{tr} \ln M$ and $\ln M$ formally via a Taylor expansions. The operator $G = \Box^{-1} = -\frac{1}{4\pi r}$ is the boundary-to-boundary propagator and can be written covariantly as

$$G(\ell,\ell') = -\frac{1}{4\pi\sqrt{-2\ell\cdot\ell'}} \, . \tag{1.54}$$

This is computed as $G(\mathbf{r},\mathbf{r}') = -\frac{1}{4\pi\sqrt{-2|\mathbf{r}-\mathbf{r}'|}}$ in the flat frame (1.6).

1.4.2 Holographic dual of the Penrose transform

To translate the CFT partition function into higher-spin-algebraic language, one replaces CFT sources with elements of the higher-spin algebra, namely twistor function,

[3] We will be discussing the partition function in more detail in Chapter 3; in this section we will be interested Z_{CFT} as the generating functional of correlation functions, and in particular we will use it to extract the expectation value of a boundary bilocal operator.

which then are combined into higher-spin invariants. This approach has been undertaken both from the boundary [55, 56] and the bulk side [57–59]. In the framework of [35] the so-called "holographic dual" of the Penrose transform is obtained by packaging the bilocal source $\Pi(\ell', \ell)$ into a twistor function

$$F(Y) = \int d^3\ell' d^3\ell \; K(\ell, \ell'; Y)\Pi(\ell', \ell)\,, \tag{1.55}$$

where the bilocal kernel (or "twistor-boundary-boundary propagator") is given by

$$K(\ell, \ell'; Y) = \frac{\sqrt{-2\ell \cdot \ell'}}{4\pi}\delta_\ell(Y) \star \delta_{\ell'}(Y) = \frac{1}{\pi\sqrt{-2\ell \cdot \ell'}} \exp \frac{iY\ell\ell' Y}{2\ell \cdot \ell'}\,, \tag{1.56}$$

and we used the boundary specialization of the two-point Gaussian (1.35).

It turns out that the twistor function $F(Y)$ as constructed above is in fact an arbitrary function of Y, *i.e.* the kernels $K(\ell, \ell'; Y)$, as a set, span the higher-spin algebra. Moreover, the twistor encoding (1.55) of the CFT sources is complete, it captures the gauge-invariant information in an optimal sense (it is free of higher-spin gauge redundancy), and it is constraint free, up to contour choice subtleties.

The rewriting of the partition function in terms of twistor functions relies on the following identities

$$K(\ell_1, \ell_1'; Y) \star K(\ell_2, \ell_2'; Y) = G(\ell_2, \ell_1')K(\ell_1, \ell_2'; Y)\,, \tag{1.57}$$

$$\mathrm{tr}_\star\, K(\ell, \ell'; Y) = -4G(\ell, \ell')\,. \tag{1.58}$$

These identities can be derived from the properties of two- and three-point products of spinor delta functions (1.35,1.36).

The single-trace products involved in the construction of Z_{CFT} can be written as

$$\mathrm{tr}(G\Pi)^n = \frac{1}{4}\,\mathrm{tr}_\star\left(F(Y) \star F(Y) \star \cdots \star F(Y)\right)$$

where $F(Y)$ appears n times. Thus, the entire partition function (1.53) can be rewritten in higher-spin language as

$$Z_{\mathrm{CFT}}\left[F(Y)\right] = \exp\left(-\frac{N}{4}\,\mathrm{tr}_\star \ln_\star\left[1 + F(Y)\right]\right) = \left(\det{}_\star\left[1 + F(Y)\right]\right)^{\frac{N}{4}}. \tag{1.59}$$

As before $\ln_\star\left[1 + F(Y)\right]$ is defined formally by substituting star products into the Taylor expansion and we define the *star determinant* as $\det_\star f = \exp \mathrm{tr}_\star \ln_\star f$.

From (1.59) one can read the expectation value of the bilocal operator $\phi^I(\ell)\bar{\phi}_I(\ell')$ as

$$\langle\phi^I(\ell)\bar{\phi}_I(\ell')\rangle = \frac{N}{4}\,\mathrm{tr}_\star\left(K(\ell, \ell'; Y) \star \left(1 - F(Y) + O(F^2)\right)\right). \tag{1.60}$$

The main achievement of [35] was to show that both the bulk and boundary pictures are encoded by the same twistor function $F(Y)$, up to discrete symmetries subtleties. Specifically, it was shown that, away from sources, the asymptotic boundary data of the linearised bulk solution (1.45) (as given by the Penrose transform of $F(Y)$) reproduces the linearised expectation values (1.60) of the CFT operators, upon translating them into local currents. Thus, the two-bilocal correlators of the CFT partition function (1.59) are in direct relation with the linearised bulk solution. Then, higher-point functions of (1.59) can be viewed as encoding bulk interactions.

Chapter 2

Spinor-helicity variables for cosmological horizons

In the quest of constructing static patch dS / CFT holography, one must contend with the fact that the boundary is unobservable; since the causal patch and the boundary only intersect at two points (the endpoints of the eternal observer's worldline) the holographic dictionary must be non-local.

One can encode boundary fields as functions of two spinors $(\Sigma_\alpha, \Delta_\alpha)$ located at the boundary endpoint n^μ of the static patch, in a way that is compatible with higher-spin symmetry. This language has been developed before as the on-shell version [60–62] of the bilocal description of the boundary vector model [53]. As will be described below, these spinors act as the square roots of boundary on-shell momenta $p_\mu = \Sigma n \gamma_\mu \Delta$, and thus will be called *spinor-helicity* variables, mirroring the Minkowski construction [63, 64]. A similar formalism has been developed in [29, 55]. Further, in our embedding twistor formalism, these spinor variables appear in a Wigner–Weyl transform between twistor functions as functions on phase space and operators in the quantum theory of the boundary particle [46]

$$F(Y) = \int d^2\Sigma d^2\Delta\, f(\Sigma, \Delta) e^{i\Delta Y} \delta(Y - \Sigma)\ . \tag{2.1}$$

It becomes a natural question to ask what are these boundary quantities correspond to in the bulk. Such spinor-helicity variables were first introduced for the Poincaré patch in [65] (see [66, 67] for similar AdS constructions). We will show that the basis coefficients $f(\Sigma, \Delta)$ correspond to the boundary limit of the free bulk master field encoded in (2.1).

Further, since n^μ is the endpoint of the future horizon H_f of the static patch observer, we will investigate the corresponding bulk field modes on this horizon and determine their symplectic structure. It turns out that $f(\Sigma, \Delta)$ also serve as a prescription to encode field data on the horizon. Furthermore, when two such horizons are specified (*eg.* by a second boundary point n'^μ, antipodally related to the initial endpoint of the observers' worldline, and with spinor-helicity basis coefficient $f'(\Xi, \Lambda)$), the spinor-helicity basis coefficients on the two horizons are related by a Fourier transform

(2.38) in the relevant spinor variables

$$f(\Sigma, \Delta) = \frac{1}{2\pi i} \int d^2\Xi d^2\Lambda \, e^{\Sigma\Xi - \Delta H} f'(\Xi, \Lambda) \, .$$

Lastly, we will use this result to construct the de Sitter S-matrix, mapping between null final $\Phi(x^\mu)$ and null initial data $\Psi(x'^\mu)$ on the two horizons. This will generalize the scalar result; at general spin-s this will read

$$\Phi^{(s,0)}(x^\mu) = \frac{2^s}{\pi u^{2s+1}} \int_{S_2} d^2\mathbf{r}' \, (1 - r_\mu r'^\mu)^s \, \frac{\partial}{\partial v} \Psi^{(s,0)}(x'^\mu) \bigg|_{uv = 2(r_\mu r'^\mu - 1)} \, . \tag{2.47}$$

This will constitute the main result of this chapter and has been published, albeit in a more straight-forward fashion, in [23]. In the present formalism, the bulk-to-boundary limit of the spinor basis coefficients has also been presented in [24].

2.1 Spinor-helicity variables

2.1.1 Boundary decomposition

Descending from the more abstract description of the boundary theory in terms of twistor variables, the spinor-helicity approach aims to describe it in terms of a plane waves basis. Such description requires choosing a particular flat conformal frame, which can be achieved by picking out a point n^μ on the S_3 boundary, to be thought of as "the point at infinity". Furthermore, in order to fix the phases of such plane-waves basis elements, we can choose a second boundary point n'^μ which amounts to choosing a "origin" of this flat frame. Recall that choosing such two boundary points amounts to choosing a bulk dS_4 observer, where we identify the past and future endpoints of this observer's worldline with $-n'^\mu$ and n^μ, respectively.

Namely, without loss of generality, we fix an observer with

$$n^\mu = \left(\frac{1}{2}, \frac{1}{2}, \mathbf{0}\right); \quad n'^\mu = \left(\frac{1}{2}, -\frac{1}{2}, \mathbf{0}\right) . \tag{2.2}$$

We can coordinatize the horizons corresponding to these boundary points as

$$\text{future horizon } H_f : \quad x^\mu = u n^\mu + (0, 0, \mathbf{r}) \, ; \tag{2.3}$$

$$\text{past horizon } H_i : \quad x'^\mu = v n'^\mu + (0, 0, \mathbf{r}) \, , \tag{2.4}$$

where $u, v \in \mathbb{R}$ are affine null times, and $\mathbf{r}$ is a three-dimensional unit vector on the S_2 horizon section. The causal patch of the observer is the region enclosed between H_i with $v < 0$ and H_f with $u > 0$. We can parameterize these half-horizons by replacing null times u, v with an observer time t

$$\begin{aligned} H_f : &\quad x^\mu = e^t n^\mu + (0, 0, \mathbf{r}) \, ; \\ H_i : &\quad x'^\mu = -e^{-t} n'^\mu + (0, 0, \mathbf{r}) \, . \end{aligned}$$

Note that the choice of this two points breaks the $O(1,4)$ dS_4 symmetry down to $SO(1,1)\times O(3)$. Thus, a twistor Y^a decomposes into $SO(3)$ spinors as follows:

$$Y^a=\begin{pmatrix} y'_\alpha \\ y^\alpha \end{pmatrix}; \quad Y_a=\begin{pmatrix} -y^\alpha \\ y'_\alpha \end{pmatrix}$$

This decomposition is consistent with the previously introduced realizations of gamma matrices (1.7) and twistor metric (1.8).

Note that our boundary points (2.2) admit twistor matrix formulations:

$$n^a{}_b=\frac{1}{2}(\gamma_0+\gamma_4)^a{}_b=\begin{bmatrix} 0 & 0 \\ -\epsilon^{\alpha\beta} & 0 \end{bmatrix}; \quad n'^a{}_b=\frac{1}{2}(\gamma_0-\gamma_4)^a{}_b=\begin{bmatrix} 0 & \epsilon_{\alpha\beta} \\ 0 & 0 \end{bmatrix},$$

where $\epsilon_{\alpha\beta}$ is the antisymmetric spinor metric and $\epsilon^{\alpha\beta}$ its inverse, $\epsilon^{\alpha\gamma}\epsilon_{\beta\gamma}=\delta^\alpha_\beta$. Thus, in this decomposition, the twistor subspace spanned by n^{ab} contains the upper-index spinors y^α, whose "squares" form the boundary vectors of the $\mathbb{R}^3$ conformal frame associated with the point n^μ. Similarly, the n'^{ab} subspace contain "primed" lower-index spinors y'_α, which square to covectors in the $\mathbb{R}^3$ frame.

Further, under this twistor decomposition, the higher-spin algebra (1.23) reduces to

$$\begin{aligned} y'_\alpha\star y'_\beta &= y'_\alpha y'_\beta\ ; \\ y^\alpha\star y^\beta &= y^\alpha y^\beta\ ; \\ y'_\alpha\star y^\beta &= y'_\alpha y^\beta - i\delta^\beta_\alpha\ ; \\ y^\alpha\star y'_\beta &= y^\alpha y'_\beta + i\delta^\alpha_\beta\ . \end{aligned}$$

To clarify the physical meaning of these spinor variables, consider the way in which the $O(1,4)$ symmetry generators Y^aY^b decompose:

$y'_\alpha y'_\beta$	$\mathbb{R}^3$ *translations*, broken by choice of "origin" n'^μ ;
$y^\alpha y^\beta$	*special conformal transformation*, broken by choice of "point at infinity" n^μ ;
$y'_\alpha y^\alpha$	*dilations* corresponding to time translations $t\to t+\tau$, $Y^a=(y'_\alpha,y^\alpha)\to\left(e^{-\frac{\tau}{2}}y'_\alpha,e^{+\frac{\tau}{2}}y^\alpha\right)$, unbroken;
traceless part of $y'_\alpha y^\beta$	$SO(3)$ *rotations*, unbroken.

Thus, since the translation generator $y'_\alpha y'_\beta$, *i.e.* the boundary momentum can be written as the square of a spinor variable, we denote this variable y'_α as a "momentum spinor". Similarly, y^α squares into the generator of spatial conformal transformations, that is, the momentum with respect to the inverted $\mathbb{R}^3$ frame upon interchanging n^μ and n'^μ.

To see this more accurately, recall the $O(1,4)$ generators $M_{\mu\nu}$ as expressed in (1.25, 1.26). As $M_{\mu\nu}$ is a simple, totally null bivector, its "direction" defines a totally null 2-plane through the origin of $\mathbb{R}^{1,4}$, that is, a projective lightray. Hence, we are left with the "magnitude" of $M_{\mu\nu}$ to parameterise the magnitude of the null momentum. Concretely, at each point ℓ^μ on the flat section (1.6) defined by $\ell\cdot n=-\frac{1}{2}$, we encode the null momentum as a vector

$$p_\mu=2M_{\mu\nu}n^\nu \tag{2.5}$$

which is consistent with the bracket in (1.26). Further, recall that generators $M_{\mu\nu}$ can be expressed in terms of twistor variables

$$M_{\mu\nu} = \frac{1}{8} Y \gamma_{\mu\nu} Y \,. \tag{1.25}$$

Thus, isolating the spatial components of the momentum (2.5)

$$p_k = 2 M_{k\nu} n^\nu = \frac{1}{4} Y \gamma_k n Y = \frac{i}{4} \left(\sigma_k\right)^{\alpha\beta} y'_\alpha y'_\beta = \frac{i}{4} y' \sigma_k y' \,. \tag{2.6}$$

Thus we have established the direct relation between momentum spinor y'_α and flat section momentum p_k.

Now, on a horizon, say the future horizon H_f (2.3), rotations act on the S_2 section in the usual way, dilations rescale null time u, while translations along a vector $\mathbf{v}$ shifts $u \to u - 2\mathbf{v} \cdot \mathbf{r}$. A fixed momentum $\mathbf{p}$ with respect to these translations describes two modes on the horizon

$$\delta^2 \left(\mathbf{r}, \pm \frac{\mathbf{p}}{|\mathbf{p}|} \right) e^{\mp \frac{iu}{2} |\mathbf{p}|} \tag{2.7}$$

of positive and negative frequency, respectively: waves with frequency $\pm\frac{|\mathbf{p}|}{2}$ with respect to null time u, with support on antipodal pair of light rays $\mathbf{r} = \pm \frac{\mathbf{p}}{|\mathbf{p}|}$.

2.1.2 Boundary quantum mechanics

We can use the above description to identify HS algebra with the quantum mechanics operator algebra in the quantum mechanics of a free massless particle in a 2 + 1-dimensional boundary spacetime. This has been realized before in [21, 68, 69], while [46] presents a version of the construction using twistor language.

Since the theory describing a free massless particle is conformal, we identify its three-dimensional Lorentzian spacetime as the projective lightcone in an embedding $\mathbb{R}^{2,d}$. Every point in the original spacetime is identified with a lightray passing through the origin of the embedding $\mathbb{R}^{2,d}$, while a lightray becomes a totally null plane. Accounting for the particle's energy, we find ourselves in the situation outlined in the previous section: identifying the phase space of the particle with totally null bivectors $M_{\mu\nu}$ which encodes the energy-momentum of the particle as in (2.5).

Recall that the Poisson brackets of the generators $M_{\mu\nu}$ read $\{M^{\mu\nu}, M_{\rho\sigma}\} = 4\delta^{[\mu}_{[\nu} M^{\nu]}{}_{\sigma]}$. This form is fixed by conformal symmetry up to a normalization constant, which has been chosen as to make (2.5) the translation generator [46]. Recall from (1.25) that a totally null bivector $M_{\mu\nu}$ can be written as the square of a twistor Y^a; this new phase space variable will have Poisson bracket $\left\{Y^a, Y^b\right\} = 2I^{ab}$, which is again fixed by conformal symmetry and normalized to match (1.25). This implies that the symplectic form reads $\Omega_{ab} = -\frac{1}{2} I_{ab}$.

To quantize the boundary particle, we upgrade the Poisson bracket of the twistor phase space variable Y^a into the commutator $\left[\hat{Y}^a, \hat{Y}^b\right] = 2iI^{ab}$. A quantum operator $\hat{f}$ is represented as a twistor function $f(Y)$, noting that the product $Y^{a_1} \cdots Y^{a_n}$ corresponds to the product of operators $\left(\hat{Y}^{a_1}, \ldots, \hat{Y}^{a_n}\right)$ symmetrized over all indices.

Further, the product of two general operators $\hat{f}\hat{g}$ corresponds to the Moyal star product $f(Y) \star g(Y)$ [46]; this is in fact the higher-spin algebra star product (1.23,1.24). Thus, higher-spin algebra is identified with the algebra of operators in the quantum mechanics of the boundary particle.

The trace operations on the two algebras are not identical, but proportional to each other

$$\mathrm{tr}\,\hat{f} = \frac{1}{4}\int d^4Y\, f(Y) = \frac{1}{4}\int \mathrm{tr}_\star\left(f(Y) \star \delta(Y)\right)\,, \tag{2.8}$$

where the proportionality constant arises from the ratio of the twistor measure d^4Y constructed directly from I_{ab} and from the symplectic form Ω_{ab}.

Having identified twistor space as a phase space we can see that the spinors components (y'_α, y^α) of a twistor Y^a play the role of configuration and momentum variables; in particular recall from (2.6) that we can identify the square of y'_α as the momentum in the flat frame defined by n^μ. Thus, pure states in the theory of the boundary particle will be expressed as wave functions $\psi(y'_\alpha)$.

This formalism can be expressed through a Wigner–Weyl transform [70], relating the representation of an operator $\hat{F}$ in the boundary quantum mechanics as a phase space function $F(Y)$ and as matrix elements $\tilde{f}(\lambda_\alpha, \mu_\alpha)$ between states $y'_\alpha = \lambda_\alpha$ and $y'_\alpha = \mu_\alpha$.

Even though this has been developed for a Lorentzian boundary, changing to Euclidean signature introduces little change: the phase space coordinates Y^a and configuration variables y'_α become complexified, and the relation (2.8) gains a minus sign which can be traced back to the measure d^4Y changing sign.

Finally, the Wigner–Weyl transform reads

$$F(Y) = \int_{P(n')} d^2\lambda d^2\mu\, K(\lambda_\alpha, \mu_\alpha; Y)\tilde{f}(\lambda_\alpha, \mu_\alpha) \tag{2.9}$$

with kernel

$$K(\lambda_\alpha, \mu_\alpha; Y) = \delta\left(y'_\alpha - \frac{\lambda_\alpha + \mu_\alpha}{2}\right) e^{\frac{i}{2}(\lambda_\alpha - \mu_\alpha)y^\alpha}\,. \tag{2.10}$$

Following [46], we impose a reality condition $\mu_\alpha = \pm i\bar{\lambda}_\alpha$ on our spinor variables, where the orientation of this real contour corresponds to positive and negative frequency modes (2.7), respectively. Note that, since the transformation kernel (2.10) satisfies

$$K(\lambda_\alpha, \mu_\alpha; Y) = -K(i\mu_\alpha, i\lambda_\alpha; iY)\,,$$

the Wigner–Weyl transform implies the following equivalence

$$\tilde{f}(\lambda_\alpha, \mu_\alpha) = \tilde{f}(i\mu_\alpha, i\lambda_\alpha) \Longleftrightarrow F(Y) = -F(iY)\,. \tag{2.11}$$

The left-hand side of (2.11) is enough to ensure the reality and positivity of the Hermitian norm on the boundary theory, while the discrete symmetry introduced by the right-hand side will restrict us to even bulk spins. One can further argue the naturalness of this reality condition by noting that, in terms of boundary momenta $\mathbf{p}, \mathbf{p}'$ it simply corresponds to $\mathbf{p}' = -\bar{\mathbf{p}}$, while the symmetry introduced by (2.11) interchanges momenta and flips their signs: $(\mathbf{p}, \mathbf{p}') \to (-\mathbf{p}', -\mathbf{p})$. Were our three-dimensional boundary Lorentzian, we would identify this with a CPT reflection. Hence, restricting to even bulk spins is equivalent to restricting to CPT-invariant boundary operators.

It will be convenient to work in redefined spinor variables

$$\Sigma_\alpha = \frac{\lambda_\alpha + \mu_\alpha}{2}\,; \quad \Delta_\alpha = \frac{\lambda_\alpha - \mu_\alpha}{2}\,, \tag{2.12}$$

in terms of which the kernel (2.10) can be written as

$$K(\Sigma_\alpha + \Delta_\alpha, \Sigma_\alpha - \Delta_\alpha; Y) = \delta(y'_\alpha - \Sigma_\alpha)\, e^{i\Delta_\alpha y^\alpha}\,, \tag{2.13}$$

and $\tilde{f}(\lambda_\alpha, \mu_\alpha) = \frac{1}{4} f(\Sigma_\alpha, \Delta_\alpha)$, so that (2.9) becomes

$$F(Y) = \int_{P(n')} d^2\Sigma d^2\Delta\, \delta(y'_\alpha - \Sigma_\alpha)\, e^{i\Delta_\alpha y^\alpha} f(\Sigma_\alpha, \Delta_\alpha)\,. \tag{2.14}$$

Note that we can perform the Σ_α spinor integral to be left with

$$F(Y) = \int_{P(n')} d^2\Delta\, f(y, \Delta_\alpha)\, e^{i\Delta_\alpha y^\alpha}\,.$$

This last form can be easily inverted as a Fourier transform to give

$$f(\Sigma, \Delta) = \int_{P(n)} d^2y\, F(y + \Sigma_\alpha)\, e^{-i\Delta_\alpha y^\alpha}\,. \tag{2.15}$$

2.2 Bulk interpretation

We will consider the bulk master field (1.45, 1.46) constructed from (2.13). First, we will show that the spinor function $f(\Sigma, \Delta)$ can be interpreted as the boundary limit of these bulk master fields. This limiting procedure, in standard higher-spin formalism, was first introduced in [29]. Next, we will specialize to a cosmological horizon, where we will conduct the Penrose transform explicitly.

To make the ensuing discussion more legible, we will mostly use the index free notation (1.10), unless indices are required for clarity. Similarly, where spinor indices are employed, we will use the equivalent twistor indices.

2.2.1 Boundary limit

A physical interpretation of the modes $f(\Sigma, \Delta)$ can be seen by investigating the boundary limit of bulk master-field corresponding to the twistor function (2.14). Namely, the bulk Penrose transform (1.46) of $F(Y)$ reads, for twistor $U = (u', u)$,

$$C(x; U) = i \int_{P_+(x)} d^2w\, F(w + u')\, e^{iwu}\,. \tag{2.16}$$

Considering the limit in which the bulk point x approaches the boundary at n, along the geodesic connected to n', we let

$$x^\mu = \lim_{z\to 0}\left(\frac{1}{z}n^\mu + zn'^\mu\right) .$$

The corresponding projector (1.14) reads

$$P_\pm{}^a{}_b(x) = \frac{1}{2}\left(\pm\frac{1}{z}n^a{}_b + \delta^a_b \pm zn'^a{}_b\right) .$$

Performing a change of variables

$$\begin{aligned} w &= 2P(x)y = (1+zn')y\,, \\ u &= P_+(x)\Delta = \frac{1}{2}\left(1+\frac{1}{z}n\right)\Delta\,, \\ u' &= P_-(x)\Sigma = \frac{1}{2}\left(1-\frac{1}{z}n\right)\Sigma\,, \end{aligned}$$

the integration measure (1.16) transforms as

$$d^2w = \frac{P^+_{ab}dw^a dw^b}{4\pi} = \frac{P^+_{ab}dy^a dy^b}{\pi} = \frac{zn'_{ab}}{2\pi} = zd^2y\,. \tag{2.17}$$

Thus, the master field (2.16) asymptotes to

$$C(x;U) \sim iz\int_{P_+(n)} d^2y\, F\left(y + zn'y - \frac{1}{2z}n\Sigma + \frac{1}{2}\Sigma\right)e^{iy\Delta} \quad \text{as } z\to 0\,.$$

To simplify the form of the argument, we shift the integration variable $y \to y + \frac{1}{2z}n\Sigma$ so to write

$$\begin{aligned} C(x;U) &\sim ize^{\frac{i\Sigma y\Delta}{2z}}\int_{P_+(n)} d^2y\, F\left(y+\Sigma\right)e^{iy\Delta} \quad \text{as } z\to 0 \\ &\sim ize^{\frac{i\Sigma y\Delta}{2z}} f(\Sigma,\Delta) \quad \text{as } z\to 0\,, \end{aligned}$$

where in the last line we have identified the boundary basis coefficients (2.15), which hence can be see boundary limit of the free bulk master field (2.16).

2.2.2 Horizon modes

Let us restrict for now to positive frequency modes, by choosing to the real contour $\mu_\alpha = i\bar\lambda_\alpha$. Thus the redefined variables (2.12) become

$$\Sigma_\alpha = \frac{\lambda_\alpha + i\bar\lambda_\alpha}{2}\;;\quad \Delta_\alpha = \frac{\lambda_\alpha - i\bar\lambda_\alpha}{2} = -i\bar\Sigma_\alpha\,. \tag{2.18}$$

Further, we specialize to the future horizon H_f, which we parametrize according to (2.3) as $x^\mu = un^\mu + r^\mu$, with null-time $u > 0$ and r^μ parameterizing the S_2 horizon section. Further, we can introduce spinors $\phi_a, {\phi'}_a$ and so that

$$r_{ab} = i\left[4\phi_{[a}\phi'_{b]} - (\phi'\phi)I_{ab}\right] .$$

Since $r^2 = (\phi'\phi)^2$ we choose normalization $\phi'\phi = 1$. Furthermore, we introduce complex conjugate spinors $(\bar\phi, \bar\phi')$, in terms of which can write $n_{ab} = 2\phi_{[a}\bar\phi_{b]}$, $n'_{ab} = 2\phi'_{[a}\bar\phi'_{b]}$, and $I_{ab} = 2\left(\phi_{[a}\phi'_{b]} + \bar\phi_{[a}\bar\phi'_{b]}\right)$.

In this notation, the twistor matrix for the position variable becomes

$$x_{ab} = un_{ab} + 4i\phi_{[a}\phi'_{b]} - iI_{ab}$$

with projector on the left spinor subspace

$$P_{-ab}(x) = \frac{1}{2}\left(I_{ab} - ix_{ab}\right) = -\frac{iu}{2}n_{ab} + 2\phi_{[a}\phi'_{b]} = -iu\phi_{[a}\bar\phi_{b]} + 2\phi_{[a}\phi'_{b]} = 2\phi_{[a}U_{b]}\,, \quad (2.19)$$

where $U = -\frac{iu}{2}\bar\phi + \phi' = \left(1 - \frac{iu}{2}n\right)\phi'$. Similarly, for the right spinor subspace

$$P_{+ab}(x) = I_{ab} - P_{-ab}(x) = 2\bar\phi_{[a}\bar\phi'_{b]} - iu\bar\phi_{[a}\phi_{b]} = 2\bar\phi_{[a}\bar U_{b]} \quad (2.20)$$

with $\bar U = \bar\phi' - \frac{iu}{2}\phi$.

Scalar modes

To determine the field strengths encoded by the twistor function $F(Y)$ we perform the Penrose transform (1.46) at a horizon point x^μ. At scalar level this reads

$$\begin{aligned}\Phi^{(0)}(x) &= \int_{P_\pm(x)} d^2Y F(Y) \\ &= \int_{P_\pm(x)} d^2Y \int_{P(n')} d^2\lambda d^2\mu\, K(\lambda, \mu\, Y) f(\lambda, \mu) \\ &= i\int_{P(n')} d^2\Sigma d^2\bar\Sigma\, C^{(0)}(\Sigma, \bar\Sigma; x) f(\Sigma, \bar\Sigma). \end{aligned} \quad (2.21)$$

where we will denote by $C^{(0)}(\Sigma, \bar\Sigma; x)$ the scalar Penrose-transformed Wigner–Weyl kernel (2.10). This takes the form

$$C^{(0)}(\Sigma, \bar\Sigma; x) = \int_{P_-(x)} d^2W\, \delta(\Sigma - w')e^{\bar\Sigma w}. \quad (2.22)$$

Any twistor W in the $P_-(x)$ subspace (2.19) can be written as $W = A\phi + B\left(1 - \frac{iu}{2}n\right)\phi'$, with A and B complex numbers; in spinor decomposition this becomes $W = (w, w') = \left(A\phi - B\frac{iu}{2}n\phi', B\phi'\right)$, with differential $dW = (dA)\phi + (dB)\left(1 - \frac{iu}{2}n\right)\phi'$, and hence measure

$$d^2W = \frac{dW\,dW}{2(2\pi)} = \frac{dA\,dB}{2\pi}\left(\phi\left(1 - \frac{iu}{2}n\right)\phi'\right) = -\frac{dA\,dB}{2\pi}.$$

Now, we can compute our Penrose-transformed kernel (2.22) as follows

$$
\begin{aligned}
C^{(0)}(\Sigma,\bar{\Sigma};x) &= \int_{P_-(x)} d^2W\, e^{\bar{\Sigma}w}\delta(\Sigma - w') \\
&= -\frac{1}{2\pi}\int dAdB\, e^{\bar{\Sigma}(A\phi - B\frac{iu}{2}n\phi')}\delta\left(B\phi' - \Sigma\right) .
\end{aligned} \tag{2.23}
$$

Decomposing Σ in the spinor basis $\left(\phi',\bar{\phi}'\right)$ as $\Sigma^a = \left(\Sigma^b\phi_b\right)\phi'^a + \left(\Sigma^b\bar{\phi}_b\right)\bar{\phi}'^a$ we can rewrite the spinor delta function in the integrand as a product of single component delta functions

$$
\delta(B\phi' - \Sigma) = \frac{1}{\phi'_a\phi^a}\delta\big(B - \Sigma^b\phi_b\big)\,\delta\big(\Sigma^b\bar{\phi}_b\big)
$$

so to perform the B-integral in (2.23) as follows

$$
\begin{aligned}
C^{(0)}(\Sigma,\bar{\Sigma};x) &= -\frac{1}{2\pi}e^{-\bar{\Sigma}_a\left(\frac{iu}{2}\Sigma^b\phi_b\bar{\phi}^a\right)}\delta\big(\Sigma^b\bar{\phi}_b\big)\int dA\, e^{A\bar{\Sigma}\phi} \\
&= -ie^{\frac{iu}{2}(\Sigma\phi)(\bar{\Sigma}\bar{\phi})}\delta\big(\Sigma\bar{\phi}\big)\,\delta\big(\bar{\Sigma}\phi\big) .
\end{aligned} \tag{2.24}
$$

Spin-s modes

For left-handed and right-handed field strengths of arbitrary spin-s we use the following formulations of the Penrose transform *cf.* (1.41), respectively

$$
C^{(2s,0)}_{a_1...a_{2s}} = -i(-1)^s\int_{P_-(x)} d^2w\, W_{a_1}...W_{a_{2s}}F(W) , \tag{2.25}
$$

$$
C^{(0,2s)}_{a_1...a_{2s}} = -i\int_{P_-(x)} d^2w\ \left.\frac{\partial^s F(W)}{\partial W^{a_1}...\partial W^{a_{2s}}}\right|_{W=w} . \tag{2.26}
$$

Recall the left-handed field strength $C^{(2s,0)}_{a_1...a_{2s}}$ is symmetric in its $2s$ spinor indices and thus has $2s+1$ independent components. Since the left-handed spinor space (2.19) is spanned by ϕ_a and U_a, we can recover the independent components of $C^{(2s,0)}_{a_1...a_{2s}}$ via contraction of the form $C^{(2s,0)}_{a_1...a_ja_{j+1}...a_{2s}}\phi^{a_1}...\phi^{a_j}U^{a_{j+1}}...U^{a_{2s}}$, for $0 \le j \le 2s$. We will focus on the contraction $C^{(2s,0)}_{a_1...a_{2s}}\phi^{a_1}...\phi^{a_{2s}}$ as it is the one playing the role of null initial. The discussion for the spin-s right-handed field-strength is identical and leads us to consider $C^{(0,2s)}_{a_1...a_{2s}}\bar{\phi}^{a_1}...\bar{\phi}^{a_{2s}}$.

For the left-handed fields, the Penrose transform is a straight-forward generalization of the scalar case. Explicitly, for a single twistor index

$$
\begin{aligned}
C^{(1,0)}_{a_1} &= \int d^2w\,(w_{a_1} + w'_{a_1})\, e^{\bar{\Sigma}w}\delta\left(w' - \Sigma\right) \\
&= -\frac{1}{2\pi}\int dAdB\, \left(A\phi_{a_1} - B\frac{iu}{2}\bar{\phi}_{a_1} + B\phi'_{a_1}\right)\, e^{\bar{\Sigma}(A\phi - B\frac{iu}{2}\bar{\phi})}\delta\left(B\phi' - \Sigma\right) ,
\end{aligned}
$$

recalling that $W^a = w^a + w'^a$. Contracting along null-direction left-handed spinor ϕ^{a_1}

$$
\begin{aligned}
C^{(1,0)}_{a_1}\phi^{a_1} &= -\frac{1}{2\pi}\int dAdB\, B\left(\phi'_{a_1}\phi^{a_1}\right) e^{A\bar{\Sigma}\phi - B\frac{iu}{2}(\bar{\Sigma}\bar{\phi})}\delta(B-\Sigma\phi)\delta\left(\Sigma\bar{\phi}\right)\\
&= -\frac{1}{2\pi}\left(\Sigma\phi\right) e^{\frac{iu}{2}(\Sigma\phi)(\bar{\Sigma}\bar{\phi})}\delta\left(\Sigma\bar{\phi}\right)\int dA\, e^{A\bar{\Sigma}\phi}\\
&= -\left(\Sigma\phi\right) e^{\frac{iu}{2}(\Sigma\phi)(\bar{\Sigma}\bar{\phi})}\delta\left(\Sigma\bar{\phi}\right)\delta\left(\bar{\Sigma}\phi\right)\,.
\end{aligned}
$$

This can be straight-forwardly generalized to arbitrary spin as

$$
C^{(2s,0)}_{a_1\dots a_{2s}}\phi^{a_1}\cdots\phi^{a_{2s}} = i(-1)^{s+1}\left(\Sigma\phi\right)^{2s} e^{\frac{iu}{2}(\Sigma\phi)(\bar{\Sigma}\bar{\phi})}\delta\left(\Sigma\bar{\phi}\right)\delta\left(\bar{\Sigma}\phi\right). \tag{2.27}
$$

Recall that the full field strength is given by integrating (2.27) against the spinor-helicity basis coefficients $f(\Sigma,\bar{\Sigma})$, namely

$$
\Phi^{(2s,0)} = \Phi^{(2s,0)}_{a_1\dots a_{2s}}\phi^{a_1}\cdots\phi^{a_{2s}} = \int d^2\Sigma d^2\bar{\Sigma}\, C^{(2s,0)}_{a_1\dots a_{2s}}\phi^{a_1}\cdots\phi^{a_{2s}} f(\Sigma,\bar{\Sigma})\,. \tag{2.28}
$$

Further, decomposing Σ as $\Sigma^a = \left(\Sigma^b\phi_b\right)\phi'^a + \left(\Sigma^b\bar{\phi}_b\right)\bar{\phi}'^a = \sigma\phi'^a + \tau\bar{\phi}'^a$, with σ and τ complex numbers, a typical integral can be performed as follows

$$
\int d^2\Sigma d^2\bar{\Sigma}\, h(\Sigma)\,\delta\bigl(\Sigma\bar{\phi}\bigr) = -\frac{1}{(2\pi)^2}\int d^2\sigma d^2\tau\, h\bigl(\sigma\phi' + \tau\bar{\phi}'\bigr)\,\delta(\tau) = -\frac{1}{(2\pi)^2}\int d^2\sigma\, h(\sigma\phi')\,.
$$

Hence we can write (2.27)

$$
C^{(2s,0)}_{a_1\dots a_{2s}}\phi^{a_1}\cdots\phi^{a_{2s}} = i(-1)^{s+1}\sigma^{2s}e^{\frac{iu}{2}\sigma\bar{\sigma}}\delta\left(\tau\right)\delta\left(-\bar{\tau}\right)\,,
$$

and thus (2.28) becomes

$$
\Phi^{(2s,0)} = -\frac{1}{(2\pi^2)}\int d^2\sigma\, i(-1)^{s+1}\sigma^{2s}e^{\frac{iu}{2}\sigma\bar{\sigma}} f(\sigma\phi',\bar{\sigma}\bar{\phi})\,.
$$

Let us find the field potentials $H^{(2s,0)} = H^{(2s,0)}_{a_1\dots a_{2s}}\phi^{a_1}\cdots\phi^{a_{2s}}$ by integrating with respect the null-time u. First note

$$
\int d^s u\, C^{(2s,0)}_{a_1\dots a_{2s}}\phi^{a_1}\cdots\phi^{a_{2s}} = -i(2i)^s\left(\frac{\sigma}{\bar{\sigma}}\right)^s e^{\frac{iu}{2}\sigma\bar{\sigma}}\delta\left(\tau\right)\delta\left(-\bar{\tau}\right)\,,
$$

and thus

$$
H^{(2s,0)} = \frac{1}{(2\pi^2)}\int d^2\sigma\, i(2i)^s\left(\frac{\sigma}{\bar{\sigma}}\right)^s e^{\frac{iu}{2}\sigma\bar{\sigma}} f(\sigma\phi',\bar{\sigma}\bar{\phi})\,. \tag{2.29}
$$

For the right-handed field strength calculation we first decompose our twistor variable in terms of both left- and right-handed spinor variables, namely $W = w_- + w_+$ where $w_- = A\phi + BU$, as before, and $w_+ = S\bar{\phi} + T\bar{U}$ with A, B, S, and T complex variables. Note that $w_+\bar{\phi} = \left(1 - \frac{iu}{2}\right)T$ and $w_+\bar{U} = -\left(1 - \frac{iu}{2}\right)S$, and hence we can write a general w_+ spinor derivative as

$$\frac{\partial}{\partial w_+^a} = \frac{1}{1-\frac{iu}{2}}\bar{U}_a\partial_S - \frac{1}{1-\frac{iu}{2}}\bar{\phi}_a\partial_T\,.$$

This allows us to write the integrand of (2.26), for a single index, as follows

$$\begin{aligned}&\frac{\partial}{\partial w_+^a}\left[\delta\left(w'-\Sigma\right)e^{\bar{\Sigma}w}\right]\Big|_{w_+=0} = \\ &\quad = \frac{1}{1-\frac{iu}{2}}\left(\bar{U}_a\partial_S - \bar{\phi}_a\partial_T\right)\left[\delta\left(B\phi'+T\bar{\phi}'-\Sigma\right)e^{\bar{\Sigma}\left(A\phi-\frac{iu}{2}B\bar{\phi}+S\bar{\phi}-\frac{iu}{2}T\phi\right)}\right]\Big|_{S=0,\,T=0}\,.\end{aligned}$$

As before, we decompose Σ in the spinor basis $\left(\phi',\bar{\phi}'\right)$, so that

$$\delta\left(B\phi'+T\bar{\phi}'-\Sigma\right) = \delta\left(B-\Sigma\phi\right)\delta\left(T-\Sigma\bar{\phi}\right)$$

and hence

$$\begin{aligned}&\frac{\partial}{\partial w_+^a}\left[\delta(w'-\Sigma)\,e^{\bar{\Sigma}w}\right]\Big|_{w_+=0} = \\ &= \frac{1}{1-\frac{iu}{2}}\Big[-\delta(B-\Sigma\phi)\,\delta\big(T-\Sigma\bar{\phi}\big)\left(\bar{\Sigma}\bar{\phi}\right)e^{\bar{\Sigma}\left(A\phi-\frac{iu}{2}B\bar{\phi}+S\bar{\phi}-\frac{iu}{2}T\phi\right)}\bar{U}_a + \\ &\qquad +\delta(B-\Sigma\phi)\left(\delta'\big(T-\Sigma\bar{\phi}\big)-\frac{iu}{2}\left(\bar{\Sigma}\phi\right)\delta\big(T-\Sigma\bar{\phi}\big)\right)e^{\bar{\Sigma}\left(A\phi-\frac{iu}{2}B\bar{\phi}+S\bar{\phi}-\frac{iu}{2}T\phi\right)}\bar{\phi}_a\Big]\Big|_{S=0,\,T=0} \\ &= \frac{1}{1-\frac{iu}{2}}\delta(B-\Sigma\phi)\,e^{\bar{\Sigma}\left(A\phi-\frac{iu}{2}B\bar{\phi}+S\bar{\phi}-\frac{iu}{2}T\phi\right)}\times \\ &\qquad\times\left[-\delta\big(T-\Sigma\bar{\phi}\big)\left(\bar{\Sigma}\bar{\phi}\right)\bar{U}_a + \left(\delta'\big(T-\Sigma\bar{\phi}\big)-\frac{iu}{2}\left(\bar{\Sigma}\phi\right)\delta\big(T-\Sigma\bar{\phi}\big)\right)\bar{\phi}_a\right]\Big|_{S=0,\,T=0} \\ &= \delta(B-\Sigma\phi)\,e^{\bar{\Sigma}\left(A\phi-\frac{iu}{2}B\bar{\phi}\right)}\left[\delta\big(\Sigma\bar{\phi}\big)\left(\bar{\Sigma}\bar{\phi}\right)\bar{U}_a + \left(-\delta'\big(\Sigma\bar{\phi}\big)+\frac{iu}{2}\left(\bar{\Sigma}\phi\right)\delta\big(\Sigma\bar{\phi}\big)\right)\bar{\phi}_a\right]\,.\end{aligned}$$

Now, contracting along null-direction right-handed spinor $\bar{\phi}^{a_1}$, and integrating over the left-handed spinor subspace, the above reduces to

$$\begin{aligned}C^{(0,1)}_{a_1}\bar{\phi}^{a_1} &= \frac{i}{2\pi}\int dAdB\,\delta(B-\Sigma\phi)\,\delta\big(\Sigma\bar{\phi}\big)\left(\bar{\Sigma}\bar{\phi}\right)e^{\bar{\Sigma}\left(A\phi-\frac{iu}{2}B\bar{\phi}\right)} \\ &= i\left(\bar{\Sigma}\bar{\phi}\right)e^{\frac{iu}{2}(\Sigma\phi)(\bar{\Sigma}\bar{\phi})}\delta\left(\Sigma\bar{\phi}\right)\delta\left(\bar{\Sigma}\phi\right)\,.\end{aligned}$$

This can be straight-forwardly generalized to arbitrary spin

$$C^{(0,2s)}_{a_1\ldots a_{2s}}\bar{\phi}^{a_1}\ldots\bar{\phi}^{a_{2s}} = i\left(\bar{\Sigma}\bar{\phi}\right)^{2s}e^{\frac{iu}{2}(\Sigma\phi)(\bar{\Sigma}\bar{\phi})}\delta\left(\Sigma\bar{\phi}\right)\delta\left(\bar{\Sigma}\phi\right)\,. \tag{2.30}$$

As for the left-handed fields, we can write (2.30) as

$$C^{(0,2s)}_{a_1 \ldots a_{2s}} \bar{\phi}^{a_1} \ldots \bar{\phi}^{a_{2s}} = i\bar{\sigma}^{2s} e^{\frac{iu}{2}\sigma\bar{\sigma}} \delta(\tau)\, \delta(-\bar{\tau}) \, .$$

Thus the right-handed field strengths

$$\Phi^{(0,2s)} = \Phi^{(0,2s)}_{a_1 \ldots a_{2s}} \bar{\phi}^{a_1} \cdots \bar{\phi}^{a_{2s}} = \int d^2\Sigma d^2\bar{\Sigma}\, C^{(0,2s)}_{a_1 \ldots a_{2s}} \bar{\phi}^{a_1} \ldots \bar{\phi}^{a_{2s}} f(\Sigma, \bar{\Sigma}) \tag{2.31}$$

evaluate to

$$\Phi^{(0,2s)} = -\frac{1}{(2\pi^2)} \int d^2\sigma\, i\bar{\sigma}^{2s} e^{\frac{iu}{2}\sigma\bar{\sigma}} f(\sigma\phi', \bar{\sigma}\bar{\phi}) \, ,$$

while the field potentials $H^{(0,2s)} = H^{(0,2s)}_{a_1 \ldots a_{2s}} \bar{\phi}^{a_1} \cdots \bar{\phi}^{a_{2s}}$ read

$$H^{(0,2s)} = \int d^2\sigma\, i(-2i)^s \left(\frac{\bar{\sigma}}{\sigma}\right)^s e^{\frac{iu}{2}\sigma\bar{\sigma}} f(\sigma\phi', \bar{\sigma}\bar{\phi}) \, . \tag{2.32}$$

2.3 Horizon symplectic form

Scalar component

We will compute the symplectic form on the future horizon H_f (2.3), which, for scalar field-strengths $\Phi_1^{(0)}(x^\mu)$ and $\Phi_2^{(0)}(x^\mu)$, reads

$$\Omega\left(\Phi_1^{(0)}, \Phi_2^{(0)}\right) = \int_{H_f} dud^2\mathbf{r}\, \Phi_1^{(0)} \overleftrightarrow{\partial_u} \Phi_2^{(0)}. \tag{2.33}$$

We want to considering $\Phi_1^{(0)}$ and $\Phi_2^{(0)}$ of positive-frequency and negative-frequency, respectively, with respect to null-time u. Recall from the construction of Section 2.2.2 that positive frequency modes correspond to choice of real contour $\mu = i\bar{\lambda}$ in the original spinor-helicity spinor variables. Thus, for the negative-frequency modes we will choose contour $\mu = -i\bar{\lambda}$; this corresponds to an exchange in the redefined variables (2.18), $\Sigma \leftrightarrow \Delta$.

Thus, expressing field-strengths via the Penrose transformed Wigner–Weyl transform (2.21), the symplectic form (2.33) reads

$$\begin{aligned}
&\Omega\left(\Phi_1^{(0)}, \Phi_2^{(0)}\right) = \\
&= -\int dud^2\mathbf{r} \left(\int d^2\Sigma d^2\bar{\Sigma}\, C^{(0)}(\Sigma, \bar{\Sigma}; x) f_1(\Sigma, \bar{\Sigma})\right) \overleftrightarrow{\partial_u} \left(\int d^2\Sigma d^2\bar{\Sigma}\, C^{(0)}(\bar{\Sigma}, \Sigma; x) f_2(\bar{\Sigma}, \Sigma)\right) \\
&= -\int dud^2\mathbf{r} \int d^2\Sigma_1 d^2\bar{\Sigma}_1 d^2\Sigma_2 d^2\bar{\Sigma}_2 C^{(0)}(\Sigma_1, \bar{\Sigma}_1; x) f_1(\Sigma_1, \bar{\Sigma}_1) \overleftrightarrow{\partial_u} C^{(0)}(\bar{\Sigma}_2, \Sigma_2; x) f_2(\bar{\Sigma}_2, \Sigma_2) \, .
\end{aligned}$$

Using the appropriate scalar Penrose-transformed kernel (2.24)

$$\Omega\left(\Phi_1^{(0)},\Phi_2^{(0)}\right)=\int dud^2\mathbf{r}\int d^8\Sigma\Big[\delta\big(\Sigma_1\bar{\phi}\big)\,\delta\big(\bar{\Sigma}_1\phi\big)\,e^{\frac{iu}{2}(\Sigma_1\phi)(\bar{\Sigma}_1\bar{\phi})}f_1(\Sigma_1,\bar{\Sigma}_1)\times$$
$$\times\,\delta\big(\Sigma_2\bar{\phi}\big)\,\delta\big(\bar{\Sigma}_2\phi\big)\,\frac{i}{2}\,(\Sigma_2\phi)\,\big(\bar{\Sigma}_2\bar{\phi}\big)\,e^{-\frac{iu}{2}(\Sigma_2\phi)(\bar{\Sigma}_2\bar{\phi})}f_2(\bar{\Sigma}_2,\Sigma_2)-\{1\leftrightarrow 2\}\Big] \quad (2.34)$$

where, for notational brevity, we abuse notation by denoting the combined spinor measures as $d^8\Sigma = d^2\Sigma_1 d^2\bar{\Sigma}_1 d^2\Sigma_2 d^2\bar{\Sigma}_2$. We then use the delta-functions to reduce the Σ-integrals as follows; decomposing Σ_i as $\Sigma_i^a = \left(\Sigma_i^b\phi_b\right)\phi'^a + \left(\Sigma_i^b\bar{\phi}_b\right)\bar{\phi}'^a = \sigma\phi'^a + \tau\bar{\phi}'^a$, with σ and τ complex numbers, the symplectic form becomes

$$\Omega\left(\Phi_1^{(0)},\Phi_2^{(0)}\right)=\frac{1}{(2\pi)^4}\int dud^2\mathbf{r}\int d^2\sigma_1 d^2\sigma_2\, e^{\frac{iu}{2}\sigma_1\bar{\sigma}_1}f_1\big(\sigma_1\phi',\bar{\sigma}_1\bar{\phi}'\big)\times$$
$$\times\,\frac{i}{2}\,(\sigma_2\bar{\sigma}_2)\,e^{-\frac{iu}{2}\sigma_2\bar{\sigma}_2}f_2\big(\bar{\sigma}_2\bar{\phi}',\sigma_2\phi'\big)-\{1\leftrightarrow 2\}\,.$$

Performing the u-integral in each of the above summands and packing the result into a single integral

$$\Omega\left(\Phi_1^{(0)},\Phi_2^{(0)}\right)=\frac{i}{16\pi^3}\int d^2\mathbf{r}d^2\sigma_1 d^2\sigma_2\,\delta(\sigma_1\bar{\sigma}_1-\sigma_2\bar{\sigma}_2)\,(\sigma_2\bar{\sigma}_2+\sigma_1\bar{\sigma}_1)\times$$
$$\times\, f_1\big(\sigma_1\phi',\bar{\sigma}_1\bar{\phi}'\big)\,f_2\big(\bar{\sigma}_2\bar{\phi}',\sigma_2\phi'\big)\,.$$

Considering the complex σ integrals over circular contours $\sigma_j = \rho_j e^{i\theta_j}$ and $\bar{\sigma}_j = \rho_j e^{-i\theta_j}$, for arbitrary radii ρ_j,

$$\Omega\left(\Phi_1^{(0)},\Phi_2^{(0)}\right)=\frac{i}{16\pi^3}\int d^2\mathbf{r}d\rho_1 d\rho_2 d\theta_1 d\theta_2\,\rho_1^2\rho_2^2\delta\big(\rho_1^2-\rho_2^2\big)\,\big(\rho_1^2+\rho_2^2\big)\times$$
$$\times\, f_1\left(\rho_1 e^{i\theta_1}\phi',\rho_1 e^{-i\theta_1}\bar{\phi}'\right)\,f_2\left(\rho_2 e^{-i\theta_2}\bar{\phi}',\rho_2 e^{i\theta_2}\phi'\right)\,.$$

Note that the integrand only has support on $\rho_1 = \rho_2 = \rho$. Thus, performing one of the ρ-integrals

$$\Omega\left(\Phi_1^{(0)},\Phi_2^{(0)}\right)=\frac{i}{8\pi^3}\int d^2\mathbf{r}d\rho d\theta_1 d\theta_2\,\rho^3 f_1\left(\rho e^{i\theta_1}\phi',\rho e^{-i\theta_1}\bar{\phi}'\right)\,f_2\left(\rho e^{-i\theta_2}\bar{\phi}',\rho e^{i\theta_2}\phi'\right)\,.$$

Recall $r_{ab} = i\left[4\phi_{[a}\phi'_{b]} - I_{ab}\right]$ and thus we can parametrize the unit sphere using spinors $\lambda_a = \rho e^{i\theta_1}\phi_a$ and $\bar{\lambda}_a$ so that $d^2\lambda d^2\bar{\lambda} = \frac{1}{2\pi}\rho^3 d^2\mathbf{r}d\rho d\theta_1$; letting $\theta_2 - \theta_1 = \theta$ we write, up to variable redefinitions,

$$\Omega\left(\Phi_1^{(0)},\Phi_2^{(0)}\right)=\frac{i}{4\pi^2}\int d^2\lambda d^2\bar{\lambda}d\theta\, f_1\left(\lambda,\bar{\lambda}\right)\,f_2\left(e^{-i\theta}\bar{\lambda},e^{i\theta}\lambda\right) \quad (2.35)$$

Spin-s component

We proceed in a similar manner for field-strengths that carry spinor indices. In this case the relevant variables are the field potentials $H^{(2s,0)}$ (2.29) and $H^{(0,2s)}$ (2.29), and their conjugates $\partial_u H^{(2s,0)}$ and $\partial_u H^{(0,2s)}$, respectively. For left-handed field strengths, the symplectic form reads

$$\Omega\left(H_1^{(2s,0)}, H_2^{(2s,0)}\right) = \int_{H_f} dud^2\mathbf{r}\, H_1^{(2s,0)} \overleftrightarrow{\partial_u} H_2^{(2s,0)}\,.$$

Plugging in (2.29) and using the same formalism as for the scalar case, this becomes

$$\Omega\left(H_1^{(2s,0)}, H_2^{(2s,0)}\right) = \frac{i}{16\pi^3}\int d^2\mathbf{r} d^2\sigma_1 d^2\sigma_2\,(-i)^2(2i)^{2s}\left(\frac{\sigma_1}{\bar\sigma_1}\right)^s\left(\frac{\sigma_2}{\bar\sigma_2}\right)^s \times$$
$$\times\,\delta(\sigma_1\bar\sigma_1-\sigma_2\bar\sigma_2)\,(\sigma_2\bar\sigma_2+\sigma_1\bar\sigma_1)\, f_1\left(\sigma_1\phi', \bar\sigma_1\bar\phi'\right) f_2\left(\bar\sigma_2\bar\phi', \sigma_2\phi'\right)\,.$$

Evaluating over the same complex contours as in the scalar case, this integral reduces to

$$\Omega\left(H_1^{(2s,0)}, H_2^{(2s,0)}\right) = \frac{i2^{2s-2}}{\pi^2}\int d^2\lambda d^2\bar\lambda d\theta\, e^{2si\theta} f_1\left(\lambda,\bar\lambda\right) f_2\left(e^{-i\theta}\bar\lambda, e^{i\theta}\lambda\right). \tag{2.36}$$

For right-handed fields, a similar computation leads to

$$\Omega\left(H_1^{(0,2s)}, H_2^{(0,2s)}\right) = \frac{i2^{2s-2}}{\pi^2}\int d^2\lambda d^2\bar\lambda d\theta\, e^{-2si\theta} f_1\left(\lambda,\bar\lambda\right) f_2\left(e^{-i\theta}\bar\lambda, e^{i\theta}\lambda\right). \tag{2.37}$$

Note that this reduces to the scalar result (2.35) when setting $s = 0$. Furthermore, the above symplectic forms (2.35, 2.36, 2.37) vanish due to the periodicity in θ of the integrands, unless the functions have the right helicity, namely, under a phase transformation $\lambda \to e^{i\theta}\lambda$, $f(\bar\lambda,\lambda) \to e^{\pm si\theta} f(\bar\lambda,\lambda)$, where $\pm s$ is the left(right)-handed spin of f.

2.4 S-matrix

In field theories on flat spacetime the scattering matrix (S-matrix) between past and future infinity is a fundamental object of study. The spinor-helicity formalism is an ideal language to study the S-matrix for massless theories such as Yang-Mills and General Relativity.

In the context of de Sitter space we are interested to study the "S-matrix" in the static patch, with an observer's past and future horizons playing the roles of past/future infinity. Thus, the de Sitter S-matrix problem is to relate the gauge-invariant field data $\Phi(x^\mu)$ on the final horizon H_f to the corresponding data $\Psi(x^\mu)$ on the initial horizon H_i. (This statement is more general than what is generally referred to as the S-matrix. Usually, one relates the quantum states obtained by acting with the fields on some vacuum state; however, by considering just the fields themselves, we avoid choosing a particular vacuum state.) We will only consider the "hard part" of the S-matrix, as we will be ignoring subtleties related to zero-frequency modes (related to the horizons' lower-dimensional boundaries).

At the core of our calculation of the S-matrix lies the realization that the spinor functions that encode the field data on the final and initial horizons, $f(\Sigma,\bar{\Sigma})$ and $f'(\Xi,\bar{\Xi})$, respectively, are simply related by a Fourier transform in our spinor variables, where $(\Xi,\bar{\Xi})$ play the same role as $(\Sigma,\bar{\Sigma})$ on the past horizon H_i. Namely, we claim that

$$f(\Sigma,\bar{\Sigma}) = \frac{1}{2\pi i}\int d^2\Xi d^2\bar{\Xi}\, e^{\Sigma\Xi-\bar{\Sigma}\bar{\Xi}} f'(\Xi,\bar{\Xi})\,. \tag{2.38}$$

To convince ourselves that this is true, plug the above into the Wigner–Weyl transform (2.14)

$$\begin{aligned}
F(Y) &= i\int d^2\Sigma d^2\bar{\Sigma}\, \delta(y'-\Sigma)e^{\bar{\Sigma}y} f(\Sigma,\bar{\Sigma}) \\
&= \frac{1}{2\pi}\int d^2\Sigma d^2\bar{\Sigma} d^2\Xi d^2\bar{\Xi}\, \delta(y'-\Sigma)e^{\bar{\Sigma}y}e^{\Sigma\Xi-\bar{\Sigma}\bar{\Xi}} f'(\Xi,\bar{\Xi}) \\
&= -\frac{1}{2\pi}\int d^2\Sigma d^2\bar{\Sigma} d^2\Xi d^2\bar{\Xi}\, \delta(y'-\Sigma)e^{i\bar{\Sigma}(-iy+i\bar{\Xi})}e^{\Sigma\Xi} f'(\Xi,\bar{\Xi}) \\
&= i\int d^2\Xi d^2\bar{\Xi}\, \delta(y-\bar{\Xi})e^{\bar{\Xi}y'} f'(\Xi,\bar{\Xi}) \\
&= F'(Y)
\end{aligned}$$

where in the last line we have constructed twistor function $F'(Y)$ through the Wigner–Weyl transform on the initial horizon H_i. Since the two twistor functions encode the same bulk field dynamics, $F'(Y) = F(Y)$, which concludes our argument for (2.38).

Scalar component

We encode scalar field data as before

$$\Phi^{(0)}(x) = i\int d^2\Sigma d^2\bar{\Sigma}\, C^{(0)}(\Sigma,\bar{\Sigma};x) f(\Sigma,\bar{\Sigma})\,, \tag{2.21}$$

$$\Psi^{(0)}(x') = i\int d^2\Xi d^2\bar{\Xi}\, C^{(0)}(\Xi,\bar{\Xi};x') f'(\Xi,\bar{\Xi})\,, \tag{2.39}$$

where the second horizon H_i, with coordinates $(0,v,\mathbf{r})$, is parameterized as $x'^\mu = vn'^\mu + r'^\mu$, while the spherical horizon section reads $r'_{ab} = i\left[4\psi_{[a}\psi'_{b]} - I_{ab}\right]$ for spinors ψ,ψ' normalized so that $\psi'\psi = 1$.

Using (2.38), we can express $\Phi^{(0)}(x^\mu)$ in terms of the spinor-helicity basis coefficient

$f(\Xi, \bar{\Xi})$ on the other horizon:

$$\begin{aligned}
\Phi^{(0)}(x) &= i \int d^2\Sigma d^2\bar{\Sigma}\, C^0(\Sigma, \bar{\Sigma}; x^\mu) f(\Sigma, \bar{\Sigma}) \\
&= \int d^2\Sigma d^2\bar{\Sigma}\, e^{\frac{iu}{2}(\Sigma\phi)(\bar{\Sigma}\bar{\phi})} \delta(\Sigma\bar{\phi})\delta(\bar{\Sigma}\phi) f(\Sigma, \bar{\Sigma}) \\
&= \frac{1}{2\pi i} \int d^2\Sigma d^2\bar{\Sigma} d^2\Xi d^2\bar{\Xi}\, e^{\frac{iu}{2}(\Sigma\phi)(\bar{\Sigma}\bar{\phi})} \delta(\Sigma\bar{\phi})\delta(\bar{\Sigma}\phi) e^{\Sigma\Xi - \bar{\Sigma}\bar{\Xi}} f'(\Xi, \bar{\Xi})\,.
\end{aligned}$$

As before, we decompose Σ as $\Sigma^a = \left(\Sigma^b\phi_b\right)\phi'^a + \left(\Sigma^b\bar{\phi}_b\right)\bar{\phi}'^a = \sigma\phi'^a + \tau\bar{\phi}'^a$ and, similarly, $\Xi^a = \left(-\Xi^b\phi'_b\right)\phi^a + \left(-\Xi^b\bar{\phi}_b\right)\bar{\phi}^a = \xi\phi^a + \chi\bar{\phi}^a$; thus the spinor measures read $d^2\Sigma d^2\bar{\Sigma} = -\frac{1}{(2\pi)^2} d^2\sigma d^2\tau$ and $d^2\Xi d^2\bar{\Xi} = -\frac{1}{(2\pi)^2} d^2\xi d^2\chi$. This allows us to perform the above delta-function integrals as follows

$$\begin{aligned}
\Phi^{(0)}(x) &= -\frac{1}{(2\pi)^3 i} \int d^2\sigma d^2\tau d^2\Xi d^2\bar{\Xi}\, e^{\frac{iu}{2}\sigma\bar{\sigma}} \delta(\tau)\delta(\bar{\tau}) e^{\sigma\xi + \tau\chi - \bar{\sigma}\bar{\xi} - \bar{\tau}\bar{\chi}} f'(\Xi, \bar{\Xi}) \\
&= -\frac{1}{(2\pi)^3 i} \int d^2\sigma d^2\Xi d^2\bar{\Xi}\, e^{\frac{iu}{2}\sigma\bar{\sigma}} e^{\sigma\xi - \bar{\sigma}\bar{\xi}} f'(\Xi, \bar{\Xi})\,.
\end{aligned}$$

Further, we regroup exponential terms

$$\Phi^{(0)}(x) = -\frac{1}{(2\pi)^3 i} \int d\sigma d\bar{\sigma} d^2\Xi d^2\bar{\Xi}\, e^{i\sigma\left(\frac{u}{2}\bar{\sigma} - i\xi\right)} e^{-\bar{\sigma}\bar{\xi}} f'(\Xi, \bar{\Xi})$$

and, noting that $\int d\sigma e^{i\sigma\left(\frac{u}{2}\bar{\sigma} - i\xi\right)} = 2\pi\delta\left(\frac{u}{2}\bar{\sigma} - i\xi\right)$ along real line contour and we use this to perform the $\bar{\sigma}$ integral as

$$\begin{aligned}
\Phi^{(0)}(x) &= -\frac{1}{(2\pi)^2 i} \int d\bar{\sigma} d^2\Xi d^2\bar{\Xi}\, \delta\left(\frac{u}{2}\bar{\sigma} - i\xi\right) e^{-\bar{\sigma}\bar{\xi}} f'(\Xi, \bar{\Xi}) \\
&= -\frac{1}{2\pi^2 iu} \int d^2\Xi d^2\bar{\Xi}\, e^{-\frac{2i}{u}\xi\bar{\xi}} f'(\Xi, \bar{\Xi}) \\
&= -\frac{1}{2\pi^2 iu} \int d^2\Xi d^2\bar{\Xi}\, e^{-\frac{2i}{u}(\Xi\phi')(\bar{\Xi}\bar{\phi}')} f'(\Xi, \bar{\Xi})\,.
\end{aligned}$$

Having expressed $\Phi^{(0)}(x^\mu)$ in terms of spinor function $f'(\Xi, \bar{\Xi})$, we will invert (2.39) as a Fourier transform in horizon time coordinate v, to write $f'(\Xi, \bar{\Xi})$ in terms of $\Psi^{(0)}$. For positive frequency modes

$$\begin{aligned}
\Psi_+^{(0)}(x') &= i \int d^2\Xi d^2\bar{\Xi}\, C^{(0)}(\Xi, \bar{\Xi}; x) f'(\Xi, \bar{\Xi}) \\
&= \int d^2\Xi d^2\bar{\Xi}\, e^{\frac{iv}{2}(\Xi\psi')(\bar{\Xi}\bar{\psi}')} \delta(\Xi\bar{\psi}')\delta(\bar{\Xi}\psi') f'(\Xi, \bar{\Xi}) \\
&= -\frac{1}{(2\pi)^2} \int d^2\xi'\, e^{\frac{iv}{2}\xi'\bar{\xi}'} f'(\Xi, \bar{\Xi})\,, \qquad (2.40)
\end{aligned}$$

where $\xi' = \Xi\psi'$. Considering the complex ξ' integrals over circular contours $\xi' = \rho e^{i\theta}$, the measure becomes $d^2\xi' = i\rho d\rho d\theta$. Further, letting $\xi'\bar{\xi}' = \rho^2 = \eta$, (2.40) becomes

$$\Psi_+^{(0)}(x') = -\frac{i}{2\pi}\int_0^\infty d\eta\, e^{\frac{iv}{2}\eta} f'(\Xi,\bar{\Xi})$$

where we performed the θ integral. Similarly for negative frequency modes, a similar calculation leads to

$$\Psi_-^{(0)}(x') = -\frac{i}{2\pi}\int_0^\infty d\eta\, e^{-\frac{iv}{2}\eta} f'(\Xi,\bar{\Xi})\,.$$

We can combine the two integrals to cover the whole real line, as

$$\Psi^{(0)}(x') = -\frac{i}{2\pi}\int_{-\infty}^\infty d\eta\, e^{-\frac{iv}{2}\eta} f'(\Xi,\bar{\Xi})\,.$$

Thus we can invert to obtain

$$f(\Xi,\bar{\Xi}) = -\frac{2\pi}{2i(2\pi i)}\int dv\, e^{iv\xi'\bar{\xi}'}\Psi^{(0)}(x') = \frac{1}{2}\int dv\, e^{-iv(\Xi\psi')(\bar{\Xi}\bar{\psi}')}\Psi^{(0)}(x')\,.$$

Now, the relation between the field-strengths on the two horizons becomes

$$\Phi^{(0)}(x) = -\frac{1}{4\pi^2 iu}\int dv d^2\Xi d^2\bar{\Xi}\, e^{-\frac{2i}{u}(\Xi\phi')(\bar{\Xi}\bar{\phi}')} e^{iv(\Xi\psi')(\bar{\Xi}\bar{\psi}')}\Psi^{(0)}(x')\,.$$

Recall from (2.40) that the delta-function integrals specialize us to spinor variables Ξ parallel to ψ; hence we parametrize $\Xi_a = \rho e^{i\theta}\psi_a$; the spinor measure becomes

$$d^2\Xi d^2\bar{\Xi} = \frac{1}{2\pi}\rho^3 d^2\mathbf{r}' d\rho d\theta$$

and $\Xi\bar{\phi} = \rho e^{i\theta}\psi\bar{\phi}$. Thus the above integral reads

$$\begin{aligned}\Phi^{(0)}(x) &= -\frac{1}{8\pi^3 iu}\int dv d^2\mathbf{r}' d\rho d\theta\, \rho^3 e^{-\frac{2i}{u}\rho^2(\psi\phi')(\bar{\psi}\bar{\phi}')} e^{iv\rho^2}\Psi(x') \\ &= -\frac{1}{4\pi^2 iu}\int dv d^2\mathbf{r}' d\rho\, \rho^3 e^{i\rho^2\left(-\frac{2}{u}(\psi\phi')(\bar{\psi}\bar{\phi}')+v\right)}\Psi^{(0)}(x') \end{aligned} \tag{2.41}$$

where we performed the integral over the spinor phase θ to obtain an overall 2π factor.

Now, note that the ρ-dependent part of (2.41) has the following general form

$$\int e^{i\rho^2(\star)}\rho^3\, d\rho\,.$$

Changing the integration variables to $\eta = \rho^2$, this becomes

$$\frac{1}{2}\int e^{i\eta(\star)}\eta\, d\eta = -\pi i\delta'(\star)\,.$$

The above integral requires η to be continued to the whole real line, and although we constructed η to be positive, we recover negative η from the $\Phi^{(0)}(x)$ negative-frequency modes, as discussed above. Thus, (2.41) becomes

$$\Phi^{(0)}(x) = \frac{1}{4\pi u}\int dv d^2\mathbf{r}'\, \delta'\Big(v - \frac{2}{u}(\psi\phi')(\bar{\psi}\bar{\phi}')\Big)\, \Psi^{(0)}(x') \\ = \frac{u}{\pi}\int dv d^2\mathbf{r}'\, \delta'\big(uv - 2(\psi\phi')(\bar{\psi}\bar{\phi}')\big)\, \Psi^{(0)}(x'). \tag{2.42}$$

Now recall, in our notation, $r^\mu = -\frac{1}{4}\gamma^\mu_{ab} r^{ab}$, where $r_{ab} = i\left[4\phi_{[a}\phi'_{b]} - I_{ab}\right]$. Thus

$$r_\mu r'^\mu = \frac{1}{16}\gamma^{ab}_\mu r_{ab}\gamma^\mu_{cd} r'^{cd} = \frac{1}{16}\left(I^{ab}I_{cd} - 4\delta^{[a}_{[c}\delta^{b]}_{d]}\right) r_{ab} r'^{cd} = -\frac{1}{4} r_{ab} r'^{ab}\,.$$

Further, we can show from direct calculation that

$$r_\mu r'^\mu - 1 = 2(\psi\phi')(\bar{\psi}\bar{\phi}') \tag{2.43}$$

Hence, we can express field-data relation (2.42) as

$$\Phi(x) = \frac{u}{\pi}\int dv d^2\mathbf{r}'\, \delta'(uv - 2(r_\mu r'^\mu - 1))\, \Psi(x')$$

Performing the integral over v

$$\Phi(x) = \frac{1}{\pi u}\int_{S_2} d^2\mathbf{r}'\ \frac{\partial}{\partial v}\Psi(x')\bigg|_{uv = 2(r_\mu r'^\mu - 1)}. \tag{2.44}$$

Note this result can be obtained directly from the more general formula

$$\Phi(x) = \int dv d^2\mathbf{r}'\, \Psi(x')\frac{\overleftrightarrow{\partial}}{\partial v} G(x,x')$$

where

$$G(x,x') = -\frac{1}{4\pi}\delta(x_\mu x'^\mu - 1)\Theta(x^0 - x'^0)$$

is the causal Green's function in dS_4. This acts as a sanity check for spinor-variable based calculation of the scalar S-matrix in dS_4 which now can be generalized to general fields.

Spin-s component

To generalize the S-matrix calculation to field strengths of general spin, we follow the same procedure, noting that the relevant Penrose-transformed kernels are (2.27) and (2.30) for left- and right-handed spin-s field strengths, respectively.

We encode the spin-s left-handed field strengths on the future horizon as in (2.31), keeping track of the correct helicity of the spinor-helicity basis coefficients:

$$\Phi^{(s,0)}(x) = i\int d^2\Sigma d^2\bar{\Sigma}\, C^{(2s,0)}(\Sigma,\bar{\Sigma};x)f^{(2s,0)}(\Sigma,\bar{\Sigma})\,,$$
$$\Psi^{(s,0)}(x') = i\int d^2\Xi d^2\bar{\Xi}\, C^{(2s,0)}(\Xi,\bar{\Xi};x')f'^{(2s,0)}(\Xi,\bar{\Xi})\,,$$

where we are using the corresponding Wigner–Weyl kernel (2.27) and, for notational brevity, the contraction along null-direction spinors,

$$C^{(2s,0)}(\Sigma,\bar{\Sigma};x) = C^{(2s,0)}_{a_1\dots a_{2s}}(\Sigma,\bar{\Sigma};x)\phi^{a_1}\dots\phi^{a_{2s}}$$

is understood. Spelling things out and performing the Fourier transform between horizons

$$\begin{aligned}\Phi^{(s,0)}(x) &= (-1)^s\int d^2\Sigma d^2\bar{\Sigma}\,(\Sigma\phi)^{2s}e^{\frac{iu}{2}(\Sigma\phi)(\bar{\Sigma}\bar{\phi})}\delta(\Sigma\bar{\phi})\delta(\bar{\Sigma}\phi)f^{(2s,0)}(\Sigma,\bar{\Sigma})\\ &= \frac{1}{(2\pi)^3 i}(-1)^{s+1}\int d\sigma d\bar{\sigma}d^2\Xi d^2\bar{\Xi}\,\sigma^{2s}e^{\frac{iu}{2}\sigma\bar{\sigma}}e^{\sigma\xi-\bar{\sigma}\bar{\xi}}f'^{(2s,0)}(\Xi,\bar{\Xi})\,.\end{aligned}$$

As before we compute the σ-integral along the real line to obtain a δ-function

$$\int d\sigma\,\sigma^{2s}e^{i\sigma\left(\frac{u}{2}\bar{\sigma}-i\xi\right)} = 2\pi(-i)^{2s}\left(\frac{2}{u}\right)^{2s+1}\delta^{(2s)}\left(\bar{\sigma}-\frac{2i}{u}\xi\right)$$

so that

$$\begin{aligned}\Phi^{(s,0)}(x) &= -\frac{2^{2s-1}}{\pi^2 i u^{2s+1}}\int d\bar{\sigma}d^2\Xi d^2\bar{\Xi}\,\sigma^{2s}\delta^{(2s)}\left(\bar{\sigma}-\frac{2i}{u}\xi\right)e^{-\bar{\sigma}\bar{\xi}}f'^{(2s,0)}(\Xi,\bar{\Xi})\\ &= -\frac{2^{2s-1}}{\pi^2 i u^{2s+1}}\int d^2\Xi d^2\bar{\Xi}\,\xi^{2s}e^{-\frac{2i}{u}\xi\bar{\xi}}f'^{(2s,0)}(\Xi,\bar{\Xi})\\ &= -\frac{2^{2s-1}}{\pi^2 i u^{2s+1}}\int d^2\Xi d^2\bar{\Xi}\,\left(\Xi\phi'\right)^{2s}e^{-\frac{2i}{u}(\Xi\phi')(\bar{\Xi}\bar{\phi}')}f'^{(2s,0)}(\Xi,\bar{\Xi})\,.\end{aligned} \tag{2.45}$$

On the second horizon, the spin-s' left-handed field strength reads

$$\begin{aligned}\Psi^{(s',0)}(x') &= i\int d^2\Xi d^2\bar{\Xi}\,C^{(2s,0)}(\Xi,\bar{\Xi};x')f'^{(2s',0)}(\Xi,\bar{\Xi})\\ &= \frac{(-1)^{s'+1}}{(2\pi)^2}\int d^2\xi\,\xi^{2s'}e^{\frac{iv}{2}\xi\bar{\xi}}f'^{(2s,0)}(\Xi,\bar{\Xi})\,.\end{aligned}$$

As in the scalar case, we invert the Fourier transform to obtain

$$\begin{aligned}f'^{(2s,0)}(\Xi,\bar{\Xi}) &= \frac{1}{2}(-1)^{s'}\xi^{-2s'}\int dv\,e^{iv\xi\bar{\xi}}\Psi^{(s',0)}(x')\\ &= \frac{1}{2}(-1)^{s'}(\Xi\psi')^{-2s'}\int dv\,e^{iv(\Xi\psi')(\bar{\Xi}\bar{\psi}')}\Psi^{(s',0)}(x')\,.\end{aligned}$$

Using the above in (2.45)

$$\Phi^{(s,0)}(x) = \frac{2^{2s-2}}{\pi^2 i u^{2s+1}}(-1)^{s'+1}\int dv d^2\Xi d^2\bar{\Xi}\,(\Xi\phi')^{2s}(\Xi\psi')^{-2s'}e^{-\frac{2i}{u}(\Xi\phi')(\bar{\Xi}\bar{\phi}')}e^{iv(\Xi\psi')(\bar{\Xi}\bar{\psi}')}\Psi^{(s',0)}(x')\,.$$

As before, letting $\Xi_a = \rho e^{i\theta}\psi_a$

$$\Phi^{(s,0)}(x) = \frac{2^{2s-3}}{\pi^3 i u^{2s+1}}(-1)^{s'+1}\int dv d^2\mathbf{r}' d\rho d\theta\,\rho^3\rho^{2s-2s'}(\psi\phi')^{2s}e^{2i\theta(s-s')}e^{i\rho^2\left(v-\frac{2}{u}(\psi\phi')(\bar{\psi}\bar{\phi}')\right)}\Psi^{(s',0)}(x')\,.$$

Note that the θ-integral fixes $s = s'$ and hence

$$\Phi^{(s,0)}(x) = \frac{2^{2s-2}}{\pi^2 i u^{2s+1}}(-1)^{s+1}\int dv d^2\mathbf{r}' d\rho\,\rho^3(\psi\phi')^{2s}e^{i\rho^2\left(v-\frac{2}{u}(\psi\phi')(\bar{\psi}\bar{\phi}')\right)}\Psi^{(s,0)}(x')\,,$$

which, as before, can be integrated over ρ as

$$\begin{aligned}\Phi^{(s,0)}(x) &= \frac{2^{2s-2}}{\pi u^{2s+1}}(-1)^s\int dv d^2\mathbf{r}'\,(\psi\phi')^{2s}\delta'\left(v-\frac{2}{u}(\psi\phi')(\bar{\psi}\bar{\phi}')\right)\Psi^{(s,0)}(x')\\ &= \frac{2^{2s}}{\pi u^{2s-1}}(-1)^s\int dv d^2\mathbf{r}'\,(\psi\phi')^{2s}\delta'(uv-2(r_\mu r'^\mu-1))\,\Psi^{(s,0)}(x')\,. \qquad (2.46)\end{aligned}$$

Note that we have the freedom to fix the relative phase of the the spinor spaces spanned by ϕ_a, ϕ'_b and ψ_a, ψ'_b; in particular choosing $\psi\phi' = \bar{\psi}\bar{\phi}'$, and, using (2.43),

$$(\psi\phi')^{2s} = \left((\psi\phi')(\bar{\psi}\bar{\phi}')\right)^s = \left(\frac{r_\mu r'^\mu - 1}{2}\right)^s = \frac{(-1)^s}{2^s}(1-r_\mu r'^\mu)^2$$

and hence, finally,

$$\Phi^{(s,0)}(x) = \frac{2^s}{\pi u^{2s+1}}\int_{S_2} d^2\mathbf{r}'\,(1-r_\mu r'^\mu)^s\,\frac{\partial}{\partial v}\Psi^{(s,0)}(x')\bigg|_{uv=2(r_\mu r'^\mu-1)}\,. \qquad (2.47)$$

This is the more general version of our scalar result (2.44), with a similar result holding for spin-s right-handed fields. These act as an effective S-matrix for spin-s data between cosmological horizons in dS_4.

Chapter 3

Boundary partition function

We investigate the decomposition of the boundary CFT partition function in terms of spherical modes in the spinor-helicity basis. Further, we observe a discrepancy between the higher-spin-algebraic calculation of the partition function and the result of the usual CFT partition function calculation.

Following [24] we will consider boundary correlators derived from the local and bilocal descriptions of the boundary theory. This will be consistent with the results we presented in Section 1.4.1 and it will allow us to probe the disagreement between the local and HS-algebraic partition functions.

We attempted to resolve the disagreement by modifying the local partition function, namely by considering the Legendre transform of the local action and accounting for contact pieces. Unfortunately, the disagreement persists even in these circumstances. We will discuss some reasons for this disagreement, as well as some of the consequences.

3.1 Correlators and boundary modes

Recall the free vector model action (1.51) we introduced in Section 1.4.1

$$S_{\text{CFT}} = -\int d^3\ell\, \bar{\phi}_I \Box \phi^I - \int d^3\ell \sum_{s=0}^{\infty} A^{(s)}_{\mu_1\cdots\mu_s}(\ell) J^{\mu_1\cdots\mu_s}_{(s)}(\ell)\,,$$

for currents $J^{(s)}$ (1.50)

$$J^{(s)}_{k_1\ldots k_s} = \frac{1}{i^s}\bar{\phi}_I \left(\sum_{m=0}^{s} (-1)^m \binom{2s}{2m} \overleftarrow{\partial}_{(k_1} \ldots \overleftarrow{\partial}_{k_m} \overrightarrow{\partial}_{k_{m+1}} \ldots \overrightarrow{\partial}_{k_s)} - \text{traces} \right) \phi^I\,.$$

For notational simplicity in the upcoming discussion we will consider the scalar null contraction of the above currents

$$J^{(s)}(\lambda, \ell) = \lambda^{\mu_1} \cdots \lambda^{\mu_s} J^{(s)}_{\mu_1\cdots\mu_s}(\ell)\,, \tag{3.1}$$

where λ^μ is a null boundary vector, such that $\lambda \cdot \ell = 0$.

The n-point correlation functions of the currents $J^{(s)}_{\mu_1\ldots\mu_s}$ are some of the most basic object of study in a CFT and can be extracted from the CFT partition function $Z_{\text{CFT}} =$

$\int \mathcal{D}\phi\mathcal{D}\bar{\phi}\, e^{-S_{\rm CFT}}$ as its derivative at zero with respect to sources inserted at n distinct points $(\ell_1,\dots,\ell_n)$. For the free vector model (1.51) the connected part of the n-point correlators for the scalar operator $J^{(0)} = \bar{\phi}_I\phi^I$ are given in terms of 1-loop Feynman diagrams in coordinate space

$$\langle J^{(0)}(\ell_1)\cdots J^{(0)}(\ell_n)\rangle_{\rm connected} = N(-1)^n \sum_{\sigma\in\tilde{S}_n}\prod_{i=1}^{n} G\left(\ell_{\sigma(i)},\ell_{\sigma(i+1)}\right)\,, \tag{3.2}$$

where we recall the propagator

$$G(\ell,\ell') = -\frac{1}{4\pi\sqrt{-2\ell\cdot\ell'}}\,. \tag{1.54}$$

The product (3.2) is cyclic and the sum is understood over the set $\tilde{S}_n$ of cyclically inequivalent permutations of the points $(\ell_1,\dots,\ell_n)$.

For currents of non-zero spin s, the correlators are best encapsulated by the bilocal formalism (1.52). For the bilocal scalar operators $\mathcal{O}(\ell,\ell') \equiv \phi^I(\ell)\bar{\phi}_I(\ell')$ the correlators become

$$\langle \mathcal{O}(\ell_1,\ell_1')\cdots\mathcal{O}(\ell_n,\ell_n')\rangle_{\rm connected} = N(-1)^n \sum_{\sigma\in\tilde{S}_n}\prod_{i=1}^{n} G\left(\ell'_{\sigma(i)},\ell_{\sigma(i+1)}\right)\,. \tag{3.3}$$

From this we can covariantly unpack the correlators for local spin-s currents via (1.50,3.1) as a differential operator

$$\begin{aligned} J^{(s)}(\ell,\lambda) &= \mathcal{D}^{(s)}\left[\mathcal{O}(\ell,\ell')\right] \equiv \\ &\equiv i^s\lambda^{\mu_1}\cdots\lambda^{\mu_s}\sum_{m=0}^{s}(-1)^m\binom{2s}{2m}\partial_{(\mu_1}\dots\partial_{\mu_m}\partial'_{\mu_{m+1}}\dots\partial'_{\mu_s)}\mathcal{O}(\ell,\ell')\Big|_{\ell'=\ell}\,, \end{aligned} \tag{3.4}$$

where we can legitimately employ the $\mathbb{R}^{1,4}$ flat derivatives $\partial_\mu = \frac{\partial}{\partial\ell^\mu}$ since the directional derivative $\lambda^\mu\partial_\mu$ do not take ℓ,ℓ' off the horizon and (3.4) is in fact invariant under arbitrary translations of λ^μ along ℓ.

3.1.1 Local description

As discussed in Section 1.4.2, the aim is to replace CFT sources with twistor function, which will then allow for the correlators to be written in higher-spin-algebraic language. In this section we will complement the *bilocal* picture of Section 1.4.2, by presenting a dictionary between *local* boundary sources and twistor language. The aim will be to find the twistor functions corresponding to boundary-to-bulk propagators, which will then be related to boundary sources.

We can expect the twistor function corresponding to scalar operator at a boundary point ℓ to be proportional to $\delta_\ell(Y)$, since this is the unique twistor function depending solely on ℓ. Thus, let

$$\kappa^{(0)}(\ell;Y) = \pm\frac{i}{4\pi}\delta_\ell(Y)\,. \tag{3.5}$$

Using the delta-star-product formula (1.37), we can write the unique higher-spin-invariant trance by considering a sequence of such twistor functions:

$$\mathrm{tr}_\star\left(\kappa^{(0)}(\ell_1;Y)\star\cdots\star\kappa^{(0)}(\ell_n;Y)\right)=-4\prod_{i=1}^{n}G(\ell_i,\ell_{i+1})\,. \tag{3.6}$$

This reproduces the terms of the correlator (3.2); the prefactor in (3.5) is chosen so that there are no n-dependent prefactors in (3.6). We can read off the related bulk-field by taking the Penrose transform (1.45) and using (1.34) as

$$\mathrm{tr}_\star\left(i\kappa^{(0)}(\ell;y)\star\delta_x(Y)\right)=\pm\frac{1}{2\pi\ell\cdot x}$$

which is proportional to the boundary-to-bulk propagator for the conformally massless scalar $C^{(0)}(x)$.

One can easily recover the results of the bilocal approach from this local construction. Note that the terms in the bilocal correlator (3.3) with n bilocal operator insertions correspond to terms in local correlator (3.2) with $2n$ local scalar operator insertions but with n propagators of the form $G(\ell_i,\ell_i')$ removed. Thus we can write the twistor kernel corresponding to bilocal sources as

$$K(\ell,\ell';Y)=\frac{1}{G(\ell,\ell')}\kappa^{(0)}(\ell;Y)\star\kappa^{(0)}(\ell';Y)\,,$$

which directly reproduces (1.56). Thus bilocal correlators (3.3) take the following form

$$\begin{aligned}&\langle\mathcal{O}(\ell_1,\ell_1')\cdots\mathcal{O}(\ell_n,\ell_n')\rangle_{\text{connected}}=\\&\qquad=\frac{N}{4}(-1)^{n+1}\sum_{\sigma\in\tilde{S}_n}\mathrm{tr}_\star\left(K(\ell_{\sigma(1)},\ell'_{\sigma(1)};Y)\star\cdots\star K(\ell_{\sigma(n)},\ell'_{\sigma(n)};Y)\right)\,.\end{aligned} \tag{3.7}$$

Recall that the local correlators for currents of general spin can be computed from bilocal ones via a differential operator (3.4). Thus the n-point correlator of the repackaged currents $J^{(s)}(\ell,\lambda)$ can be written in higher-spin-algebraic fashion as

$$\begin{aligned}&\langle J^{(s_1)}(\ell_1,\lambda_1)\cdots J^{(s_n)}(\ell_n,\lambda_n)\rangle_{\text{connected}}=\\&\qquad=\frac{N}{4}(-1)^{n+1}\sum_{\sigma\in\tilde{S}_n}\mathrm{tr}_\star\left(\kappa^{\sigma(1)}(\ell_{\sigma(1)},\lambda_{\sigma(1)};Y)\star\cdots\star\kappa^{\sigma(n)}(\ell_{\sigma(n)},\lambda_{\sigma(n)};Y)\right)\,,\end{aligned} \tag{3.8}$$

for $\kappa^{(s)}(\ell,\lambda;Y)$ the twistor function corresponding to a spin-s insertion $J^{(s)}(\ell,\lambda)$, namely

$$\kappa^{(s)}(\ell,\lambda;Y)=D^{(s)}[K(\ell,\ell';Y)]\,. \tag{3.9}$$

Direct computation of (3.9) is made difficult by the singular limit $\lim_{\ell'\to\ell}K(\ell,\ell';Y)$. Thus this has been computed [24] by first Penrose-transforming $K(\ell,\ell';Y)$ into the bulk, where it is easier to apply the $D^{(s)}$ operator, and transforming the resulting fields back into twistor space. Here we will just quote the result of this calculation for later convenience. (Note that we will encounter the bulk field corresponding to $K(\ell,\ell';Y)$ in its full form in Chapter 4 as the bilocal master field (4.1).) The twistor functions (3.9) read

$$\kappa^{(s)}(\ell,\lambda;Y) = \pm\frac{iM^{a_1}\cdots M^{a_{2s}}}{8\pi s!}\left(Y_{a_1}\cdots Y_{a_{2s}} + (-1)^s\frac{\partial^{2s}}{\partial Y^{a_1}\cdots\partial Y^{a_{2s}}}\right)\delta_\ell(Y)\,, \quad (3.10)$$

where the polarization twistor M^a satisfies $(\ell M)^a(\ell M)^b = \gamma^{ab}_{\mu\nu}\ell^\mu\lambda^\nu$.

3.1.2 Constructing the partition functions

Having written the correlation functions in a higher-spin-algebraic manner it would be natural to use them to construct the relevant partition function. Note that we will be ignoring contact corrections in our construction since in the following we will restrict ourselves to spin-0 modes; we will remark on this later.

Using the standard CFT construction, we can build the CFT partition function as a functional of sources $A^{(s)}_{\mu_1\dots\mu_s}(\ell)$, namely

$$Z_{\text{local}}[A^{(s)}(\ell)] = \exp\Bigg[\sum_{n=2}^\infty\frac{1}{n!}\int d^3\ell_1\sum_{s_1=0}^\infty A^{(s_1)}_{\mu_1\dots\mu_{s_1}}(\ell_1)\cdots\int d^3\ell_n\sum_{s_1=0}^\infty A^{(s_n)}_{\mu_1\dots\mu_{s_n}}(\ell_n)$$
$$\langle J^{\mu_1\dots\mu_{s_1}}_{(s_1)}(\ell_1)\cdots J^{\mu_1\dots\mu_{s_n}}_{(s_n)}(\ell_n)\rangle_{\text{connected}}\Bigg]\,. \quad (3.11)$$

where the subscript "local" indicates that the partition function was build from the local correlators (3.8). For the free vector model, the partition function can be expressed as a functional determinant; for a scalar source $\sigma(\ell)$ it reads

$$Z_{\text{local}}[\sigma(\ell)] = (\det(\Box+\sigma))^{-N} \sim (\det(1+\sigma G))^{-N}\,, \quad (3.12)$$

where G is the usual propagator (1.54) for the fundamental field ϕ^I, $G=\Box^{-1} = -\frac{1}{4\pi r}$. Recall from Section 1.4.1 that the sources of all spins are concisely captured by the bilocal formalism, from which gives a similarly compact expression (1.53)

$$Z_{\text{bilocal}}\left[\Pi(\ell',\ell)\right] = \left(\det\left(\Box+\Pi\right)\right)^{-N} \sim \left(\det\left(1+G\Pi\right)\right)^{-N}$$

from which the local partition function (3.11) can be obtained as a limit.

Now, we have seen in Section 1.4.1 how the twistor kernel $K(\ell,\ell';Y)$ can be used to package the *bilocal* sources into a twistor function and thus upgrade the bilocal partition function into its higher-spin-algebraic form (1.59). A similar process can be deployed for the *local* spin-s insertion twistor functions $\kappa^{(s)}(\ell,\lambda;Y)$ given in (3.10). Specifically, unraveling polarization indices

$$\kappa^{(s)}(\ell,\lambda;Y) \equiv \lambda_{\mu_1}\cdots\lambda_{\mu_s}\kappa^{\mu_1\dots\mu_s}_{(s)}(\ell;Y)$$

and packaging finite sources $A^{(s)}_{\mu_1\dots\mu_s}(\ell)$ into the twistor function

$$F(Y) = \int d^3\ell \sum_{s=0}^{\infty} A^{(s)}_{\mu_1\dots\mu_s}(\ell)\kappa^{\mu_1\dots\mu_s}_{(s)}(\ell;Y) \tag{3.13}$$

one can integrate local higher-spin-algebraic correlators (3.8) and sum into the Taylor series (3.11) to obtain the higher-spin-algebraic partition function

$$Z_{\text{HS}}\left[F(Y)\right] = \exp\left(\frac{N}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\operatorname{tr}_\star\left(\underbrace{F(Y)\star\cdots\star F(Y)}_{n\text{ terms}}\right)\right) \tag{3.14}$$

which reproduces the previous result

$$Z_{\text{HS}}\left[F(Y)\right] = \exp\left(\frac{N}{4}\operatorname{tr}_\star \ln_\star\left[1+F(Y)\right]\right) = \left(\det_\star\left[1+F(Y)\right]\right)^{\frac{N}{4}}\ . \tag{1.59}$$

3.1.3 Boundary modes

Having expressed the partitions in higher-spin-algebraic language we can forgo sources $A^{(s)}_{\mu_1\cdots\mu_s}$ in favor of twistor function source distributions. In this section we will consider two related twistor bases for the twistor function argument of partition function (1.59).

Spinor-helicity modes

Recall the spinor-helicity formalism that was introduced in Section 2.1 for field data on the cosmological horizon determined by a boundary point ℓ. In slightly different notation ($u_+ = \Sigma,\ u_- = \Delta,\ \ell = n,\ \ell' = n'$) the basis functions (2.13) read

$$K(u_+,u_-;Y) = e^{iu_-Y}\delta_\ell(Y-u_+)\,. \tag{3.15}$$

Considering again the scalar boundary insertion (3.5) proportional to $\delta_\ell(Y)$, we can read the basis (3.15) as a four-parameter basis extending the single twistor function $\delta_\ell(Y)$. This combines the generators $\delta_\ell(Y-M)$ and $e^{iMY}\delta_\ell(Y)$, with $M_a \in P^*(\ell)$, of the right-handed and left-handed parts of the general spin-s boundary-to-bulk propagators, respectively. In our notation, spinor parameters u_+ and u_- indicate the right- and left-handedness of the aforementioned propagators, respectively. Recall from our earlier discussion in Section 1.1.3, that when fixing two boundary points, we can take both $u_+, u_- \in P(\ell')$.

Thus, as in (2.14), we decompose our twistor function $F(Y)$ into basis coefficients $f(u_+,u_-)$ as

$$F(Y) = -i\int_{P(\ell')} d^2u_+d^2u_-\, f(u_+,u_-)\, e^{iu_-Y}\delta_\ell(Y-u_+)\,, \tag{3.16}$$

where we have introduced the prefactor $-i$ for later convenience. Note that we can perform the u_+ spinor integral to be left with

$$F(Y) = -i\int_{P(\ell')} d^2u_+\, f(y,u_-)\, e^{iu_-y}\,,$$

where the twistor Y is decomposed into spinors as $Y = y + y'$ with $y \in P(\ell)$ and $y' \in P(\ell')$. This last form can be easily inverted as a Fourier transform to give

$$f(u_+,u_-) = i\int_{P(\ell)} d^2y\, F(y+u_+)\, e^{iyu_-}\,. \tag{3.17}$$

Recall from our construction of the Wigner–Weyl transform in Section 2.1.2 that this was originally introduced in terms of variables $u = u_+ + u_-$ and $\bar{u} = u_+ - u_-$. Here we will take u and $\bar{u}$ independent, and not related by complex conjugation; we will come back to the issue of choosing real contours at the end of this section. In terms of these variables the modes decomposition (3.16) becomes

$$F(Y) = \int_{P(\ell')} d^2u d^2\bar{u}\, \tilde{f}(u,\bar{u})\, k(u,\bar{u};Y)\,. \tag{3.18}$$

with modes coefficients and kernel reading

$$\tilde{f}(u,\bar{u}) = -\frac{i}{4} f\left(\frac{u+\bar{u}}{2}, \frac{u-\bar{u}}{2}\right)\,, \tag{3.19}$$

$$k(u,\bar{u};Y) = \exp\left(i\frac{u-\bar{u}}{2}Y\right)\delta_\ell\left(Y - \frac{u+\bar{u}}{2}\right)\,, \tag{3.20}$$

where the numerical factors have been chosen for convenience. One of the advantages of using this basis is its behavior under the star product. Namely, using the star-product integral formula (1.24),

$$\begin{aligned} k(u,\bar{u};Y) \star k(v,\bar{v};Y) &= \int d^4X d^4Z\, k(u,\bar{u};Y+X)\, k(v,\bar{v};Y+Z)\, e^{-iXZ} \\ &= \int d^4X d^4Z\, \exp\left(iu_-(Y+X)\right)\delta_\ell((Y+X)-u_+) \\ &\qquad \exp\left(iv_-(Y+Z)\right)\delta_\ell((Y+Z)-v_+)\, e^{-iXZ}\,. \end{aligned}$$

Expressing the two delta-function as integrals over spinor subspace $P(\ell)$, the above becomes

$$\begin{aligned} &k(u,\bar{u};Y) \star k(v,\bar{v};Y) = \\ &\quad = \int d^4X d^4Z \int_{P(\ell)} d^2\zeta d^2\eta\, e^{iu_-(Y+X)} e^{i\zeta(Y+X-u_+)} e^{iv_-(Y+Z)} e^{i\eta(Y+Z-v_+)} e^{-iXZ} \end{aligned}$$

Regrouping the exponentials as

$$k(u,\bar{u};Y)\star k(v,\bar{v};Y) = \\ = \int d^4X d^4Z \int_{P(\ell)} d^2\zeta d^2\eta \, e^{i(u_-+v_-)Y} e^{i\zeta(Y-u_+)} e^{i\eta(Y-v_+)} e^{i(u_-+\zeta+Z)X} e^{i(v_-+\eta)Z}$$

allows us to rewrite the X-integral as a δ-function, using (1.12), as

$$\int dX^4 \, e^{i(u_-+\zeta+Z)X} = \delta(u_- + \zeta + Z)$$

and thus, performing the Z-integral,

$$k(u,\bar{u};Y)\star k(v,\bar{v};Y) = \int_{P(\ell)} d^2\zeta d^2\eta \, e^{i(u_-+v_-)Y} e^{i\zeta(Y-u_+)} e^{i\eta(Y-v_+)} e^{-i(v_-+\eta)(u_-+\zeta)} \, .$$

Now, changing to new spinor variables $\phi = \zeta + \eta$ and $\psi = \zeta - \eta$ the above integral rearranges itself as

$$k(u,\bar{u};Y)\star k(v,\bar{v};Y) = \int_{P(\ell)} d^2\phi d^2\psi \, e^{i\phi\left(Y-\frac{u+\bar{v}}{2}\right)} e^{i\psi\frac{v-\bar{u}}{2}} e^{i(u_-+v_-)Y} e^{-i(v_-u_-+\phi\psi)}$$

which, upon noting that $e^{i(u_-+v_-)Y} = e^{\frac{i}{2}(u-\bar{v})Y} e^{\frac{i}{2}(\bar{u}-v)Y}$, can be reexpressed as

$$k(u,\bar{u};Y)\star k(v,\bar{v};Y) = \delta_\ell(\bar{u}-v) e^{\frac{i}{2}(u-\bar{v})Y} \delta_\ell\left(Y - \frac{1}{2}(u+\bar{v})\right) \, .$$

Now, recalling the form of the spinor kernel (3.20), the above is simply

$$k(u,\bar{u};Y)\star k(v,\bar{v};Y) = \delta_\ell(\bar{u}-v)\, k(u,\bar{v};Y) \, . \tag{3.21}$$

Further, taking the trace in (3.20),

$$\mathrm{tr}_\star \, k(u,\bar{u};Y) = 4\delta_\ell(u+\bar{u}) \, . \tag{3.22}$$

Thus, we note that the kernels $k(u,\bar{u};Y)$ have the an analogous algebraic structure to that of the bilocal kernels $K(\ell,\ell';Y)$ (1.57,1.58)that were used to construct the twistor-partition function (1.59). It is then tempting to attempt a similar construction. First note that in the $(u,\bar{u})$ decomposition (3.18) the star product becomes a product modes coefficients

$$\begin{aligned} F(Y) \star F(Y) &= \int_{P(\ell')} d^2u d^2\bar{u} d^2v d^2\bar{v}\, \tilde{f}(u,\bar{u})\tilde{f}(v,\bar{v}) k(u,\bar{u};Y) \star k(v,\bar{v};Y) \\ &= \int_{P(\ell')} d^2u d^2\bar{u} d^2v d^2\bar{v}\, \tilde{f}(u,\bar{u})\tilde{f}(v,\bar{v}) \delta_\ell(\bar{u}-v)\, k(u,\bar{v};Y) \\ &= \int_{P(\ell')} d^2u d^2\bar{v}\, d^2\bar{u}\, \tilde{f}(u,\bar{u})\tilde{f}(\bar{u},\bar{v})\, k(u,\bar{v};Y)\,, \end{aligned} \tag{3.23}$$

whereas taking the trace gives

$$\mathrm{tr}_\star\, F(Y) = 4 \int_{P(\ell')} d^2u\, \tilde{f}(u,-u)\,. \tag{3.24}$$

Note that in (3.23) we obtain a correspondence between an operator product on the right-hand-side and a star product on the left-hand side, as we mentioned in Section 2.1.2 when discussing the Moyal star product.

Thus, the traced star-products that constitute the higher-spin algebraic partition function (3.14) become

$$\mathrm{tr}_\star \Bigg(\underbrace{F(Y) \star \cdots \star F(Y)}_{n \text{ terms}} \Bigg) = 4 \int_{P(\ell')} d^2u_1 \cdots d^2u_n\, \tilde{f}(u_1,u_2)\tilde{f}(u_2,u_3)\cdots \tilde{f}(u_n,-u_1)\,. \tag{3.25}$$

Even though the above looks like a promising way of explicitly computing the partition function (3.14), this formulation is troubled by the complex nature of spinor spaces $P(\ell)$ and $P(\ell')$ on the Euclidean three-dimensional boundary. Specifically, the delta-functions appearing in (3.21, 3.22) do not have well-defined support. A similar issue appears when we consider the products of basis coefficients in (3.25) — the required spinor integrals are over contours in the complex plane. For the $n = 2$ case one can specify such contours, since the spinor variables can be related by complex conjugation; however this is not as straightforward for general n.

One solution is to change signature to Lorentzian AdS_4 where spinors and twistors are real, and issues pertaining to contours and converges have been studied [55]. However, remaining in EAdS_4 we can circumvent the contour ambiguities by considering the S_3 boundary conformal frame instead of the flat, $\mathbb{R}^3$ one, which has the advantage of being compact and thus modes in this frame are discrete. In the next section we will adapt our spinor-helicity variables to such a spherical conformal frame.

Spherical modes

To construct a S_3 conformal frame for the boundary, we first choose a bulk point $x^\mu \in \mathrm{EAdS}_4$, which singles out a timelike direction in the embedding space. The spin-weighted spherical harmonics (the natural modes in this frame) are arranged into integer-spin irreducible representations of the residual $SO(4)$. Namely, for a scalar source the modes are spherical harmonics of angular momentum j. This matches the structure of a linearized master field $C(x;Y)$: the corresponding master field can be spanned by monomials $C(x;Y) = (m_+y_+)^j(m_-y_-)^j$, where the polarization spinors $m_\pm \in P(\pm x)$. This generalizes to spin-s modes as unbalanced monomials $C(x;Y) = (m_+y_+)^{2s+j}(m_-y_-)^j$ and $C(x;Y) = (m_+y_+)^j(m_-y_-)^{2s+j}$. All of these modes can be arranged as the Taylor expansion of a single master field $C(x;Y) = e^{iMY}$, where the polarization spinors are combined as $M^a = m_+^a + m_-^a$. Finally, we obtain the corresponding twistor function by taking the inverse Penrose transform as

$$\varkappa_x(M;Y) = -ie^{iMY} \star \delta_x(Y) = -ie^{iMY}\delta_x(Y-M) = -ie^{im_-y_-}\delta_x(y_+ - m_+)\ . \quad (3.26)$$

Now, we could take the star-products of the above modes directly, however we will take a detour and consider the above modes on the boundary. This is how these modes were conceived off originally; further, they enable us to choose a reality condition and can act as a consistency check between the two mode bases.

We will want to further restrict the polarization vectors by introducing a "reality condition". In order to do so, we break down further the spacetime symmetry by choosing a bulk direction at x^μ, *i.e.* a spacelike unit vector $v^\mu \in \mathbb{R}^{1,4}$, $x \cdot v = 0$. The residual symmetry is $SO(3)$ and we can set $m_- \propto v\bar{m}_+$.

Note that choosing bulk direction v^μ is equivalent to choosing a geodesic passing through x^μ, with endpoints

$$\ell^\mu = \frac{1}{2}(x^\mu + v^\mu)\,; \quad \ell'^\mu = \frac{1}{2}(x^\mu - v^\mu)\,.$$

Using these two points we can proceed to construct a spinor-helicity modes as we did earlier in this section. Decomposing (3.26) into modes (3.17)

$$f(u_+,u_-) = 4\exp\left(iu_-\ell u_+ + im(u_+ + u_-) + im'\ell(u_- - u_+) + im'm\right)\,,$$

where m, m' are the decompositions of M in the $P(\ell), P(\ell')$ subspaces. In terms of our preferred variables $(u,\bar{u})$ this becomes

$$\tilde{f}(u,\bar{u}) = -i\exp\left(\frac{1}{2}iu\ell\bar{u} + imu - im'\ell\bar{u} + im'm\right)\,. \quad (3.27)$$

Thus we can compute star products that appear in the partition function (3.14) using the matrix-product-like formula (3.25). The single star-product (3.23) reads

$$F_1(Y) \star F_2(Y) = \int_{P(\ell')} d^2u_1 d^2u_2 d^2u_3\, \tilde{f}_1(u_1,u_2)\tilde{f}_2(u_2,u_3)\, k(u_1,u_3;Y)\,.$$

For our modes (3.27) parameterized by polarization spinors m_1, m_2, the inner most integral becomes

$$\int_{P(\ell')} d^2u_2\, \tilde{f}_1(u_1,u_2)\tilde{f}_2(u_2,u_3) =$$
$$= \int_{P(\ell')} d^2u_2\, (-i)^2 e^{\frac{1}{2}iu_1\ell u_2 + im_1u_1 - im_1'\ell u_2 + im_1'm_1} e^{\frac{1}{2}iu_2\ell u_3 + im_2u_2 - im_2'\ell u_3 + im_2'm_2}\,.$$

Grouping multiples of u_2 allows us to perform this integral as a delta-function, namely

$$\int_{P(\ell')} d^2u_2\, \tilde{f}_1(u_1,u_2)\tilde{f}_2(u_2,u_3) =$$
$$= -\delta_{\ell'}\left(-\frac{1}{2}\ell u_1 + \frac{1}{2}\ell u_3 + \ell m_1' + m_2\right) \exp\left(im_1u_1 + i\ell m_2' u_3 + im_1'm_1 + im_2'm_2\right) .$$

Now, as in (3.22, 3.24), taking the trace of the star-product (3.1.3) introduces a delta-function with respect to the out-most integration variables

$$\begin{aligned} \mathrm{tr}_\star\left(F_1(Y)\star F_2(Y)\right) &= 4\int_{P(\ell')} d^2u_1d^2u_2d^2u_3\, \tilde{f}_1(u_1,u_2)\tilde{f}_2(u_2,u_3)\,\delta_\ell(u_1+u_3) \\ &= -4\int_{P(\ell')} d^2u\, \delta_{\ell'}(-\ell u + \ell m_1' + m_2)\exp\left(im_1u + i\ell m_2'u + im_1'm_1 + im_2'm_2\right) \\ &= -4\exp(im_1\ell' m_2 + im_1'\ell m_2') . \end{aligned}$$

In the case of a double star-product, a similar calculation leads us to

$$\mathrm{tr}_\star\left(F_1(Y)\star F_2(Y)\star F_3(Y)\right) =$$
$$= 4(-i)^3\delta_{\ell'}(-m_1-m_1'+m_2+m_2'-m_3-m_3')\exp\left(i\sum_{1\le p<q\le 3}\left(m_p\ell' m_q + (-1)^{q-p}m_p'\ell m_q'\right)\right)$$

noting that the delta-function arises from the fact that there is an even number of integrals to perform in the calculation, as opposed to an odd number in the previous case. The general result reads

$$\mathrm{tr}_\star\Big(\underbrace{F(Y)\star\cdots\star F(Y)}_{n\ \text{terms}}\Big) =$$
$$= 4(-i)^n\exp\left(i\sum_{1\le p<q\le n}\left(m_p\ell' m_q + (-1)^{q-p}m_p'\ell m_q'\right)\right)\times\begin{cases}\delta_{\ell'}\left(\sum_{p=1}^{n}(-1)^pM_p\right) & n\ \text{odd;}\\ 1 & n\ \text{even.}\end{cases} \tag{3.28}$$

While the even case is well-behaved, we have to treat the odd case more carefully, since the the δ-function is not well-defined. However, as argued in [24], this contribution can be shown to vanish by symmetry consideration. Letting $\tilde{m} = \sum_{p=1}^{n}(-1)^pm_p$ note the we formally obtain the delta-functions in the last step of computing (3.28), when taking the overall trace; thus let us write the delta-function as

$$\delta_{\ell'}(\tilde{m}) = \mathrm{tr}_\star\,\delta_{\ell'}(Y+\tilde{m}) = \sum_{k=0}^{\infty}\frac{1}{k!}\tilde{m}^{a_1}\cdots\tilde{m}^{a_n}\,\mathrm{tr}_\star\left(\partial_{a_1}\cdots\partial_{a_n}\delta_{\ell'}(Y)\right) , \tag{3.29}$$

where in the last equality we consider the Taylor expansion of the traced δ-function and we denote $\partial_{a_i} = \frac{\partial}{\partial Y^{a^i}}$. Note that the non-zero k terms must vanish by rotational symmetry; the zeroth-order term, $\operatorname{tr}_\star \delta_{\ell'}(Y)$, can be shown to vanish by considering discrete symmetries, either complex conjugation or spin parity. By complex conjugation symmetry $\operatorname{tr}_\star \delta_{\ell'}(Y)$ should be a real quantity, however were it nonzero, the partition function would be complex due to the $(-i)^n$ prefactor in (3.28). Alternatively, recall that higher-spin algebra separates even from odd spins; moreover, even spins correspond to twistor function $F(Y)$ of homogeneity degree $\deg_{\text{hom}} F(Y) = 2 \mod 4$, whereas odd spins have corresponding twistor functions with $\deg_{\text{hom}} F(Y) = 0 \mod 4$. Now, $\deg_{\text{hom}} \delta_{\ell'}(Y) = -2$, as expected for a scalar quantity; however, the trace operation $\operatorname{tr}_\star F(Y) = F(0)$ picks out the zero-homogeneity component of $F(Y)$. Hence, we conclude that the zeroth-order contribution to the Taylor series (3.29) also vanishes.

Lastly, re-expressing (3.28) in terms of polarization spinors $m^\pm \in P(\pm x)$, the traces (3.28) read

$$\operatorname{tr}_\star \Big(\underbrace{F(Y) \star \cdots \star F(Y)}_{n \text{ terms}} \Big) =$$

$$= 4(-i)^n \exp\Bigg(i \sum_{1 \le p < q \le n} \big(m_p^- m_q^- + (-1)^{q-p} m_p^x m_q^x \big) \Bigg) \times \begin{cases} 0 & n \text{ odd;} \\ 1 & n \text{ even.} \end{cases} \tag{3.30}$$

The aim of calculating the above traces was to compute the higher-spin algebraic partition function as in (3.14). In particular, if we set all polarization twistors equal to each other, say $M_n \equiv M$, the M dependence in (3.30) becomes trivial, namely

$$\operatorname{tr}_\star \Big(\underbrace{F(Y) \star \cdots \star F(Y)}_{n \text{ terms}} \Big) = 4(-i)^n \begin{cases} 0 & n \text{ odd;} \\ 1 & n \text{ even.} \end{cases}$$

Thus, we can easily evaluate the partition function Z_{HS} on a single mode of the form (3.26), $F(Y) = c\varkappa_x(M;Y)$, where c is a scalar coefficient setting the magnitude of the mode. Using (3.14),

$$Z_{\text{HS}}[F(Y)] = \exp\left(\frac{N}{4} \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{2n} (-ic)^{2n} \right) = (1 + c^2)^{\frac{N}{8}} . \tag{3.31}$$

Note that we have performed this calculation over a single scalar mode, and hence it was indeed valid to ignore contact corrections when construction the partition functions.

3.2 Partition function disagreement

We want to compare the higher-spin-algebraic partition function (3.31) with the standard local construction of the boundary CFT. We focus to the case of a scalar source, for which local correlators do not require contact correction. Further, we specialize to a scalar source σ, which we choose to be constant in a S_3 conformal frame. This corresponds to a twistor function $F(Y) \propto -i\delta_x(Y)$, that is, the $M = 0$ element of the spherical basis (3.26). To fix the normalization, recall from (3.5) that we wrote the local insertion of the scalar operator as a twistor function $\kappa^{(0)}(\ell;Y) = \pm\frac{i}{4\pi}\delta_\ell(Y)$. Thus, for constant source σ,

$$F(Y) = \int_{S_3} d^3\ell\, \kappa^{(0)}(\ell;Y) = \pm\frac{i\sigma}{4\pi}\int_{S_3} d^3\ell\, \delta_\ell(Y)\,.$$

Since it is unclear how to calculate the last integral directly, we will perform it by considering the corresponding bulk fields at a point x. The Penrose transform of $\kappa^{(0)}(\ell;Y)$ can be read off from the star product (1.34) as

$$i\kappa^{(0)}(\ell;Y)\star\delta_x(Y) = \mp\frac{1}{4\pi}\delta_\ell(Y)\star\delta_x(Y) = \pm\frac{1}{2\pi(\ell\cdot x)}\exp\frac{iY\ell xY}{2(\ell\cdot x)} = \mp\frac{1}{2\pi}\exp\frac{Y\ell xY}{2i}\,.$$

Now, by spherical symmetry, integrating the above over boundary point $\ell\in S_3$, makes away with all non-zero powers of Y^a, order by order, leaving only a Y-independent contribution of $\mp\frac{1}{2}$. Hence, the Penrose transform of $F(Y)$ reads

$$iF(Y)\star\delta_x(Y) = \mp\pi\sigma\,,$$

which can be immediately inverted to give

$$F(Y) = \pm i\pi\sigma\delta_x(Y) = \mp\pi\sigma\varkappa_x(0;Y)\,. \tag{3.32}$$

Finally, the higher-spin-algebraic partition function (3.31) becomes

$$Z_{\mathrm{HS}} = (1+\pi^2\sigma^2)^{N/8}\,. \tag{3.33}$$

We perform the corresponding local CFT calculation, following [24, 25]. Recall from (3.12) that the local partition function Z_{local} can be expressed as a functional determinant. Taking σ to be a constant, this becomes the S_3 partition function for N free scalar fields of mass $m^2=-\sigma$. Recalling that formally $\det M = \exp\operatorname{tr}\ln M$ and removing the low-order divergent terms, we write

$$\ln Z_{\mathrm{local}} = -N\operatorname{tr}\left[\ln\left(1+\frac{\sigma}{\Box}\right)-\frac{\sigma}{\Box}\right]\,. \tag{3.34}$$

Next we decompose the scalar fields into S_3 spherical harmonics ϕ_j of angular momentum j, *i.e.* the $\left(\frac{j}{2},\frac{j}{2}\right)$ $SO(4)$ representations, which have dimension $(j+1)^2$. On these harmonics the conformal Laplacian $\Box$ has eigenvalues

$$\Box\phi_j = \left(\nabla^2-\frac{3}{4}\right)\phi_j = \left(-j(j+2)-\frac{3}{4}\right)\phi_j = \left(-(j+1)^2+\frac{1}{4}\right)\phi_j\,.$$

Hence, the local partition function (3.34) becomes

$$\begin{aligned}\ln Z_{\text{local}} &= \sum_{j=0}^{\infty}(j+1)^2\left[\ln\left(1-\frac{\sigma}{(j+1)^2-\frac{1}{4}}\right)+\frac{\sigma}{(j+1)^2-\frac{1}{4}}\right]\\ &= -N\sum_{k=1}^{\infty}k^2\left[\ln\left(1-\frac{\sigma}{k^2-\frac{1}{4}}\right)+\frac{\sigma}{k^2-\frac{1}{4}}\right].\end{aligned}$$

This can be expressed in integral form as

$$\ln Z_{\text{local}} = -\frac{N\pi}{8}\int_{1}^{\sqrt{1+4\sigma}} dt\, t^2 \cot\frac{\pi t}{2}\,, \tag{3.35}$$

which was further evaluated [25] in terms of polylogarithm functions as

$$\begin{aligned}\ln Z_{\text{local}} = -\frac{N}{48\pi^2}\Big(&6\pi^2(1+4\sigma)\ln\left(1-e^{-i\pi(1+4\sigma)}\right)+12\,\mathrm{Li}_3\left(e^{-i\pi(1+4\sigma)}\right)+\\ &+i\pi\sqrt{1+4\sigma}\left(\pi^2(1+4\sigma)+12\,\mathrm{Li}_2\left(e^{-i\pi(1+4\sigma)}\right)\right)-3\pi^2\ln 4+9\zeta(3)\Big).\end{aligned} \tag{3.36}$$

Beyond the explicit form (3.36) of the local partition function, the main observation is that it differs from the higher-spin-algebraic one (3.33). To make this apparent, consider the Taylor series expansions of $\ln Z$ with respect to source σ

$$\ln Z_{\text{HS}} = \frac{N\pi^2}{8}\left(\sigma^2-\frac{\pi^2}{2}\sigma^4+\frac{\pi^4}{3}\sigma^6+O\left(\sigma^8\right)\right)\,; \tag{3.37}$$

$$\ln Z_{\text{local}} = \frac{N\pi^2}{8}\left(\sigma^2+\frac{2}{3}\sigma^3+\left(\frac{\pi^2}{6}-1\right)\sigma^4-2\left(\frac{\pi^2}{15}-1\right)\sigma^5+O\left(\sigma^6\right)\right). \tag{3.38}$$

3.3 Attempts at resolving the disagreement

In this section we will briefly describe some attempts at resolving the disagreement between Z_{HS} and Z_{local}, by modifying the latter. Unfortunately, these attempts have not proven fruitful.

First, note that since in twistor language, we construct Z_{HS} in terms of gauge invariant structures, it makes sense to look at Z_{local} in terms of gauge invariant current expectations rather than the fields themselves. Thus we consider the Legendre transformed action

$$\tilde{Z}[\rho] = \int \mathcal{D}\sigma\, e^{\ln Z_{\text{local}}[\sigma]+i\rho\sigma} \tag{3.39}$$

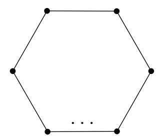
where we introduce the conjugate variable ρ the vacuum expectation value of $\bar{\phi}(\ell)\phi(\ell)$. By virtue of the Legendre transform, this action encodes the same physical information as the original description. The factor of i in the Legendre transform (3.39) is unusual, however it was introduced so that the signs in the expansion of (3.39) match those of the higher-spin calculation order by order. This approach does manage to reproduce the highest powers of π at each order, but the lowest powers of π still do not agree.

The next attempt was to consider the effect of contact pieces; as we will outline below, we noted that the third order term in (3.38), which is missing in (3.37), is in fact a pure contact term. This led to the idea that the difference between the two descriptions might be in fact contact pieces.

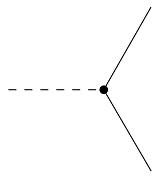
To investigate this we will construct the first couple of n-point correlators of the theory. First, by expanding $\ln Z[\sigma]$ as

$$\ln Z[\sigma] = \sum_{n=2}^{\infty} \frac{1}{n!} \int d^3\ell_1 \cdots d^3\ell_n \, \sigma(\ell_1) \cdots \sigma(\ell_n) K_n(\ell_1, \ldots, \ell_n) \tag{3.40}$$

we can extract the n-point connected correlation functions order-by-order as

$$\ln Z\Big|_{n\text{th order}} = \frac{1}{n!} \int d\ell_1^3 \cdots d\ell_n^3 \, K_n(\ell_1, \ldots, \ell_n) = \frac{1}{2n} \int d\ell_1^3 \cdots d\ell_n^3$$  .

The two-point kernel reads $K_2(x_1, x_2) = \frac{1}{32\pi^2|\ell_1-\ell_2|^2}$ which can be inverted to give the propagator $K_2^{-1}(r) = -\frac{16}{\pi^2 r^4}$, where $r = |\ell_1 - \ell_2|$. We will take this as an effective propagator for σ; this follows as an operator of conformal weight 2. Thus we can establish Feynman rules[1] as follows:

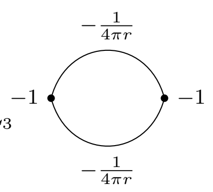
ϕ propagator	————	$-G(r) = \frac{1}{4\pi r}$
σ propagator	- - - - - - - -	$-K_2^{-1}(r) = \frac{16}{\pi^2 r^4}$
vertex		-1
ρ insertion	$\times$	i

We can see that the internal propagator $G(r)$ is correctly normalized by comparing with the two-point function as read-off from the expansion (3.38). Specializing to a flat conformal frame, we can write the distance between two arbitrary points as

$$|\mathbf{r}_1 - \mathbf{r}_2|^2 = \ell_1 \cdot \ell_2 = (1, \mathbf{n}_1) \cdot (1, \mathbf{n}_2) = 1 - \mathbf{n}_1 \cdot \mathbf{n}_2 = 1 - \cos\theta \,,$$

where $\mathbf{n}_1, \mathbf{n}_2$ are directions normal to the sphere and θ their relative angle. Thus, we parameterize the distance $r = |\mathbf{r}_1 - \mathbf{r}_2| = 2\sin\frac{\theta}{2}$, for $\theta \in [0, \pi]$.

Thus, we can compute the first-loop Feynman diagram as

$$\frac{N}{2}\int_{S^3}$$ 

$$= \frac{N}{2(-4\pi)^2}(2\pi^2)^2 = \frac{\pi^2 N}{8},$$

[1]Feynman diagrams are drawn using the TikZ-Feynman package [71].

which agrees with the coefficient of σ^2 in (3.38). We are accounting for symmetry factors in the integral prefactor. Also note that one of the $2\pi^2$ factors comes from a surface integral, while the second results from integrating $\frac{1}{r^2}$ over the sphere

$$\int_{S^3} d^3x\, \frac{1}{r^2} = \int_0^\pi d\theta\, 4\pi^2 \frac{\sin^2\theta}{4\sin^2\frac{\theta}{2}} = 4\pi^2 \int_0^\pi d\theta\, \cos^2\frac{\theta}{2} = 2\pi^2\,.$$

We will want to know the value of of the external leg $-K_2^{-1}(r)$ integrated over our boundary sphere. Employing dimensional regularization techniques, we compute the integral of $k(r) = \frac{1}{r^4}$ over the d-dimensional sphere S_d as

$$\begin{aligned}\int_{S^d} dx^d\, k(r) &= \int_{S^d} d\Omega_{d-1} \int_0^\pi d\theta\, \frac{\sin^{d-1}\theta}{\left(2\sin\frac{\theta}{2}\right)^4} \\ &= \frac{2^{d-4}\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \int_0^\pi d\theta\, \sin^{d-5}\frac{\theta}{2}\cos^{d-1}\frac{\theta}{2}\,.\end{aligned}$$

Changing integration variable to $\xi = \sin^2\frac{\theta}{2}$ the above becomes

$$\begin{aligned}\int_{S^d} dx^d\, k(r) &= \frac{2^{d-4}\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \int_0^1 d\xi\, \xi^{\frac{d}{2}-3}\left(1-\xi\right)^{\frac{d}{2}-1} \\ &= 2^{d-4}\pi^{\frac{d}{2}} \frac{\Gamma\left(\frac{d}{2}-2\right)}{\Gamma\left(d-2\right)}\,,\end{aligned}$$

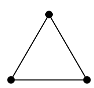
where in the last line we note that the integral evaluates to a Beta function $B\left(\frac{d}{2}-2, \frac{d}{2}\right)$. Thus, for our three-dimensional boundary, this gives

$$\int_{S^3} dx^3\, \frac{1}{r^4} = -\pi^2\,.$$

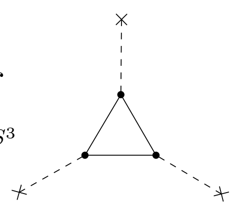
Now, the external leg can be integrated over the sphere to give

$$\int_{S^3} dx^3\, K_2^{-1}(r) = -16\,.$$

From partition function (3.38) at third-order

$$\int_{S^3}$$  $$= \frac{\pi^2}{4} \qquad (3.41)$$

and thus the three-point function reads

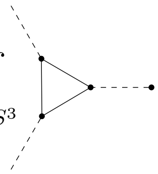
$$\frac{N}{6}\int_{S^3}$$  $$= \frac{N}{6} i^3 (-16)^3 (-1)^3 \frac{\pi^2}{4} = -i\frac{2^9}{3} N\pi^2\,. \qquad (3.42)$$

Independently, it can be shown that the integrand of (3.42) is proportional to a term $\delta(\ell_1 - \ell_2)\delta(\ell_2 - \ell_3)$, where ℓ_1, ℓ_2, ℓ_3 are the three external insertions, that is, a pure contact term.

Since this third-order term does not appear in Z_{HS}, we will investigate at higher order whether Z_{HS} is in fact blind to contact pieces; as expressed earlier, this was hoped to provide an account for the difference between Z_{HS} and Z_{local}.

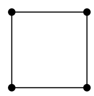
From here onwards we will neglect factors of N coming from loops in the Feynman diagrams, for computational simplicity. They can however be accounted for by redefining the external leg $K_2^{-1}(r)$.

It will be useful to consider the triangular diagram with one internal vertex

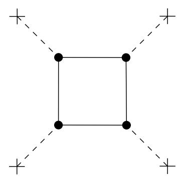
$$\int_{S^3}$$

$$= (-16)^3(-1)^3 \frac{\pi^2}{4} \frac{1}{2\pi^2} = 2^9 \,.$$

Proceeding to forth order, we have two contributions: a one-loop that we read off from the partition function, and a two-loop contribution coming from gluing to copies of the diagram in (3.42).

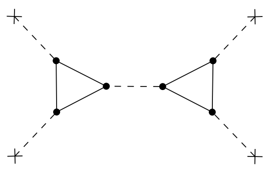
From partition function (3.38) at fourth-order

$$\int_{S^3}$$

$$= \frac{\pi^2}{2} \left(\frac{\pi^2}{6} - 1 \right) \tag{3.43}$$

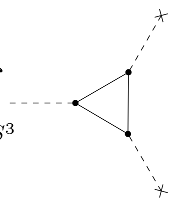
and hence

$$\frac{1}{8} \int_{S^3}$$

$$= \frac{1}{8} i^4 (-16)^4 (-1)^4 \frac{\pi^2}{2} \left(\frac{\pi^2}{6} - 1 \right) = 2^{12}\pi^2 \left(\frac{\pi^2}{6} - 1 \right) \,. \tag{3.44}$$

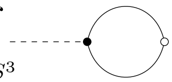
Next,

$$\frac{1}{8} \int_{S^3}$$

$$= \frac{1}{8} i^4 (-16)^2 \frac{\pi^2}{4} 2^9 (-1)^3 = -2^{12}\pi^2, \tag{3.45}$$

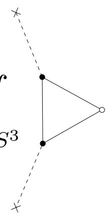
which unfortunately has the wrong sign to remove the lowest powers of π in (3.45). To take into account contact pieces, we will first consider the following generic diagram, with two external insertions

$$\int_{S^3}$$

$$= i^2 \frac{\pi^2}{4} (-16)^3 (-1)^3 = -2^{10}\pi^2 \,. \tag{3.46}$$

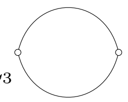
The contact piece corresponding to (3.46) can be computed in momentum space, by taking the infinite momentum limit on the internal leg connected to both external insertions. This limit extracts a term proportional to a spacetime function $\delta(\ell_1 - \ell_2)$ with ℓ_1, ℓ_2 the external insertions. This term is independent of the momentum on the internal leg and thus we take it is a contact piece. We can describe this as an effective vertex in spacetime, labeled by an empty circle; we calibrate its value by comparing it to (3.46):

$$\int_{S^3}$$

$$= \beta \frac{16}{\pi^2}(-\pi^2)\frac{1}{(4\pi)^2}(2\pi^2)^2(-1) = 4\beta\pi^2 \,.$$

Thus we can read the proportionality coefficient as $\beta = -2^8$. Now the contact piece corresponding to the four-point correlator (3.44) reads as follows

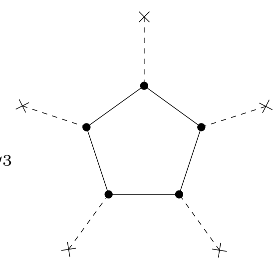
$$\frac{1}{2}\int_{S^3}$$

$$= \frac{1}{2} i^2 (-16)^2 \frac{\pi^2}{4}(-1)^2(-2^8) = 2^{13}\pi^2 \,;$$

Note, however, that we have to subtract the following contact piece which we have double counted

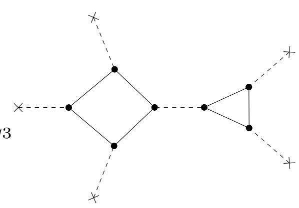
$$\frac{1}{4}\int_{S^3}$$

$$= \frac{1}{4}(-2^8)^2\frac{\pi^2}{4} = 2^{12}\pi^2 \,.$$

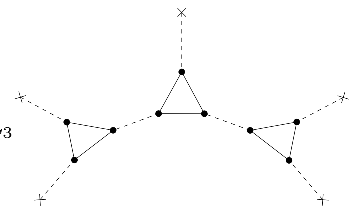
Thus, the total contribution from contact pieces reads $2^{12}\pi^2$ which cancels the diagram (3.45); hence, it does not account for the lowest order in π in (3.44).

At fifth-order the single-loop diagram reads

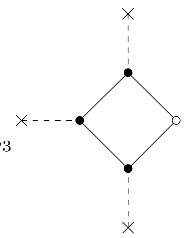
$$\int_{S^3}$$

$$= -(-1)^5 i^5 (-16)^5 \frac{4!}{2}\frac{\pi^2}{4}\left(\frac{\pi^2}{15} - 1\right) = i2^{18}\pi^2\left(\frac{4}{5}\pi^2 - 12\right) . \tag{3.47}$$

We also consider the contributions of composite loop diagrams

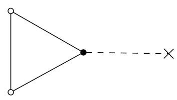
$$\frac{1}{4}\int_{S^3}$$

$$= \frac{1}{4} i^5 (-1)^4 2^9 (-16)^3 \frac{\pi^2}{2}\left(\frac{\pi^2}{6} - 1\right) = -i2^{18}\pi^2\left(\frac{\pi^2}{6} - 1\right) ; \tag{3.48}$$

$$\frac{1}{8}\int_{S^3}$$

$$= \frac{1}{8} i^5 (-1)^3 (2^9)^2 \frac{\pi^2}{4} (-16) = i 2^{17} \pi^2 \,. \tag{3.49}$$

The contact piece of the single-loop diagram (3.47) reads

$$\frac{1}{2}\int_{S^3}$$

$$= \frac{1}{2} i^3 (-1)^3 (-16)^3 \frac{\pi^2}{2} \left(\frac{\pi^2}{6} - 1\right) (-2^8) = i 2^{18} \pi^2 \left(\frac{\pi^2}{6} - 1\right) , \tag{3.50}$$

where we have double counted the term

$$\frac{1}{2}\int_{S^3}$$

$$= \frac{1}{2} i (-16) \frac{\pi^2}{4} (-1)(-2^8)^2 = i 2^{17} \pi^2 \,. \tag{3.51}$$

Thus note that at fifth order, the contact piece (3.50) cancels the contribution of composite diagram (3.48), whereas the double counted (3.51), as it has to be subtracted, will cancel (3.49). Hence, contact piece do not account for the lowest powers of π in (3.47).

The same situation is encountered at sixth order: contact pieces of diagrams that do not contain triangular components (which are contact pieces themselves) cancel out the composite loop diagrams, and do not alter the lowest power of π in the correlator.

Thus we conclude that the disagreement between Z_{local} and Z_{HS} persists despite our best efforts and must be understood on its own terms.

3.4 Conflict resolution

Having understood that the local and higher-spin-algebraic partition functions genuinely differ, even though they are constructed from the same correlators, we would like to better understand how this arises and what implications this might have. These issues have been explored in detail in [24]; in this section we will present a brief summary, including technical and ontological explanations for the disagreement, and an argument for choosing Z_{HS} over its local counterpart and spin-locality over locality as the new guiding principle for constructing the boundary theory. We refer the interested reader to [24] for the full discussion.

3.4.1 Understanding the disagreement

Technical aspects

Complex conjugation and spin-parity are discrete symmetries obeyed by the high-spin algebra when acting on polynomial twistor functions, and formally can be extended to general functions and distributions.

Note that under twistor complex conjugation, since gamma matrices $(\gamma_\mu)^a{}_b$ are real, so are the spinor space projectors $P_{ab}(\ell)$, $P^{\pm}_{ab}(x)$ and corresponding δ-functions $\delta_\ell(Y)$, $\delta_{\pm x}(Y)$. Thus the local scalar boundary insertion (3.5) proportional to $i\delta_\ell(Y)$, and spherical mode function (3.32) proportional to $i\delta_{\pm x}(Y)$ are in fact imaginary quantities; the same holds true for their non-zero spin analogues. Further, since the star product (1.23) (up to a sign in non-commutative term) and the trace operation (1.27) are also preserved by complex conjugation, one can see that the higher-spin-algebraic correlators (3.8) at an odd number of points, and the odd part of the partition function Z_{HS} should be purely imaginary. In the S_3 basis for example, Z_{HS} indeed obeys this symmetry since it vanishes at odd orders. However the local correlators and related partition function Z_{local} fail to do so; notably, the non-vanishing three-point correlator is real, and the local partition function (3.35) for constant scalar source is also real, but neither even nor odd. This is due to a spontaneous breaking of symmetry induced by the sign ambiguity in the three-point star product (1.36), where the symmetrized left-hand-side is real, while the right-hand-side is imaginary. This is enabled by the sign ambiguity of the RHS which is imaginary, however it vanishes upon averaging the two possible signs.

Even without invoking complex conjugation, we witness a similar phenomenon when considering spin parity. Under higher-spin algebra the space of even-spin twistor functions is closed under commutation (hence the legitimacy of the even-spin truncation of higher-spin gravity). However in the correlators (3.8) and partition function (3.14) we are dealing with anti-commutators, which map even-spin twistor functions into odd ones. This problem manifests itself, again, at odd orders. Similarly to before, by spin-parity we would expect that correlators (3.8) to vanish for even spins and odd n, and that Z_{HS} should be even at even sources. This is indeed the case for Z_{HS} in the spherical basis, but fails to hold for local correlators. This mismatch can, again, be traced back to the sign ambiguity of the three-point star product (1.36).

It is worth nothing that the sign ambiguity in (1.36), which seems to be the culprit of the partition function disagreement, is the result of performing a Gaussian integral over a complex spinor space. This can usually be avoided by a change of signature to Lorentzian AdS_4, where twistors and boundary spinors have in fact a real structure, and thus the spinor and twistor integrals within star product do not suffer from contour ambiguities. However, even though in such a setup the sign ambiguity no longer appear, the disagreement between Z_{local} and Z_{HS} persists. It can be shown [24] that higher-spin-algebraic correlators have signs that are different from those required to correctly reproduce the CFT ones.

Ontological aspects

Beyond the technical aspects that we have explored in the previous section, one can understand the disagreement between Z_{local} and Z_{HS} at a different level. Namely, the CFT path integral is concerned with off-shell boundary particles, whereas higher-spin algebra deals with on-shell ones.

At an algebraic level we can see that an equality between the local partition function (3.12), restricted to scalar sources, and the higher-spin-algebraic one (3.14) would amount to an isomorphism between the algebra of infinite-dimensional $\Pi(\ell, \ell')$ over the space of boundary fields $\phi^I(\ell)$ and the the higher-algebra of twistor functions $F(Y)$, with linear mapping (3.13) reading, for scalar sources,

$$F(Y) = \int d^3\ell\, \sigma(\ell) \kappa^{(0)}(\ell; Y)\,.$$

Note that we can restrict to scalar sources without loss of generality, since the entire algebra of bilocal insertions can be retrieved by considering pairs of scalar ones.

However, by counting degrees of freedom, we can see that the bilocal algebra consists of functions $\Pi(\ell, \ell')$ of 6 spacetime coordinates, whereas higher-spin algebra consists of functions $F(Y)$ of 4 twistor components. Hence, this makes the disagreement seem rather natural, as the underlying algebras have different dimensions.

Further, the dimensional mismatch itself is an issue of guage redundancy: succinctly put, the twistor function $F(Y)$ contains only physical degrees of freedom, whereas the sources $A^{(s)}_{\mu_1 \cdots \mu_s}(\ell)$ for $s > 0$ and hence the bilocal sources $\Pi(\ell, \ell')$ are gauge-redundant, due to the conservation of associated currents.

We can understand this mismatch in an equivalent way, but considering again the boundary with Lorentzian signature. There higher-spin algebra can be identified as the operator algebra of a free massless particle in the 2+1d boundary spacetime [21, 46, 68, 69]. We this perspective, we can see that, while Z_{local} and Z_{HS} are similar in the sense of both calculating functional determinants over boundary fields, the higher-spin algebra sees on-shell boundary fields, *i.e.* solutions to the homogeneous field equations $\Box\phi = 0$, describing states of the free boundary particle. On the other hand, the CFT path integral is performed over off-shell fields, using propagators $G = \Box^{-1}$, *i.e.* solutions to the inhomogeneous field equation $\Box G(\ell, \ell') = \delta(\ell, \ell')$.

3.4.2 Picking a side

Spin-locality

Having described and understood the origins of the disagreement between Z_{local} and Z_{HS}, one is left with a genuine choice between the two: one can stay with Z_{local} and keep the connection to the local boundary theory, or go with Z_{HS} and gain manifest global higher-spin symmetry. In this section we will sketch an argument for choosing the latter.

The orthodox choice from the perspective of holography is to choose Z_{local} since the CFT is supposed to provide the very definition of a theory of quantum gravity in the bulk. However usual arguments for the locality of the boundary theory in AdS/CFT do not obviously apply in the context of higher-spin gravity, since it remains non-local at all scales [45]. Furthermore, in the context of pure dS_4, if we consider its causal structure, the boundary is not observable within the causal patch, and therefore its locality not of utmost relevance.

Forgoing locality is not without consequence, however. One must accept the most general form of the partition function compatible with higher-spin symmetry, namely, replacing the higher-spin-algebraic partition function (3.14) with

$$Z_{\text{HS}}\left[F(Y)\right] = \exp\left(\sum_{n=1}^{\infty} c_n \operatorname{tr}_\star\left(\underbrace{F(Y)\star\cdots\star F(Y)}_{n \text{ terms}}\right)\right),$$

for arbitrary coefficients c_n. However, this is rather problematic, since, although restrictive, higher-spin symmetry leaves enough freedom to non-linearly redefine $F(Y)$ so to arbitrarily modify coefficients c_n, rendering the theory empty. This is similar to the situation outlined in [45] for bulk fields.

As a way out of this impasse, we can use *spin-locality* in place of spacetime locality to control the freedom of redefinitions, mirroring recent developments in the bulk theory [72–75].

In bulk language, spin-locality refers to locality with respect to spinor arguments $y_\pm \in P(\pm x)$, $Y = y_- + y_+$ of the master field $C(x;Y) = C(x;y_+,y_-)$. Now, recall from our construction of the spinor-helicity modes in Section 2.2.1 that, in the boundary limit, the master field $C(x;Y) = C(x;y_+,y_-)$ corresponds to functions $f(u_+,u_-)$ of boundary spinors $u_\pm \in P(\ell')$. Hence, we will take the boundary version of spin-locality to be locality with respect to these spinor variables $u_\pm$, or equivalently, their linear combinations $u, \bar{u}$.

In turns out that the restrictions of spin-locality allow for the higher-spin-algebraic partition function (3.14) to be reproduced from first principle, modulo some subtle sign choices that have to be made by hand.

dS/CFT

In the context of the problem of quantum gravity in de Sitter space, the partition function of the either past or future boundary CFT has been interpreted [17, 18] as the Hartle–Hawking wavefunction [76] of quantum higher-spin gravity in dS_4. For this to make sense, the partition function should have a global maximum on empty de Sitter space, namely when the sources vanish. However the partition function of the boundary theory (3.11) has a local minimum at the origin. A proposed solution [18] was to change the fundamental fields of the boundary vector model from commuting $\phi^I(\ell)$ with internal $O(2N)$ symmetry group, to anti-commuting ones with internal $Sp(2N)$ symmetry group. Upon restricting to even fields, this indeed flips the sign of the effective action and gives the partition function the required local maximum.

However, it was further shown [25] that this maximum is not global. Specifically, for a constant scalar source σ on S_3, as reviewed in Section 3.1.3, the result is as given in (3.38), with flipped overall sign as discussed above. For this partition function, the local maximum at $\sigma = 0$ is joined by a series of higher-valued maxima.

There have been new foundations proposed for dS/CFT that would produce a well-behaved boundary partition function [22, 77]; we believe that our construction of Z_{HS} also provides a resolution to this problem. For a constant scalar source since, upon careful flipping of the sign in the exponent, (3.33) becomes

$$\Psi_{\text{Hartle–Hawking}}[\sigma] = Z_{\mathrm{HS}}[\sigma] = \frac{1}{(1+\pi^2\sigma^2)^{\frac{N}{8}}} \ ,$$

which has a global maximum at $\sigma = 0$ as required.

Chapter 4

Higher-spin black hole from boundary bilocals

Finding solutions in higher-spin theory is notoriously difficult, in part due to the highly non-local behavior of the field equations in twistor space, which is translated to space-time non-locality. Since higher-spin theory is an extension of General Relativity, the natural question arises if there exists counterparts to black hole solutions in higher-spin gravity. In fact, various black-hole-like solutions to the Vasiliev equations have been constructed [26, 78–81] over AdS backgrounds, however their precise physical nature (presence of horizons, physicality of curvature singularity, thermodynamic properties, etc.) is difficult to discern.

In this chapter we will show that the linearized version of the Didenko–Vasiliev black hole solves the Fronsdal field equations with a particle-like source. Furthermore, these fields are precisely the linearized bulk higher-spin fields corresponding to a bilocal source on the boundary. We will also show that the boundary correlator of such two bilocal operators agrees with the bulk action describing the two corresponding particles interacting in the bulk. These results are currently being prepared for publication [82].

Let us start by noting that the holographic boundary bilocal sources employed in the bilocal formulation of the CFT, as described in Section 1.4.2 have the property that they are local in the bulk. In particular, for a bilocal pair of sources located at ℓ and ℓ', $\Pi(l,l') = \delta^{\frac{5}{2},\frac{1}{2}}(l,\ell)\delta^{\frac{5}{2},\frac{1}{2}}(l',\ell')$ the master field at a point x in EAdS_4 is calculated [35] to be

$$C(x;Y) = \frac{1}{\pi\sqrt{2\left[\ell\cdot\ell' + 2(\ell\cdot x)(\ell'\cdot x)\right]}}\exp\frac{iY\left[\ell\ell' + 2(\ell'\cdot x)\ell x\right]Y}{2\left[\ell\cdot\ell' + 2(\ell\cdot x)(\ell'\cdot x)\right]}. \tag{4.1}$$

From this master field we want to extract the field strength and find related potentials for each spin s around the singular world line determined by boundary points ℓ, ℓ'.

At scalar level, the field strength is simply

$$C^{(0,0)}(x) = C(x;0) = \frac{1}{\pi\sqrt{2\left[\ell\cdot\ell' + 2(\ell\cdot x)(\ell'\cdot x)\right]}}\,.$$

Specializing to lightcone coordinates in embedding space $x^\mu = (u,v,\mathbf{r})$ with line element $ds^2 = -du\,dv + d\mathbf{r}^2$ and picking, without loss of generality, $\ell^\mu = (0,1,\mathbf{0})$ and $\ell'^\mu =$

$(1, 0, \mathbf{0})$, then

$$r = |\mathbf{r}| = \sqrt{4(x\cdot\ell)(x\cdot\ell') - 1}\,, \tag{4.2}$$

along the geodesic determined by boundary points ℓ, ℓ'. Thus scalar field strength becomes $C^{(0,0)}(x) = \frac{1}{\pi r}$, as expected.

For non-zero spins, one computes the field strengths by taking appropriately projected Y-derivatives of the master field (4.1) and evaluate them at $Y = 0$. For instance, the left-handed spin-1 field strength is

$$C^{(2,0)}_{ab}(x) = \frac{1}{\pi r^3}\left(\ell\ell' + (\ell'\cdot x)\ell x - (\ell\cdot x)\ell' x - i\ell\ell' x\right)_{ab}\,.$$

Summing left and right-handed components, the potential A^ν for the spin-1 field strength has to satisfy

$$\begin{aligned}\nabla^{[\mu}A^{\nu]} &= \frac{1}{\pi r^3}\left(\ell^{[\mu}\ell'^{\nu]} + (\ell'\cdot x)\ell^{[\mu}x^{\nu]} - (\ell\cdot x)\ell'^{[\mu}x^{\nu]}\right)\\ &= \frac{1}{\pi r^3}\left(\ell^{[\rho}\ell'^{\mu}x^{\nu]}x_\rho\right).\end{aligned}$$

Note that the simpler, "radial" part of the equation

$$\nabla^{[\mu}{A_0}^{\nu]} = \ell^{[\rho}\ell'^{\mu}x^{\nu]}x_\rho$$

is solved by

$$A^\nu_0 = \ell^\nu(\ell'\cdot x) - \ell'^\nu(\ell\cdot x)\,. \tag{4.3}$$

By considering ansatz of the form $A^\nu = f(r)A^\nu_0$, a general solution is found to be

$$A^\nu = \frac{1}{\pi r}\left(\ell^\nu(\ell'\cdot x) - \ell'^\nu(\ell\cdot x)\right)\,.$$

The aim was to generalize this procedure to find the higher spin $s \geq 2$ potentials. As mentioned before, the linear field strengths as derived above for each spin are the same as the linear solutions to the Didenko–Vasiliev black hole in 4d higher-spin theory [26].

4.1 Kerr–Schild formalism

The Didenko–Vasiliev "black hole" [26] (generalized in [79]) is a spherically symmetric solution of the Vasiliev equation. Its construction is similar to the Kerr–Schild procedure for arriving at the Kerr black hole. As mentioned in the introduction of this chapter, it is not clear whether the similarity to General Relativity black holes goes beyond this formal construction.

The metric of a black hole of mass m admits the Kerr-Schild metric form

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2m}{r}k_\mu k_\nu\,,$$

in both flat [83] and curved backgrounds [84], where $\eta_{\mu\nu}$ is the background metric and

k^μ, the so-called the Kerr–Schild vector, satisfies

$$k_\mu k^\nu = 0\,, \quad k^\mu \nabla_\mu k_\nu = 0\,. \tag{4.4}$$

It was shown [26, 78] that the Kerr–Schild AdS_4 black hole solution admits a higher-spin black hole generalization for massless bosonic fields of any spin s

$$\phi_{\mu_1\dots\mu_s}(x) = \frac{2m}{r} k_{\mu_1}\dots k_{\mu_s} \tag{4.5}$$

which satisfies the linearized spin-s Fronsdal equation (1.22) away from sources

$$F_{\mu_1\dots\mu_s} \equiv \left(\Box - (s^2 - 2s - 2)\right)\phi_{\mu_1\dots\mu_s} - s\nabla_{(\mu_1}\nabla^\nu \phi_{\mu_2\dots\mu_s)\nu} = 0\,. \tag{4.6}$$

Formalizing the preamble discussion, we will construct the linearized version of the Didenko–Vasiliev black hole and show that it satisfies the Fronsdal field equation with a particle-like source.

Similar to the construction of A_0^μ (4.3), consider the "time-like" vector quantity

$$T^\mu = \frac{1}{2}\left(\frac{\ell'^\mu}{x\cdot\ell'} - \frac{\ell^\mu}{x\cdot\ell}\right) \tag{4.7}$$

with norm $T_\mu T^\mu = \frac{1}{1+r^2}$. Further, note that

$$\nabla_\mu T_\nu = -2R_{(\mu}T_{\nu)}\,,$$

where we define the "radial" vector

$$R^\mu = x^\mu + \frac{1}{2}\left(\frac{\ell^\mu}{x\cdot\ell} + \frac{\ell'^\mu}{x\cdot\ell'}\right)\,. \tag{4.8}$$

This vector has norm $R_\mu R^\mu = \frac{r^2}{1+r^2}$ and it is tangent to the world-line (4.2) since

$$\nabla_\mu r = \frac{1+r^2}{r} R_\mu\,.$$

Further, note that these two vectors are orthogonal, $R_\mu T^\nu = 0$. Thus, we can now construct the null linear combination

$$k^\mu = \frac{1}{2}\left(T^\mu + \frac{i}{r}R^\mu\right)\,. \tag{4.9}$$

We will use this vector to construct a Kerr–Schild type solution (4.5) in $EAdS_4$. We will obtain the explicit form of the Fronsdal operator (4.6) by rewriting all covariant derivatives of the field (4.5) in terms of R^μ, T^μ, and k^μ. First note

$$\begin{aligned}\nabla_\mu R_\nu &= g_{\mu\nu} - T_\mu T_\nu - R_\mu R_\nu \\ &= g_{\mu\nu} - 4k_\mu k_\nu + \frac{4i}{r}k_{(\mu}R_{\nu)} + \frac{1-r^2}{r^2}R_\mu R_\nu\end{aligned}$$

and thus
$$\nabla_\mu k_\nu = \frac{i}{2r} g_{\mu\nu} - \frac{2i}{r} k_\mu k_\nu - 2\frac{1+r^2}{r^2} k_{(\mu} R_{\nu)}. \tag{4.10}$$
Hence, noting that $k_\mu R^\mu = \frac{ir}{2(1-r^2)}$, it follows that $k^\nu \nabla_\nu k^\mu = 0$, as required.

We can now proceed to show that the field $\phi_{\mu_1 \dots \mu_s}(x)$ (4.5) constructed from such a vector k_μ indeed satisfies the linearized Fronsdal equation (4.6), away from sources. For simplicity, we will set $m = 1$ without loss of generality.

At spin $s = 0$ the Frosndal operator becomes $F = (\Box - 2)\,\phi$, for $\phi(x) = \frac{2}{r}$. Since $\nabla_\mu \frac{1}{r} = \frac{1-r^2}{r^3} R_\mu$ it follows that $\Box\phi = \nabla_\nu \nabla^\nu \phi = \frac{2}{r}$ and thus $F = 0$ away from $r = 0$.

For spin $s = 1$ the Frosndal operator reads $F_\mu = (\Box - 3)\,\phi_\mu - \nabla_\mu \nabla^\nu \phi_\nu$ for a spin-1 field $\phi_\nu(x) = \frac{2}{r} k_\nu$. In particular, $\Box\phi_\mu = \Box\left(\frac{2}{r}\right) k_\mu + 2\nabla^\nu\left(\frac{2}{r}\right)\nabla_\nu k_\mu + \frac{2}{r}\Box k_\mu$. Noting the following identities, $\nabla^\nu\left(\frac{1}{r}\right)\nabla_\nu k_\mu = \frac{1}{r} k_\mu$, $\Box k_\mu = -3k_\mu - i\frac{1-r^2}{r^3} R_\mu$, and $\nabla_\mu\left(\frac{1}{r} k^\mu\right) = \frac{i}{2r^2}$, it follows that $F_\mu = 0$ away from $r = 0$.

For spin $s \geq 2$ the calculation follows in a similar matter, noting that the symmetrization factors have to be treated carefully. First, the field reads $\phi_{\mu_1 \dots \mu_s}(x) = \frac{2}{r} k_{\mu_1} \cdots k_{\mu_s}$ and note that

$$\begin{aligned}\frac{1}{2}\Box\phi_{\mu_1 \dots \mu_s} = \Box\frac{1}{r}\, k_{\mu_1} \cdots k_{\mu_s} + 2\nabla_\nu \frac{1}{r}\, \underbrace{k_{\mu_1} \cdots \nabla^\nu k_{\mu_i} \cdots k_{\mu_s}}_{s} + \\ + \frac{1}{r}\underbrace{k_{\mu_1} \cdots \Box k_{\mu_i} \cdots k_{\mu_s}}_{s} + \frac{1}{r}\underbrace{k_{\mu_1} \cdots \nabla_\nu k_{\mu_i} \nabla^\nu k_{\mu_j} \cdots k_{\mu_s}}_{s(s-1)},\end{aligned}$$

where the subindices i, j run from 1 to s and the underbraced indicates the number of terms of each type. Noting that $\nabla_\nu k_{\mu_i} \nabla^\nu k_{\mu_j} = -\frac{1}{4r^2} g_{\mu_i \mu_j} + \frac{1+r^2}{r^2} k_{\mu_i} k_{\mu_j} - i\frac{1+r^2}{r^3} k_{(\mu_1} k_{\mu_j)}$, the above reduces to

$$\begin{aligned}\frac{1}{2}\Box\phi_{\mu_1 \dots \mu_s} = \left(-\frac{2+s}{r} + s(s-1)\frac{1+r^2}{r^3}\right) k_{\mu_1} \dots k_{\mu_s} + \\ - is^2 \frac{1+r^2}{r^4} R_{(\mu_1} k_{\mu_2} \dots k_{\mu_s)} - \frac{s(s-1)}{4r^2} g_{(\mu_1 \mu_2} k_{\mu_3} \dots k_{\mu_s)}\,.\end{aligned}$$

Similarly, since $\nabla_\mu k^\mu = \frac{i}{r}$, and $\frac{1}{2}\nabla^\nu \phi_{\mu_2 \cdots \mu_s \nu} = \frac{i}{2r^2} k_{\mu_2} \dots k_{\mu_s}$, it follows that

$$\begin{aligned}\nabla_{(\mu_1} \nabla^\nu \phi_{\mu_2 \dots \mu_s)\nu} = \frac{2(s-1)}{r^3} k_{\mu_1} \cdots k_{\mu_s} + \\ - 2is\frac{1+r^2}{r^4} R_{(\mu_1} k_{\mu_2} \cdots k_{\mu_s)} - \frac{s-1}{2r^3} g_{(\mu_1 \mu_2} k_{\mu_3} \cdots k_{\mu_s)}\,.\end{aligned}$$

Hence, putting everything together, we can see that the various contributions cancel so that the full Fronsdal operator $F_{\mu_1 \dots \mu_s} = 0$ away from $r = 0$, for any spin s.

To account for the source term at $r = 0$ we integrate the full Fronsdal operator (4.6) over a sphere centered at origin and of infinitesimal radius r. Thus, by the divergence theorem, at leading order in r,

$$F_{\mu_1\dots\mu_s} = \delta(\mathbf{r}) \int_{S_2} dS\, rR^\nu \left(\nabla_\nu \phi_{\mu_1\dots\mu_s} - s\nabla_{(\mu_1}\phi_{\mu_2\dots\mu_s)\nu}\right) . \tag{4.11}$$

Then, term-wise, the integrand reads

$$R^\nu \nabla_\nu \phi_{\mu_1\dots\mu_s} = -\left(1 + s\frac{r^2}{1+r^2}\right)\phi_{\mu_1\dots\mu_s} ,$$

and hence, denoting by $q_{\mu\nu} = g_{\mu\nu} - T_\mu T_\nu$ the metric on a spatial slice,

$$\begin{aligned} R^\nu \nabla_{(\mu_1}\phi_{\mu_2\dots\mu_s)\nu} = \frac{r^2}{1+r^2}\phi_{\mu_1\dots\mu_s} - i\left(2\frac{s-1}{1+r^2} + \frac{1}{r^2}\right) R_{(\mu_1} k_{\mu_2}\cdots k_{\mu_s)} + \\ - \frac{s-1}{1-r^2}\left(\frac{1}{2r} q_{(\mu_1\mu_2} k_{\mu_3}\cdots k_{\mu_s)} - \frac{1+2r^2}{2r^3} R_{(\mu_1} R_{\mu_2}\cdots k_{\mu_s)}\right) . \end{aligned}$$

Finally, at leading order in r,

$$rR^\nu \left(\nabla_\nu \phi_{\mu_1\dots\mu_s} - s\nabla_{(\mu_1}\phi_{\mu_2\dots\mu_s)\nu}\right) = -r\phi_{\mu_1\dots\mu_s} + \frac{is}{r} R_{(\mu_1} k_{\mu_2}\cdots k_{\mu_s)} + \frac{s(s-1)}{2}\Omega_{(\mu_1\mu_2} k_{\mu_3}\cdots k_{\mu_s)} , \tag{4.12}$$

where $\Omega_{\mu_1\mu_2} = q_{\mu_1\mu_2} - \frac{1}{r^2} R_{\mu_1} R_{\mu_2}$ is the spherical metric tensor.

Recall that the integral (4.11) is performed over a the sphere and thus any contribution from terms in the integrand proportional to odd powers of R_{μ_i} will vanish since they are odd. We claim that the even contributions to the integral will be of the form

$$\int_{S_2} dS\; n_{\mu_1}\cdots n_{\mu_{2i}} = A_i\, q_{(\mu_1\mu_2}\cdots q_{\mu_{2i-1}\mu_{2i})} \tag{4.13}$$

where we denote $n_i = \frac{R_i}{r}$ as the normal direction to the sphere. The tensor structure on the right-hand side of (4.13) follows from symmetry considerations: since the left-hand side integral is spherically symmetric and symmetric in its indices the spatial metric provides the only compatible tensor structure. To fix the over-all numerical factor, note that for $i = 0$ the above reduces to the area of a unit sphere, and hence $A_0 = 4\pi$. For $i = 1$, (4.13) reduces to

$$\int_{S_2} dS\, n_{\mu_1} n_{\mu_2} = A_2\, q_{\mu_1\mu_2}$$

and thus, taking the trace on both sides, $A_1 = \frac{4\pi}{3}$. Proceeding by induction, at order $i+1$, (4.13) reads

$$\int_{S_2} dS\; n_{\mu_1}\cdots n_{\mu_{2i}} n_{\mu_{2i+1}} n_{\mu_{2i+2}} = A_{i+1}\, q_{(\mu_1\mu_2}\cdots q_{\mu_{2i-1}\mu_{2i}} q_{\mu_{2i+1}\mu_{2i+2})} . \tag{4.14}$$

Contracting indices μ_{2i+1} and μ_{2i+2}, the left-hand side is given by (4.13), whereas on the right-hand side, due to the symmetrization of the indices, we count the following possible contributions: $2(i+1)$ terms in which the indices are contracted within the same metric tensor; $4i(i+1)$ cases in which the indices contract two different metric tensors. Accounting for the three-dimensional trace and overall symmetrization factors, (4.14) becomes

$$A_i\, q_{(\mu_1\mu_2} \dots q_{\mu_{2i-1}\mu_{2i})} = 2(2i+3)(i+1)\frac{(2i)!}{(2i+2)!} A_{i+1}\, q_{(\mu_1\mu_2} \dots q_{\mu_{2i-1}\mu_{2i})}\,,$$

and thus

$$(2i+1)A_i = (2i+3)A_{i+1}\,.$$

Accounting for the base cases, we read the overall numerical factor in (4.13) to be

$$A_i = \frac{4\pi}{2i+1}\,.$$

Hence, performing (4.11) order-by-order in powers of R_{μ_i} and grouping the result order-by-order in powers of $q_{\mu\nu}$ we claim that

$$F_{\mu_1\dots\mu_s} = -4\pi\delta(\mathbf{r})\frac{1}{2^{s-1}}\sum_{i=0}^{\lfloor \frac{s}{2}\rfloor}\binom{s}{2i} q_{(\mu_1\mu_2} \cdots q_{\mu_{2i-1}\mu_{2i}} T_{\mu_{2i+1}} \cdots T_{\mu_s)}\,. \tag{4.15}$$

To zero-order in $q_{\mu\nu}$, the only contribution comes from the $-r\phi_{\mu_1\dots\mu_s}$ term in the integrand (4.12); recalling that $\phi_{\mu_1\dots\mu_s} = \frac{2}{r}k_{\mu_1}\cdots k_{\mu_s}$ and that $k_\mu = \frac{1}{2}(T_\mu + in_\mu)$, this contribution reads

$$\tilde{F}^{(0)}_{\mu_1\dots\mu_s} = -\frac{4\pi}{2^{s-1}} T_{\mu_1}\cdots T_{\mu_s}\,.$$

To first-order in $q_{\mu\nu}$ we count a contribution considering the $q_{\mu_1\mu_2}T_{\mu_3}\cdots T_{\mu_s}$ factor from the last term in the integrand and contributions from all the terms in integrand considering factors proportional to $n_{\mu_1}n_{\mu_2}T_{\mu_3}\cdots T_{\mu_s}$. Accounting for symmetrization factors, this becomes

$$\tilde{F}^{(1)}_{\mu_1\dots\mu_s} = \left[-\frac{4\pi}{2^{s-1}} + \frac{4\pi}{3}\left(-\frac{1}{2^{s-1}}\binom{s}{2} + 2\frac{s(s-1)}{2^{s-1}}\right)\right] q_{(\mu_1\mu_2}T_{\mu_3}\cdots T_{\mu_s)}\,.$$

The combinatorial prefactors combine rather elegantly to give

$$\tilde{F}^{(1)}_{\mu_1\dots\mu_s} = -\frac{4\pi}{2^{s-1}}\binom{s}{2} q_{(\mu_1\mu_2}T_{\mu_3}\cdots T_{\mu_s)}\,.$$

Similarly, at a general order i in $q_{\mu\nu}$, we count the same contributions as above: one contribution from the $q_{\mu_1\mu_2}n_{\mu_3}\cdots n_{\mu_{2i}}T_{\mu_{2i+1}}\cdots T_{\mu_s}$ factor in the last term of the integrand, and a further three contributions from the $n_{\mu_1}\cdots n_{\mu_{2i}}T_{\mu_{2i+1}}\cdots T_{\mu_s}$ proportional terms. Namely,

$$\tilde{F}^{(i)}_{\mu_1\dots\mu_s} = \frac{1}{2^{s-1}}\left[-s(s-1)\binom{s-2}{2i-2}\frac{4\pi}{2i-3} + \frac{4\pi}{2i+1}\left(-\binom{s}{2i} + s\binom{s-1}{2i-1} + s(s-1)\binom{s-2}{2i-2}\right)\right]\times$$
$$\times\, q_{(\mu_1\mu_2}\cdots q_{\mu_{2i-1}\mu_{2i}}T_{\mu_{2i+1}}\cdots T_{\mu_s)}\,.$$

As before, the prefactors simplify beautifully to give

$$\tilde{F}^{(i)}_{\mu_1\ldots\mu_s} = -\frac{4\pi}{2^{s-1}}\binom{s}{2i} q_{(\mu_1\mu_2}\cdots q_{\mu_{2i-1}\mu_{2i}}T_{\mu_{2i+1}}\cdots T_{\mu_s)}\,.$$

Summing the above results order-by-order we obtain the full form (4.15), as claimed.

In our upcoming calculations will be interested in the following tensor

$$G_{\mu_1\ldots\mu_s} = F_{\mu_1\ldots\mu_s} - \frac{s(s-1)}{4} g_{(\mu_1\mu_2}F^{\nu}{}_{\mu_3\ldots\mu_s)\nu}\,. \tag{4.16}$$

This is constructed for doubly traceless fields as a generalization of the Einstein tensor; note its divergence is pure trace trace. For the Fronsdal operator (4.15), this reduces to

$$G_{\mu_1\ldots\mu_s} = -4\pi\delta(\mathbf{r})\left[T_{\mu_1}\cdots T_{\mu_s} - \text{double traces}\right]\,, \tag{4.17}$$

where by double traces we refer to terms proportional to $g_{(\mu_1\mu_2}\cdots g_{\mu_{2i-1}\mu_{2i}}T_{\mu_{2i+1}}\cdots T_{\mu_s)}$ for $0 < i < \left\lfloor \frac{s}{2}\right\rfloor$ which will be irrelevant in upcoming calculations.

Now, we claim that the field strengths $\tilde{C}_{\mu_1\nu_1\ldots\mu_s\nu_s}$ derived from the antisymmetrized, traceless part of $\nabla_{\mu_1}\ldots\nabla_{\mu_s}\phi_{\nu_1\cdots\nu_s}$, read

$$\tilde{C}_{\mu_1\nu_1\ldots\mu_s\nu_s} = (-1)^s\frac{(2s)!}{s!}\frac{1}{r^{2s+1}}S_{\mu_1\nu_1}\cdots S_{\mu_s\nu_s} - \text{traces}\,, \tag{4.18}$$

where we define

$$S_{\mu\nu} = (1+r^2)\,R_{[\mu}k_{\nu]}\,. \tag{4.19}$$

At spin $s=1$, recall $\phi_\rho(x) = \frac{2}{r}k_\rho$; since

$$\nabla_\mu\phi_\rho = -2\frac{1+r^2}{r^3}R_\mu k_\nu + \frac{2}{r}\nabla_\mu\phi_\rho$$

and noting from (4.10) that $\nabla_\mu\phi_\rho$ is a symmetric quantity,

$$\tilde{C}_{\mu\rho} = \nabla_{[\mu}\phi_{\rho]} = -\frac{2}{r^3}S_{\mu\rho}\,.$$

For spin $s=2$, $\phi_{\rho\sigma}(x) = \frac{2}{r}k_\rho k_\sigma$, and thus

$$\nabla_{[\mu}\phi_{\rho]\sigma} = -\frac{4}{r^3}S_{\mu\rho}k_\sigma + \frac{1}{r^2}k_{[\rho}g_{\mu]\sigma}\,.$$

It then follows that

$$\tilde{C}_{\nu\mu\rho\sigma} = g_{\nu\lambda}g_{\sigma\tau}\nabla^{[\lambda}\nabla_{[\mu}\phi_{\rho]}{}^{\tau]} = \frac{12}{r^5}S_{\mu\rho}S_{\nu\sigma} - \text{traces}$$

where the traces terms are proportional to $K_{[\nu}g_{\sigma][\mu}K_{\rho]}$, $R_{[\nu}g_{\sigma][\mu}K_{\rho]}$, and $\delta^{[\nu}{}_{[\mu}\delta_{\rho]}{}^{\nu]}$.

For general spin $s>2$, we can then proceed by induction on the number of pairs on antisymmetrized indices to show that

$$\begin{aligned}\tilde{C}_{\mu_1\nu_1\ldots\mu_s\nu_s} &= (-1)^s \frac{(2s-1)!}{(s-1)!}\frac{2}{r^{2s+1}}S_{\mu_1\nu_1}\cdots S_{\mu_s\nu_s} - \text{traces} \\ &= (-1)^s \frac{(2s)!}{s!}\frac{1}{r^{2s+1}}S_{\mu_1\nu_1}\cdots S_{\mu_s\nu_s} - \text{traces}\end{aligned} \tag{4.20}$$

Recall the original bi-local master field (4.1) reads

$$C(x;Y) = \frac{1}{\pi r}\exp\frac{iY\,[\ell\ell' - 2(\ell'\cdot x)\ell x]\,Y}{2r^2}\,,$$

where we have identified the radial amplitude r as in (4.2). Now recall that $S_{\mu\nu} = (1+r^2)R_{[\mu}k_{\nu]}$, as defined in (4.9). Using the explicit expressions of T^μ and R^μ, (4.7) and (4.8) respectively, we can see that

$$S_{\mu\nu} = -\ell_{[\mu}\ell'_{\nu]} - (x\cdot\ell')x_{[\mu}\ell_{\nu]} + (x\cdot\ell)x_{[\mu}\ell'_{\nu]}$$

and hence

$$C(x;Y) = \frac{1}{\pi r}\exp\frac{iY\,(-\gamma_{\mu\nu}S^{\mu\nu})\,Y}{r^2}\exp\frac{iY\,[-(\ell'\cdot x)\ell x - (\ell\cdot x)\ell' x]\,Y}{r^2}\,. \tag{4.21}$$

In particular, the spin-s component reads

$$C_{\mu_1\nu_1\ldots\mu_s\nu_s} = \frac{(2i)^s(2s)!}{\pi s!\,r^{2s+1}}S_{\mu_1\nu_1}\cdots S_{\mu_s\nu_s} - \text{traces}\,, \tag{4.22}$$

where the first exponential term in (4.21) gives the first term in (4.22), while the second exponential contributes with traces terms due to the direct x-dependence in the exponential.

Thus note by comparing (4.21) with (4.20) we can identify the bulk field corresponding to the boundary bilocal sources and the field of a particle source moving along a geodesic.

4.2 Fixing the normalization

The solutions that we have described above are constructed up to an overall constant. To fix this normalization constant let us look at the easy case of field strengths encoded in terms of local boundary sources, as in [85]. Antipodally even spin-s boundary-to-bulk propagators can be encoded in terms of a boundary point ℓ^μ and a symmetric and traceless polarization tensor on $\mathcal{I}^+$; without loss of generality, this tensor can be taken to have the form $\lambda_{\mu_1}\ldots\lambda_{\mu_s}$, for λ_μ a null complex vector on $\mathcal{I}^+$ orthogonal to ℓ^μ. We can encode this data as a totally null bivector $M_{\mu\nu} = 2\ell_{[\mu}\lambda_{\nu]}$.

The propagators were originally described for EAdS_4 in [86] and adapted to the current framework in [85]. They read

$$\varphi_{\mu_1\ldots\mu_s}(x;\ell,M) = \frac{1}{(x\cdot\ell)^{2s+1}}M_{\mu_1}\cdots M_{\mu_s}\,,$$

where $M_\mu = M_{\mu\nu}x^\nu$. This has corresponding scalar field strengths

$$\mathcal{C}_{\mu_1\nu_1\ldots\mu_s\nu_s} = \frac{(2s-1)!!}{2(x\cdot\ell)^{2s+1}} M^\perp_{\mu_1\nu_1}\cdots M^\perp_{\mu_s\nu_s} - \text{traces}\,, \tag{4.23}$$

where $M^\perp_{\mu\nu} = M_{\mu\nu} - 2M_{[\mu}x_{\nu]} = -\nabla_\mu M_\nu$ and $(2s-1)!! = 1\cdot 3\cdot 5\cdot\ldots\cdot(2s-1)$.

To make contact with the bi-local picture, we send the the bulk point x^μ to a boundary point ℓ'^μ as an extreme boost limit $x^\mu \to \frac{1}{z}\ell'^\mu$ as $z\to 0$. In this limit the polarization vector reads $M_\mu \to \frac{1}{z}m_\mu$ where $m_\mu = (\ell'\cdot\lambda)\ell_\mu - (\ell\cdot\ell')\lambda_\mu$, while the propagator becomes, to leading order,

$$\varphi_{\mu_1\ldots\mu_s} \to z^{2-s}A_{\mu_1\ldots\mu_s} + z^{s+1}J_{\mu_1\ldots\mu_s} + \cdots\,,$$

where the field potential and current terms read

$$A_{\mu_1\ldots\mu_s} = -\frac{2\pi^2(2s-3)!!}{s!}\delta(\ell\cdot\ell')\lambda_{\mu_1}\cdots\lambda_{\mu_s}\;;\quad J_{\mu_1\ldots\mu_s} = \frac{m_{\mu_1}\cdots m_{\mu_s}}{(\ell\cdot\ell')^{2s+1}}\,.$$

Thus, noting that $m_\mu\lambda'^\mu = -\frac{1}{2}M_{\mu\nu}M'^{\mu\nu}$, the action reads [48]

$$S = -\frac{2s-1}{2}A_{\mu_1\ldots\mu_s}J'^{\mu_1\ldots\mu_s} = \begin{cases} \frac{2\pi^2}{\ell\cdot\ell'}\,, & s=0 \\ (-1)^s\frac{\pi^2(2s-1)!!}{2^s s!}\frac{(M_{\mu\nu}M'^{\mu\nu})^s}{(\ell\cdot\ell')^{2s+1}}\,, & s>0 \end{cases}.$$

For the scalar case, the relevant local twistor function is simply the spin-0 insertion $F(Y) = i\delta_\ell(Y)$ which gives rise to master-field $C(x;Y) = -\frac{2}{x\cdot\ell}e^{\frac{iY\ell xY}{2\ell\cdot x}}$. The scalar component of the master-field reads simply as the bulk-to-boundary scalar propagator $C(x;0) = -\frac{2}{x\cdot\ell}$. Thus a scalar field

$$\varphi(x) = A_0 C(x;0) = -\frac{2A_0}{x\cdot\ell}$$

will induce an action

$$S = -\frac{8\pi^2A_0^2}{\ell\cdot\ell'}.$$

On the other side, we can also read the action from the boundary bilocal encoding, namely

$$S = -\frac{N}{8}\,\mathrm{tr}_\star\,(F\star F') = \frac{N}{8}\int d^4Y\,\delta_\ell(Y)\star\delta_{\ell'}(Y) = -\frac{N}{2}\,.$$

Thus, given that for our particular choice of boundary points $\ell\cdot\ell' = -\frac{1}{2}$, we can fix the proportionality constant, for scalar fields, to be $A_0 = \frac{\sqrt{N}}{4\pi\sqrt{2}}$.

For spin $s>0$, we introduce twistor functions $\delta_\ell(Y-M)$ and $e^{iMY}\delta_\ell(Y)$, with $M^a\in P^*(\ell)$; these functions generate, respectively, the left- and right-handed parts of the spin $s>0$ boundary-to-bulk propagators. We thus consider

$$F(Y) = \frac{1}{2}\left(\delta_\ell(Y-M) + \delta_\ell(Y+M)\right) + \frac{1}{2}\delta_\ell(Y)\left(e^{iMY} + e^{-iMY}\right) \tag{4.24}$$

which transforms into the master-field

$$C(x;Y)=-\frac{2}{x\cdot\ell}e^{\frac{iY\ell xY}{2\ell\cdot x}}\left(\cos\frac{M\ell P_xY}{x\cdot\ell}+\cos\frac{M\ell P_{-x}Y}{x\cdot\ell}\right)\ .$$

In twistor notation

$$M_{\mu\nu}=\frac{1}{4}\gamma^{ab}_{\mu\nu}(\ell M)_a(\ell M)_b\ ;\ M_{\mu\nu}M'^{\mu\nu}=\frac{1}{2}\left(M'\ell'\ell M\right)^2$$

and thus, at spin $s>0$, the field strengths read

$$\tilde{\mathcal{C}}_{\mu_1\nu_1\dots\mu_s\nu_s}=\frac{2(-1)^{s+1}}{(x\cdot\ell)^{2s+1}}M^{\perp}_{\mu_1\nu_1}\dots M^{\perp}_{\mu_s\nu_s}-\text{traces}\ .$$

Thus, comparing with (4.23), since the appropriately normalized field-strength should have the form $\tilde{\mathcal{C}}_{\mu_1\nu_1\dots\mu_s\nu_s}=A_s\mathcal{C}_{\mu_1\nu_1\dots\mu_s\nu_s}$, it follows that the corresponding field potential reads

$$\varphi_{\mu_1\dots\mu_s}=\frac{4(-1)^{s+1}}{(2s-1)!!}\frac{A_s}{(x\cdot\ell)^{2s+1}}M_{\mu_1}\dots M_{\mu_s}\ .$$

Hence, the spin-s contribution to the action, can be computed as

$$S^{(s)}=(-1)^s\frac{16\pi^2}{(2s-1)!!\,s!\,2^s}\frac{A_s^2}{(\ell\cdot\ell')^{2s+1}}\left(M_{\mu\nu}M'^{\mu\nu}\right)^s\ . \tag{4.25}$$

From the twistor picture, as before, the action reads $S=-\frac{N}{8}\operatorname{tr}_\star(F\star F')$; in particular, for twistor function (4.24), this becomes

$$S=-\frac{N}{32}\Big[\operatorname{tr}_\star\Big(\delta_\ell(Y\pm M)\star\delta_{\ell'}(Y\pm M')\Big)+\operatorname{tr}_\star\Big(\delta_\ell(Y)e^{\pm iMY}\star\delta_{\ell'}(Y)e^{\pm iM'Y}\Big)+$$
$$+\operatorname{tr}_\star\Big(\delta_\ell(Y\pm M)\star\delta_{\ell'}(Y)e^{\pm iM'Y}\Big)+\operatorname{tr}_\star\Big(\delta_\ell(Y)e^{\pm iMY}\star\delta_{\ell'}(Y\pm M')\Big)\Big]$$

where, for notational convenience, we each $\pm$ is understood independently, *i.e.* there are four star-trace terms of each kind. Using star-product and spinor-delta formulae (1.31), and (1.32) we compute the above to

$$S=-N\left(\cos(M'\ell'\ell M)+1\right)=\frac{N}{2\ell\cdot\ell'}\left(\cos\left(\frac{M'\ell'\ell M}{2\ell\cdot\ell'}\right)+1\right) \tag{4.26}$$

where in the last term we introduced the relative normalization $\ell\cdot\ell'=-\frac{1}{2}$ in order to facilitate comparison with (4.25).

The spin-s contribution to the action (4.26) reads

$$S^{(s)}=\frac{(-1)^sN}{2^{2s}(2s)!}\frac{(M'\ell'\ell M)^{2s}}{(\ell\cdot\ell')^{2s+1}}=\frac{(-1)^sN}{2^{2s}(2s)!}\frac{(M'_{\mu\nu}M^{\mu\nu})^{2s}}{(\ell\cdot\ell')^{2s+1}}\ .$$

Thus, by direct comparison with (4.25) we can determine the proportionality constant as

$$A_s = \frac{1}{4\pi\sqrt{2}}\sqrt{\frac{N}{2^s}} \tag{4.27}$$

in direct generalization of the scalar result.

Thus, taking into account the proportionality constants A_s we can finally arrive at correct normalization of our solutions (4.5) to the Fronsdal field equations with source:

$$\phi_{\mu_1\dots\mu_s}(x) = \frac{\sqrt{2^s N}}{2\sqrt{2}\pi^2}\frac{(-i)^s}{r}k_{\mu_1}\cdots k_{\mu_s}\ , \tag{4.28}$$

with the scalar case reading

$$\phi(x) = \frac{\sqrt{N}}{4\sqrt{2}\pi^2 r}\ . \tag{4.29}$$

4.3 The worldline action

In this section we will consider two boundary bilocal sources, localized at ℓ_1, ℓ_1' and ℓ_2, ℓ_2', respectively. We will show that the CFT correlators of these operators reproduce the action of the two corresponding bulk particles, as described in the previous section, interacting via their higher-spin gauge fields. The bulk action for a particle moving in the field sourced by a second particle reads

$$S_{12} = \frac{1}{2}\int dx^4 \sum_{s=0}^{\infty} \phi_{1\,\mu_1\dots\mu_s} G_2{}^{\mu_1\dots\mu_s}\ . \tag{4.30}$$

The object that acts as a source is in fact the generalized Einstein tensor (4.16), which reads, for the correctly normalized scalar field $\phi(x^\mu)$ (4.29) and spin-s fields $\phi_{\mu_1\dots\mu_s}(x^\mu)$ (4.28), respectively,

$$\begin{aligned}
G(x^\mu) &= -\frac{\sqrt{N}}{\sqrt{2}\pi}\delta(\mathbf{r})\ ;\\
G_{\mu_1\cdots\mu_s}(x^\mu) &= -\frac{(-i)^s\sqrt{2^s N}}{\sqrt{2}\pi}\delta(\mathbf{r})\left(T_{\mu_1}\cdots T_{\mu_s} - \text{double traces}\right)\ .
\end{aligned}$$

Thus, the bulk action (4.30) becomes

$$\begin{aligned}
S_{12} &= -\frac{1}{2}\int dx^4\,\delta(\mathbf{r})\left(\frac{N}{8\pi^3 r} + \frac{N}{4\pi^3 r}\sum_{s=1}^{\infty}(-1)^s 2^s\left(k_1\cdot T_2\right)^s\right)\\
&= \frac{N}{8\pi^3}\int d\eta\,\frac{1}{r}\left(\frac{1}{2} - \frac{1}{1+2k_1\cdot T_2}\right)
\end{aligned} \tag{4.31}$$

for worldline affine parameter $\eta \in \mathbb{R}$.

We will parametrize our boundary points as

$$\begin{aligned} \ell_1 \cdot \ell_1' = \ell_2 \cdot \ell_2' = \ell_1 \cdot \ell_2 = \ell_1' \cdot \ell_2 = -\frac{1}{2}\,, \\ \ell_1 \cdot \ell_2' = -\frac{a^2}{2}\,, \quad \ell_1' \cdot \ell_2' = -\frac{b^2}{2}\,, \end{aligned} \tag{4.32}$$

so that, along the worldline with endpoint ℓ_2, ℓ_2',

$$x^\mu = \sqrt{ab}\, e^\eta\, \ell_2^\mu + \frac{1}{\sqrt{ab}}\, e^{-\eta}\, {\ell_2'}^\mu\,.$$

In this parameterization, the minimum separation between the worldlines $r_{\rm min}$ is achieved at $\eta = 0$, where $x^\mu_{\rm min} = \sqrt{ab}\,\ell_2^\mu + \frac{1}{\sqrt{ab}}\,{\ell_2'}^\mu$. In particular,

$$r_{\rm min}^2 = 4\,(\ell_1 \cdot x_{\rm min})\,(\ell_1' \cdot x_{\rm min}) - 1 = (a+b)^2 - 1\,.$$

Similarly, at a general point,

$$r^2 = a^2 + b^2 + ab\left(e^{2\eta} + e^{-2\eta}\right) - 1 = (a+b)^2 - 1 + 4ab\,\mathrm{sh}^2\,\eta\,. \tag{4.33}$$

In this parameterization, at minimal separation,

$$T_1 \cdot T_2\Big|_{r_{\rm min}} = \frac{1}{4}\left(\frac{{\ell_1'}^\mu}{x_{\rm min}\cdot \ell_1'} - \frac{\ell_1^\mu}{x_{\rm min}\cdot \ell_1}\right)\left(\frac{\ell'_{2\,\mu}}{x_{\rm min}\cdot \ell_2'} - \frac{\ell_{2\,\mu}}{x_{\rm min}\cdot \ell_2}\right) = \frac{a-b}{a+b} = \frac{a^2-b^2}{r_{\rm min}^2+1}\,,$$

and similarly, at a general point,

$$T_1 \cdot T_2 = \frac{a^2-b^2}{a^2+b^2+ab\,(e^{2\eta}+e^{-2\eta})} = \frac{a^2-b^2}{r^2+1}\,.$$

Analogously, from direct calculation we see that

$$R_1 \cdot T_2 = \frac{2ab\sinh(2\eta)}{r^2+1}\,.$$

The full action is obtained by symmetrizing with respect to the two worldlines, which in the current construction is equivalent to taking the real part of S_{12}. The full action thus reads

$$S = \frac{N}{16\pi^3}\int d\eta\, \frac{1}{r}\left(1 - \frac{1}{1+2k_1\cdot T_2} - \frac{1}{1+2\bar{k}_1\cdot T_2}\right)\,, \tag{4.34}$$

where

$$k_1 \cdot T_2 = \frac{1}{2}\left(T_1\cdot T_2 + \frac{i}{r} R_1 \cdot T_2\right) = \frac{1}{2}\left(\frac{a^2-b^2}{r^2+1} + \frac{i}{r}\,\frac{2ab\sinh(2\eta)}{r^2+1}\right)\,.$$

While the integral (4.34) might not be fully tractable, to gain some insight into the above expression, consider the restriction to scalar contribution

$$\tilde{S}^{(0)} = \int d\eta \, \frac{1}{r} \, . \tag{4.35}$$

Rewriting the relative radial distance (4.33) as

$$\begin{aligned} r^2 &= (a+b)^2 - 1 + 4ab \sinh^2 \eta \\ &= \big((a+b)^2 - 1\big) \cosh^2 \eta - \big((a-b)^2 - 1\big) \sinh^2 \eta \end{aligned}$$

and thus the scalar contribution to the action (4.35) becomes

$$\tilde{S}^{(0)} = \int_{-\infty}^{\infty} d\eta \, \frac{1}{\operatorname{ch}\eta \sqrt{(a+b)^2 - 1 - ((a-b)^2 - 1) \operatorname{th}^2 \eta}} \, . \tag{4.36}$$

Further, introducing integration variable $\phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ so that $\operatorname{th}\eta = \sin\phi$, $\operatorname{ch}\eta = \frac{1}{\cos\phi}$, and $\operatorname{sh}\eta = \tan\phi$, with line element $d\phi = \frac{d\eta}{\operatorname{ch}\eta}$, the action (4.36) becomes

$$\tilde{S}^{(0)} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi \, \frac{1}{\sqrt{(a+b)^2 - 1 - ((a-b)^2 - 1) \sin^2 \phi}} = \frac{2}{(a+b)^2 - 1} K\left(\frac{(a-b)^2 - 1}{(a+b)^2 - 1}\right) \tag{4.37}$$

where $K(\kappa)$ is the complete elliptic integral of the first kind

$$K(\kappa) = \int_0^{\frac{\pi}{2}} d\phi \, \frac{1}{\sqrt{1 - \kappa^2 \sin^2 \phi}} \, ,$$

noting that the integrand in (4.37) is even, and thus the integration range can be reduced to the positive half-interval.

For the full action integral (4.34), even though we have not been able to make progress analytically, as announced, we expect it to agree with the correlator of the boundary bilocals, which reads

$$S_{\text{bnd}} = -\frac{N}{2} \frac{1}{(4\pi)^2} \frac{1}{\sqrt{-2\ell_1' \cdot \ell_2} \sqrt{-2\ell_1 \cdot \ell_2'}} \, ,$$

In our parametrization (4.32), this becomes

$$S_{\text{bnd}} = -\frac{N}{32\pi^2 a} \, . \tag{4.38}$$

By numerical integration we have seen evidence that indeed the full action (4.34) agrees with the expected (4.38). In figure 4.1 we plot the difference $S - S_{\text{bnd}}$ for the range of parameters $a, b \in (0, 4)$.

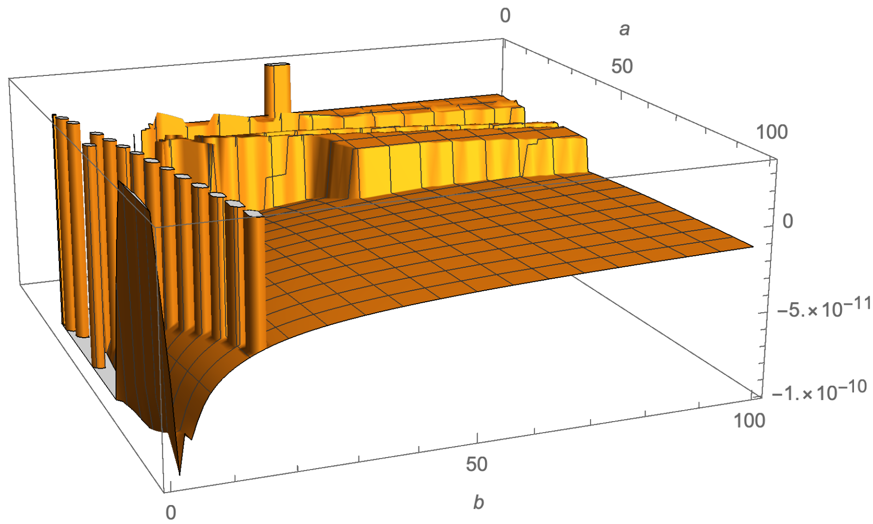


Figure 4.1: Numerical integration of $S - S_{\text{bnd}}$ for parameters in the range $a, b \in (0, 4)$, discretized over an array of size 100×100.

Conclusion

This body of work was motivated by the quest of constructing a theory of quantum gravity inside the cosmological horizons of an observer. A way forward is to consider a holographic duality between higher-spin theory in the bulk of dS_4 and a vector model on the conformal boundary.

The causal structure of dS_4 implies that the boundary is unobservable from the causal patch and it only intersects it at two points (the endpoints of the eternal observer's worldline). Thus we were led to consider variables living at these two points that could be used to construct the non-local holographic dictionary between bulk and boundary theories. These spinor-helicity variables were used to encode null data on the cosmological horizons, and hence we were able to write a simple expression for the free field “S-matrix” between two such cosmological horizons. There is hope that this could be used to construct perturbation theory for interacting fields in the causal patch.

Further, the spinor-helicity variables were used to describe the boundary theory. This led to the observation that there exists a persistent disagreement between the higher-spin-algebraic construction of the boundary partition function and the result of the result of a direct CFT calculation. A way out of this impasse is to make away with locality as the guiding principle for constructing the boundary theory and consider instead spin-locality, similar to recent developments in the higher-spin literature. This higher-spin-algebraic, spin-local construction of the boundary partition function also provides a better-behaved Hartle–Hawking wave function.

Lastly, we realized that the linearized bulk higher-spin corresponding to a bilocal boundary source are essentially the same as the linearized version of the Didenko–Vasiliev black hole; moreover these fields solve the Fronsdal field equations with a source that takes the form of a massive particle traveling along the geodesic determined by pair of boundary points of the bilocal source. If we consider two such particles, we saw that their interaction the bulk is reproduced by the boundary CFT correlator of the corresponding bilocals. All of this makes Didenko–Vasiliev black hole a potential higher-spin gravity analog of D-branes, objects that were fundamental in the development of string theory and necessary for the theory's consistency. This might raise interesting opportunities in the development of higher-spin theory.

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