

PAPER • OPEN ACCESS

Conditions for the classicality of the center of mass of many-particle quantum states

To cite this article: Xavier Oriols and Albert Benseny 2017 *New J. Phys.* **19** 063031

View the [article online](#) for updates and enhancements.

Related content

- [Quantum Mechanics: The role of Hermitian operators](#)
M Saleem
- [Quantum Mechanics: The failure of classical physics and the advent of quantum mechanics](#)
M Saleem
- [Bohmian mechanics, collapse models and the emergence of classicality](#)
Marko Toroš, Sandro Donadi and Angelo Bassi

**PAPER**

Conditions for the classicality of the center of mass of many-particle quantum states

OPEN ACCESS**RECEIVED**

14 January 2017

REVISED

9 April 2017

ACCEPTED FOR PUBLICATION

8 May 2017

PUBLISHED

26 June 2017

Original content from this work may be used under the terms of the [Creative Commons Attribution 3.0 licence](#).

Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.

Xavier Oriols¹ and Albert Benseny² ¹ Departament d'Enginyeria Electrònica, Universitat Autònoma de Barcelona, E-08193-Bellaterra (Barcelona), Spain² Quantum Systems Unit, Okinawa Institute of Science and Technology Graduate University, Onna, Okinawa 904-0495, JapanE-mail: xavier.oriols@uab.cat**Keywords:** quantum-to-classical transition, Bohmian mechanics, quantum mechanics, decoherence, open systems**Abstract**

We discuss the conditions for the classicality of quantum states with a very large number of identical particles. By defining the center of mass from a large set of Bohmian particles, we show that it follows a classical trajectory when the distribution of the Bohmian particle positions in a single experiment is always equal to the marginal distribution of the quantum state in physical space. This result can also be interpreted as a single experiment generalization of the well-known Ehrenfest theorem. We also demonstrate that the classical trajectory of the center of mass is fully compatible with a quantum (conditional) wave function solution of a classical non-linear Schrödinger equation. Our work shows clear evidence for a quantum–classical inter-theory unification, and opens new possibilities for practical quantum computations with decoherence.

1. Introduction

Since the beginning of quantum theory a century ago, the study of the frontier between classical and quantum mechanics has been a constant topic of debate [1–8]. Despite great efforts, the quantum-to-classical transition still remains blurry and certainly much more puzzling and intriguing than, for example, the frontier between classical mechanics and relativity. The relativistic equations of motion just tend to the classical ones when the velocities are much slower than the speed of light [3].

The difficulties in finding a simple explanation for the classical-to-quantum transition have their roots in the so-called *measurement problem* that requires getting rid of quantum superpositions [8–10]. Possible quantum states of a particle are represented by vectors in a Hilbert space, and linear combinations of them (such as superpositions of macroscopically distinguishable states) also correspond to valid states of the Hilbert space. However, such a superposition of states is not always compatible with measurements [10, 11]. The measurement problem can be formulated as the impossibility for a physical quantum theory (in empirical agreement with experiments) to satisfy simultaneously the following three assumptions [9]. First, the wave function always evolves deterministically according to the linear and unitary Schrödinger equation. Second, a measurement always finds the physical system in a localized state, not in a superposition of macroscopically distinguishable states. Third, the wave function is a complete description of a quantum system. Different quantum theories (interpretations) appear depending on which assumption is ignored [3].

The first type of solution argues that the unitary and linear evolution of the Schrödinger equation is not always valid. For instance, in the instantaneous collapse theories [12] (like the GRW interpretation [13]), a new stochastic equation is used that breaks the superposition principle at a macroscopic level, while still keeping it at a microscopic one. Another possibility is substituting the linear Schrödinger equation by a non-linear collapse law only when a measurement is performed [1, 14]. This is the well-known orthodox (or Copenhagen) solution, and most of the attempts to reach a quantum-to-classical transition have been developed under this last approach [4–8, 15–17].

A second type of solution ignores the assumption that a measurement always find the physical system in a localized state. One then assumes that there are different worlds where different states of the superposition are

found. This is the many worlds solution [18–20], in which the famous Schrödinger’s cat is found alive in one world and dead in another. Explanations of the quantum-to-classical transition have also been attempted within this interpretation [20].

There is a final kind of solution that assumes that the wave function alone does not provide a complete description of the quantum state, i.e. additional elements (hidden variables) are needed. The most widespread of these approaches is Bohmian mechanics [10, 23–28], where, in addition to the wave function, well-defined trajectories are needed to define a complete (Bohmian) quantum state. In a spatial superposition of two disjoint states, only the one whose support contains the position of the particle becomes relevant for the dynamics. Previous attempts to study the quantum-to-classical transition with Bohmian mechanics mainly focused on single-particle problems [28–31]. In this paper, we generalize such works by analyzing under which conditions the center of mass of a many-particle quantum system follows a classical trajectory.

The use of the center of mass for establishing the classicality of a quantum state has some promising advantages. The first one is related to the description of the initial conditions. Fixing the initial position and velocity of a classical particle seems unproblematic, while it is forbidden for a quantum particle due to the uncertainty principle [1, 14]. The use of the center of mass relaxes this contradiction: it is reasonable to expect that two experiments with the same preparation for the wave function will give quite similar values for the initial position and velocity of the center of mass when a large number of particles is considered, although the microscopic distribution of the positions and velocities for all (Bohmian) particles will be quite different in each experiment.

The second advantage is that it provides a *natural* coarse-grained definition of a classical trajectory that coexists with the underlying microscopic quantum reality. One can reasonably expect that the Bohmian trajectory of the center of mass of a large number of particles can follow a classical trajectory, without implying that each individual particle becomes classical. Therefore, the use of the center of mass allows a definition of the quantum-to-classical transition, while keeping a pure quantum behavior for each individual particle.

This article is structured as follows. We begin by studying the conditions under which the center of mass of a quantum state behaves classically. We then present a type of wave function that always fulfills these conditions, and show the equation that guides the wave function of the center of mass. Next, we discuss examples of quantum states whose center of mass does not behave classically. To finish, we summarize the main results, contextualize them within previous approaches and comment on further extensions of this work.

2. Conditions for a classical center of mass

2.1. Evolution of the center of mass in an ensemble of identical experiments

Throughout the article, we will consider a quantum system composed of N particles of mass m governed by the wave function $\Psi(\vec{r}_1, \dots, \vec{r}_N, t)$ solution of the many-particle non-relativistic Schrödinger equation,

$$i\hbar \frac{\partial \Psi}{\partial t} = \left(-\frac{\hbar^2}{2m} \sum_{i=1}^N \nabla_i^2 + V \right) \Psi, \quad (1)$$

where \vec{r}_i is the position of the i th particle, ∇_i^2 its associated Laplacian operator, and the potential $V = V(\vec{r}_1, \dots, \vec{r}_N, t)$ contains an external and an interparticle component,

$$V = \sum_{i=1}^N V_{\text{ext}}(\vec{r}_i) + \frac{1}{2} \sum_{i=1}^N \sum_{f=1; i \neq f}^N V_{\text{int}}(\vec{r}_i - \vec{r}_f). \quad (2)$$

In particular, we are interested in the evolution of one specific degree of freedom, the center of mass, defined as

$$\vec{r}_{\text{cm}} = \frac{1}{N} \sum_{i=1}^N \vec{r}_i. \quad (3)$$

Our aim in this paper is to analyze under which circumstances the observable associated with the operator \vec{r}_{cm} follows a classical trajectory in a single experiment.

We first consider an ensemble of experiments realized with the same (prepared) wave function, whose average ensemble value of the center of mass is given by

$$\langle \vec{r}_{\text{cm}} \rangle(t) = \int d^3\vec{r}_1 \dots \int d^3\vec{r}_N |\Psi(\vec{r}_1, \dots, \vec{r}_N, t)|^2 \vec{r}_{\text{cm}}. \quad (4)$$

From Ehrenfest’s theorem [32], it is well-known that the time derivative of $\langle \vec{r}_{\text{cm}} \rangle$ is

$$\frac{d\langle \vec{r}_{\text{cm}} \rangle}{dt} = \frac{1}{N} \sum_{i=1}^N \frac{d\langle \vec{r}_i \rangle}{dt} = \frac{1}{N} \sum_{i=1}^N \langle \vec{p}_i \rangle = \langle \vec{p}_{\text{cm}} \rangle. \quad (5)$$

We can follow the same procedure for the time derivative of the momentum of the center of mass,

$$\frac{d\langle\vec{p}_{\text{cm}}\rangle}{dt} = \frac{1}{N} \sum_{i=1}^N \frac{d\langle\vec{p}_i\rangle}{dt} = -\frac{1}{N} \sum_{i=1}^N \langle\nabla_i V_{\text{ext}}(\vec{r}_i)\rangle. \quad (6)$$

When the spatial extent of the many-particle wave function is much smaller than the variation length-scale of the potential, we can assume $\langle\nabla_i V_{\text{ext}}(\vec{r}_i)\rangle = \nabla V_{\text{ext}}(\langle\vec{r}_{\text{cm}}\rangle)$, and write

$$\frac{d^2\langle\vec{r}_{\text{cm}}\rangle}{dt^2} = -\nabla V_{\text{ext}}(\langle\vec{r}_{\text{cm}}\rangle). \quad (7)$$

This classical behavior of the average $\langle\vec{r}_{\text{cm}}\rangle$ is a very well-known result [1, 2, 32]. The types of V_{ext} that satisfy the condition $\langle\nabla_i V_{\text{ext}}(\vec{r}_i)\rangle = \nabla V_{\text{ext}}(\langle\vec{r}_{\text{cm}}\rangle)$ will be discussed further later.

2.2. Evolution of the center of mass in a single experiment

In order to satisfy our classical intuition, we need to certify that the observable associated with \vec{r}_{cm} follows a classical trajectory in each experiment (not in an average over several experiments). This problem could be analyzed within the orthodox formalism [4, 5, 7, 15–17, 33]. The typical approach would be to construct a reduced density matrix of the center of mass by tracing out the rest of the degrees of freedom interpreted as the environment. The effect of decoherence, i.e. the entanglement between the environment and the system, then leads to a diagonal (or nearly diagonal) density matrix. Finally, after invoking the collapse law, one obtains the observable result for the operator \vec{r}_{cm} by selecting one element of the diagonal at each measuring time. In this work, however, we will approach the problem using Bohmian mechanics [23, 25–28]. This alternative formalism will allow us to reach the quantum-to-classical transitions without dealing with the reduced density matrix and without specifying the collapse law (this law is not needed in the Bohmian postulates [25, 26, 28]).

As indicated in the introduction, in Bohmian mechanics, a quantum state is completely described by two elements: the many-particle wave function $\Psi(\vec{r}_1, \dots, \vec{r}_N, t)$, solution of the usual Schrödinger equation, and the trajectory $\{\vec{r}_i^j(t)\}$ of each $i = 1 \dots N$ particle. Hereafter, each Bohmian quantum state will refer to a wave function and to a particular set of trajectories labeled by the superindex j that correspond to a single experiment. The velocity of each particle is given by

$$\vec{v}_i^j(t) = \frac{d\vec{r}_i^j(t)}{dt} = \frac{\vec{J}_i(\vec{r}_1^j(t), \dots, \vec{r}_N^j(t), t)}{|\Psi(\vec{r}_1(t), \dots, \vec{r}_N(t), t)|^2}, \quad (8)$$

where $\vec{J}_i = \hbar \text{Im}(\Psi^* \nabla_i \Psi) / m$. Thus, the configuration of particles reproduces all quantum features while evolving ‘choreographed’ by the wave function [26–28, 34, 35].

By construction, Bohmian predictions are as uncertain as the orthodox ones [36]: it is not possible to know the initial positions in a particular experiment (unless the wave function is a position eigenstate). The best we can know about the particle positions in the j -experiment, $\{\vec{r}_i^j(t)\}$, is that they are found in locations where the wave function has a reasonable presence probability. In particular, the set of positions in M different experiments (prepared with the same wave function) are distributed according to

$$|\Psi(\vec{r}_1, \dots, \vec{r}_N, t)|^2 = \lim_{M \rightarrow \infty} \frac{1}{M} \sum_{j=1}^M \prod_{i=1}^N \delta(\vec{r}_i - \vec{r}_i^j(t)). \quad (9)$$

If the set of N positions follows this distribution at some time $t=0$, it is easy to demonstrate that (9) will also be satisfied at any other time t , provided that the many-particle wave function evolves according to (1) and that the particles move according to (8). This property is known as equivariance [39] and it is key for the empirical equivalence between Bohmian mechanics and other quantum theories. Equation (9) says that Born’s law is always satisfied by counting particles [23, 25, 27, 28] and that the quantum results are unpredictable [36]. Several authors assume that, as a postulate of the Bohmian theory, the initial configuration of the particles satisfies (9), while others argue that it is just a consequence of being in a ‘typical’ Universe [21, 39]³.

³ In principle, one could postulate (9) (at some initial time) in the Bohmian theory in the same way that Born’s law is a postulate in the orthodox theory. However, some authors argue that this is not necessary [21]. Probably the most accepted view against taking (9) as a postulate comes from the seminal work by Dürr *et al* [39], where the equivariance in any system is discussed from the initial configurations of (Bohmian) particles in the Universe. Using Bohmian mechanics to describe the wave function of the whole Universe, then the wave function associated with any (sub)system is an effective (conditional) wave function of the universal one. Using typicality arguments, Dürr *et al* showed that the overwhelming majority of the possible selections of initial positions of particles in the Universe will satisfy condition (9) in a subsystem [39]. Other authors [22] have attempted to dismiss (9) as a postulate by showing that any initial configuration of Bohmian particles will relax, after some time, to a distribution very close to (9) for a subsystem.

After selecting one initial position of the particles from (9) in a single j -experiment, we can then define the trajectory for the center of mass of the Bohmian quantum state as

$$\vec{r}_{\text{cm}}^j(t) = \frac{1}{N} \sum_{i=1}^N \vec{r}_i^j(t). \quad (10)$$

As discussed above, in general $\vec{r}_{\text{cm}}^j(t) \neq \vec{r}_{\text{cm}}^h(t)$ for any two different experiments j and h because the Bohmian positions have an intrinsic uncertainty coming from (9).

2.3. Classical center of mass in a single experiment

A classical trajectory for the center of mass $\vec{r}_{\text{cm}}^j(t)$ of a quantum state in a single experiment is obtained when the following two conditions are satisfied:

- **Condition 1**—For the overwhelming majority of experiments associated with the same wave function, the same trajectory for the center of mass is obtained. That is to say, for (almost) any two different experiments, j and h , we obtain $\vec{r}_{\text{cm}}^j(t) = \vec{r}_{\text{cm}}^h(t)$.
- **Condition 2**—The spatial extent of the (many-particle) wave function in each direction is much smaller than the variation length-scale of the external potential V_{ext} .

According to condition 1, since $\vec{r}_{\text{cm}}^j(t) = \vec{r}_{\text{cm}}^{j_0}(t)$ for all M experiments, the empirical evaluation of $\langle \vec{r}_{\text{cm}} \rangle$ (that we know is classical if condition 2 is also satisfied) will be equal to the trajectory of the center of mass $\vec{r}_{\text{cm}}^{j_0}(t)$ in a single experiment:

$$\langle \vec{r}_{\text{cm}} \rangle(t) = \lim_{M \rightarrow \infty} \frac{1}{M} \sum_{j=1}^M \vec{r}_{\text{cm}}^j(t) = \vec{r}_{\text{cm}}^{j_0}(t). \quad (11)$$

Moreover, we notice that $\vec{r}_{\text{cm}}^j(t)$ in such a quantum state has the same well-defined initial conditions (position and velocity) as in the overwhelming majority of experiments. While condition 1 might seem very restrictive, we will show, in what follows, that quantum states that satisfy it are more natural than expected when the number of particles is very large.

A better understanding of condition 2 can be found from a Taylor expansion of the external potential $V_{\text{ext}}(\vec{r}_i)$ in (6). One can easily realize that the condition $\langle \nabla V_{\text{ext}}(\vec{r}_i) \rangle = \nabla V_{\text{ext}}(\langle \vec{r}_i \rangle)$ is directly satisfied by constant, linear or quadratic potentials. Where V_{ext} can be approximated by potentials with such dependence requires a discussion of its physical meaning. $V_{\text{ext}}(\vec{r}_i)$ in (2) describes the interaction of particle i with some distant ‘source’ particles located elsewhere. Moreover, the fact that this potential is felt identically by all N system particles (i.e. $V_{\text{ext}}(\vec{r}_i)$ is a single particle potential) is due to the large distance between our system and the potential sources. We can then assume that V_{ext} is generated by some kind of long-range force, such as an electromagnetic or gravitational one. Such external long-range potentials will usually have a small spatial variation along the support of $\Psi(\vec{r}_1, \dots, \vec{r}_N, t)$, and a linear or quadratic approximation for V_{ext} would seem enough in most macroscopic scenarios. In any case, scenarios where higher orders of the series expansion of V_{ext} are relevant are possible in the laboratory. Then, if condition 1 is applicable, it will guarantee a unique trajectory $\langle \vec{r}_{\text{cm}} \rangle(t) = \vec{r}_{\text{cm}}^{j_0}(t)$ in all experiments with well-defined initial conditions; however, its acceleration will not only be given by the gradient of V_{ext} , but it will also depend on the wave function.

3. Quantum states with a classical center of mass

3.1. A quantum state full of identical particles

We define here a type of quantum state with a very large number of indistinguishable particles (either fermions or bosons) that we name *quantum state full of identical particles*. We will show that the center of mass of these states always follows a classical trajectory. Our definition will revolve around the concept of marginal probability distribution, i.e. the spatial distribution for the i th particle independently of the position of the rest of the particles,

$$D(\vec{r}_i, t) = \int \dots \int |\Psi(\vec{r}_1, \dots, \vec{r}_N, t)|^2 \prod_{f=1; f \neq i}^N d^3\vec{r}_f. \quad (12)$$

Empirically, this distribution can be calculated from a very large number M of experiments as

$$D(\vec{r}_i, t) = \lim_{M \rightarrow \infty} \frac{1}{M} \sum_{j=1}^M \delta(\vec{r}_i - \vec{r}_i^j(t)). \quad (13)$$

Since our definition of a quantum state full of identical particles always involves indistinguishable particles, the subindex i is superfluous, and all particles will have the same marginal distribution. We notice that, while all Bohmian particles $\vec{r}_i(t)$ are ontologically distinguishable (through the index i), the Bohmian dynamical laws, equations (1) and (8), ensure that they are empirically indistinguishable⁴.

We define a quantum state full of identical particles as a state whose distribution of the positions of the N particles in a single experiment is always equal to the marginal distribution of a single variable obtained from averaging over different experiments,

$$D(\vec{r}, t) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \delta(\vec{r} - \vec{r}_i^{j_0}(t)) = \lim_{M \rightarrow \infty} \frac{1}{M} \sum_{j=1}^M \delta(\vec{r} - \vec{r}_{i_0}^j(t)). \quad (14)$$

For the practical application of this definition in systems with a finite (but very large) number of particles, one can impose that the condition in (14) has to be satisfied for the overwhelming majority of experiments (see appendix A).

The selection of the initial position of the particles, $\vec{r}_1^{j_0}(0), \vec{r}_2^{j_0}(0) \dots \vec{r}_N^{j_0}(0)$, in a single experiment (labeled here j_0) can be done from (9). One would start by first selecting $\vec{r}_1^{j_0}(0)$ (independently of the rest of the positions). Then, selecting $\vec{r}_2^{j_0}(0)$ conditioned to the fact the $\vec{r}_1^{j_0}(0)$ is already selected. This procedure is repeated until the last position is chosen, $\vec{r}_N^{j_0}(0)$, conditioned to all previous selected positions. The probability distribution for selecting $\vec{r}_i^{j_0}(0)$, when the previous positions $\vec{r}_1^{j_0}(0), \dots, \vec{r}_{i-1}^{j_0}(0)$ are already chosen, can be defined from a combination of conditional and marginal probabilities as:

$$D^{j_0,i}(\vec{r}_i, 0) = \frac{\bar{D}^i(\vec{r}_1^{j_0}(0), \dots, \vec{r}_{i-1}^{j_0}(0), \vec{r}_i, 0)}{\int \bar{D}^i(\vec{r}_1^{j_0}(0), \dots, \vec{r}_{i-1}^{j_0}(0), \vec{r}_i, 0) d\vec{r}_i} \quad (15)$$

with

$$\bar{D}^i(\vec{r}_1, \dots, \vec{r}_i, 0) = \int \dots \int |\Psi(\vec{r}_1, \dots, \vec{r}_i, \dots, \vec{r}_N, 0)|^2 d^3\vec{r}_{i+1} \dots d^3\vec{r}_N. \quad (16)$$

By construction, the probability distribution function in (14) has a total probability equal to unity. On the contrary, a normalization constant is explicitly included in the definition of (15) to ensure that it is a probability distribution function properly normalized to unity. In particular, for any j_0 -experiment, we get $D^{j_0,1}(\vec{r}_1, 0) \equiv D(\vec{r}_1, 0)$ and $D^{j_0,N}(\vec{r}_N, 0) \equiv |\Psi(\vec{r}_1^{j_0}(0), \dots, \vec{r}_i^{j_0}(0), \dots, \vec{r}_{N-1}^{j_0}(0), \vec{r}_N, 0)|^2$. Therefore, a quantum state full of identical particles can be alternatively defined as the wave function satisfying the global distribution of the $i = 1, \dots, N$ particles in a single j_0 -experiment constructed from (15) and (16), and is equal to $D(\vec{r}, 0)$ in (12) for the overwhelming majority of experiments. A trivial example of a quantum state full of identical particles is one where the corresponding distribution for selecting the $i = 1, \dots, N$ particles in the overwhelming majority of experiments satisfies $D^{j_0,i}(\vec{r}_i, 0) = D(\vec{r}_i, 0)$.

The equivalence between both expressions in (14) implies the equivalence between two sets of positions: first, the positions of particle i_0 in M different experiments, $\{\vec{r}_{i_0}^j(t)\}$ for $j = 1, \dots, M$, and, second, the positions of the N particles in the same j_0 -experiment, $\{\vec{r}_i^{j_0}(t)\}$ for $i = 1, \dots, N$. Because of this equivalence, a position in the first set, say $\vec{r}_{i_0}^j(t)$, is equal to another position in the second set, $\vec{r}_{i_0}^{j_0}(t)$. Any position of one set has another identical position in the other set. Therefore, since the exchange of positions of identical particles does not exchange their velocity [37], we obtain that $\vec{v}_{i_0}^j = \vec{v}_{i_0}^{j_0}$, which implies that $\vec{r}_{i_0}^j(t) = \vec{r}_{i_0}^{j_0}(t)$ at any time. Therefore, we conclude that if (14) is satisfied at a particular time, such as $t = 0$, then the quantum state will be full of identical particles at any other time.

At this point, using (14) for any time t , we can certify that the trajectory of the center of mass of a quantum state full of identical particles satisfies

⁴ The empirical indistinguishability of the Bohmian trajectories means that the \vec{r}_2 -observable computed from $\vec{r}_2^j(t)$ is identical to the \vec{r}_1 -observable computed from $\vec{r}_1^j(t)$. This property can be easily understood from the symmetry of the wave function (see also [25, 28, 37]). Consider a set of trajectories $\{\vec{r}_1^j(t), \vec{r}_2^j(t), \dots, \vec{r}_N^j(t)\}$ assigned to an experiment j . We construct another set of trajectories $\{\vec{r}_1^h(t), \vec{r}_2^h(t), \dots, \vec{r}_N^h(t)\}$ whose initial conditions are $\vec{r}_1^h(0) = \vec{r}_2^j(0)$ and $\vec{r}_2^h(0) = \vec{r}_1^j(0)$, while $\vec{r}_i^h(0) = \vec{r}_i^j(0)$ for $i = 3, \dots, N$. Due to the symmetry of the wave function (and of velocity (8)), $\vec{r}_1^h(t) = \vec{r}_2^j(t)$ and $\vec{r}_2^h(t) = \vec{r}_1^j(t)$ (the rest of the trajectories are identical in j and h). Any observable related to \vec{r}_1 (or \vec{r}_2) is evaluated over an ensemble of different experiments. For each j -element of the ensemble, we can construct its corresponding h -set of trajectories and evaluate the \vec{r}_2 -observable using $\vec{r}_2^h(t)$ instead of $\vec{r}_2^j(t)$. By construction, since $\vec{r}_2^h(t) = \vec{r}_1^j(t)$, the \vec{r}_2 -observable is identical to the \vec{r}_1 -observable.

$$\begin{aligned}
\vec{r}_{\text{cm}}^{j_0}(t) &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \vec{r}_i^{j_0}(t) = \lim_{M \rightarrow \infty} \frac{1}{M} \sum_{j=1}^M \vec{r}_{i_0}^j(t) \\
&= \lim_{N, M \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \frac{1}{M} \sum_{j=1}^M \vec{r}_i^j(t) = \lim_{N, M \rightarrow \infty} \frac{1}{M} \sum_{j=1}^M \frac{1}{N} \sum_{i=1}^N \vec{r}_i^j(t) \\
&= \lim_{M \rightarrow \infty} \frac{1}{M} \sum_{j=1}^M \vec{r}_{\text{cm}}^j = \langle \vec{r}_{\text{cm}} \rangle(t),
\end{aligned} \tag{17}$$

where we have used

$$\vec{r}_{\text{cm}}^{j_0}(t) = \int \vec{r} D(\vec{r}, t) d\vec{r} \tag{18}$$

with $D(\vec{r}, t)$ given by any of the two expressions in (14). In summary, a quantum state full of identical particles satisfies condition 1, and, if condition 2 also holds, its center of mass will be a classical trajectory.

The arguments we have presented here are for a system of indistinguishable particles. For a macroscopic object composed of several types of particles, we can apply the same reasoning and obtain a classical center of mass for each type of particle subsystem, such that the global center of mass is also classical.

3.2. Example 1: a quantum state with all particles sharing the same single-particle wave function

Here we show the simplest example of a quantum state full of identical particles. We consider an N -particle wave function given by

$$\Psi(\vec{r}_1, \dots, \vec{r}_N, t) = \prod_{j=1}^N \psi(\vec{r}_j, t). \tag{19}$$

It corresponds, for example, to a system of non-interacting bosons, all with the same single-particle wave function $\psi(\vec{r}, t)$ solution of a single-particle Schrödinger equation under the external potential $V_{\text{ext}}(\vec{r})$.

The quantum state in the j -experiment is completed with the set of trajectories $\{\vec{r}_i^j(t)\}$ for $i = 1, \dots, N$ selected according to $|\Psi|^2$. Since (19) corresponds to a separable system, each position $\vec{r}_i^j(0)$ has to be selected according to its own probability distribution in (15) and (16) with $D^{j_0, i}(\vec{r}_i, 0) = |\psi(\vec{r}_i, 0)|^2$. The marginal distribution in (12) satisfies $D(\vec{r}_i, 0) = |\psi(\vec{r}_i, 0)|^2$, which is exactly the same distribution mentioned above for selecting the particles. Therefore, this quantum state trivially satisfies (14) when $N \rightarrow \infty$, i.e. $D^{j_0, i}(\vec{r}, 0) = D(\vec{r}, 0)$. As a result, the (Bohmian) trajectory of the center of mass will follow a classical trajectory when condition 2 about V_{ext} is also satisfied.

Numerical example

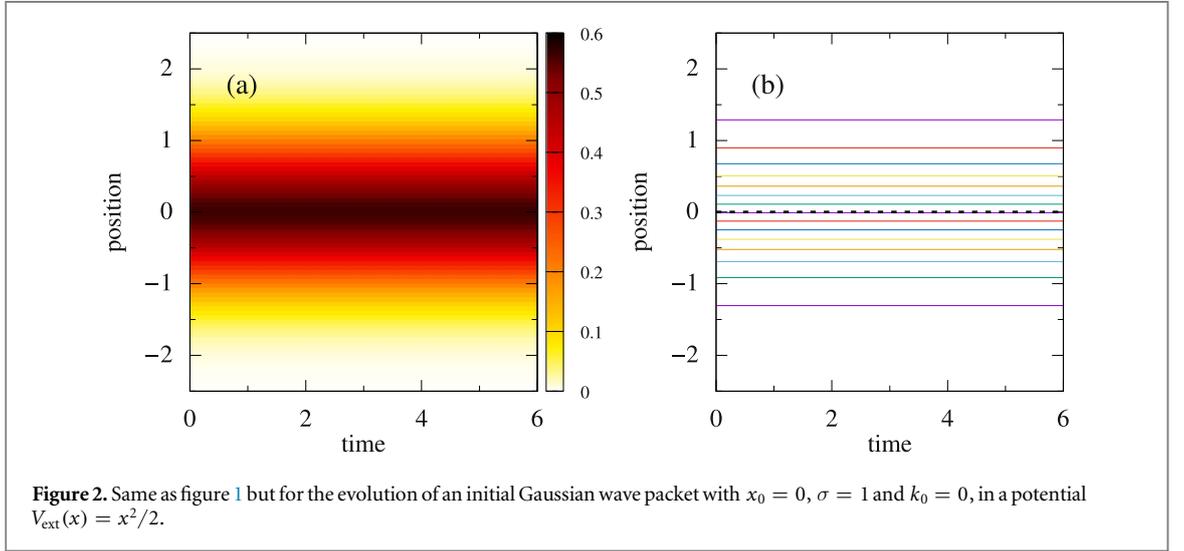
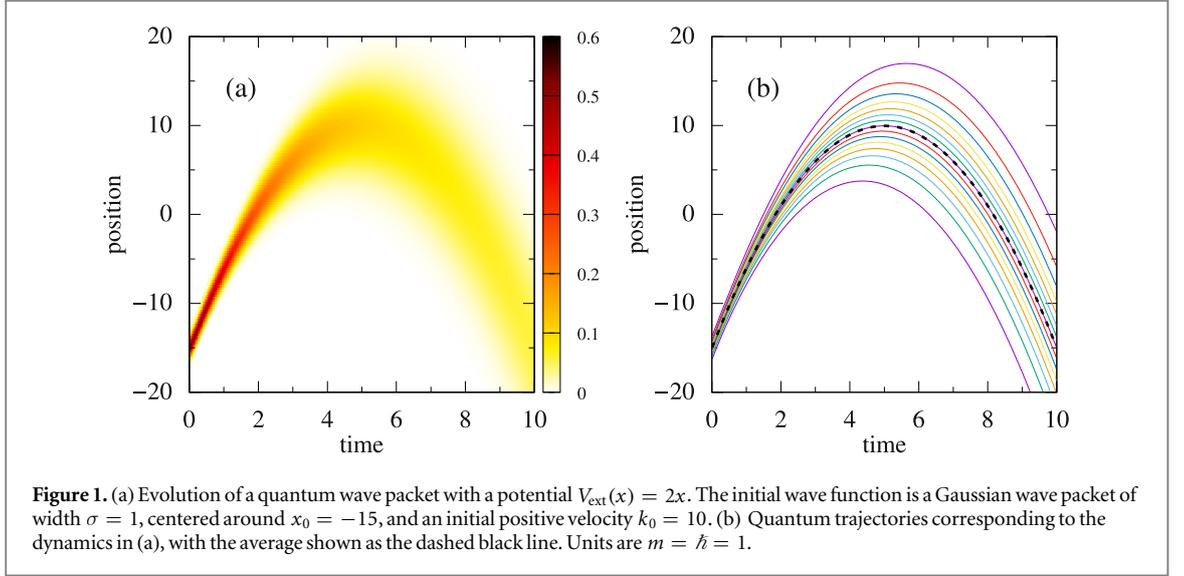
For simplicity, we consider a 1D physical space to numerically test the properties of the above state. As the initial single-particle wave function we select a wave packet of the form

$$\psi(x, 0) = \frac{1}{\sqrt{\sigma\sqrt{\pi}}} \exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right) \exp(ik_0x), \tag{20}$$

with σ the dispersion of the wave-packet, x_0 the initial position and k_0 the initial momentum. Then, since the particles are independently selected, the central limit theorem [38] ensures that the center of mass of the quantum state will be normally distributed with a dispersion $\sigma_{\text{cm}} = \sigma/\sqrt{N} \rightarrow 0$, confirming that the center of mass has the same well-defined position in all the experiments (see appendix A).

In the first example in figure 1 we use a linear potential $V_{\text{ext}}(x) = 2x$ emulating a particle in free fall under a gravity force. The quantum wave packet increases its width over time and its center follows a typical parabolic movement. The second example in figure 2 corresponds to a harmonic potential $V_{\text{ext}}(x) = x^2/2$. In this case, because the wave function corresponds to the ground state of the quantum harmonic oscillator, it does not show any dynamics and the trajectories remain static at their initial positions. In any case, the center of mass (dashed black line in figure 2) corresponds to the classical trajectory at the position of the minimum of the harmonic potential with zero velocity.

Now, we confirm the classicality of the center of mass of a quantum state defined by (19) using simpler arguments. Since there is no correlation between different trajectories $x_i^j(t)$, the Bohmian trajectories plotted in figures 1 and 2 can be interpreted in two different ways. The first interpretation is the one explained above where they correspond to different $i = 1, \dots, N$ trajectories in the same experiment described by the many-particle wave function given by (19). In this case, the average value of the trajectories (dashed black lines in figures 1(b) and 2(b)) is understood as the trajectory for the center of mass in that particular experiment. The second interpretation is that the trajectories correspond to different experiments of a single particle system defined by the wave function $\psi(x, t)$. In this interpretation, $\langle x_{\text{cm}} \rangle$ corresponds to a classical trajectory (for large enough N



and V_{ext} satisfying condition 2), as shown by Ehrenfest's theorem [32] discussed in section 2.1. Since the trajectories in both interpretations are mathematically identical, we conclude that the (Bohmian) trajectory of the center of mass in a single experiment follows a classical trajectory $x_{\text{cm}}^j(t) = \langle x_{\text{cm}} \rangle$, as anticipated in the discussion above on how these quantum states satisfy the condition in (14), i.e. $D^{j,i}(x, 0) = D(x, 0)$.

3.3. Example 2: a many-particle quantum state with exchange and inter-particle interactions

In the following, we consider a more general example of a quantum state full of identical particles with exchange and inter-particle interactions. We consider here a quantum wave function Ψ which, at time $t = 0$, is built from permutations of N single-particle wave functions, $\psi_i(\vec{r}, 0)$. We define $\Psi(\vec{r}_1, \dots, \vec{r}_N, 0)$ as

$$\Psi(\vec{r}_1, \dots, \vec{r}_N, 0) = \sum_{\vec{p} \in S_N} \prod_{i=1}^N \psi_{p_i}(\vec{r}_i, 0) s_{\vec{p}}, \quad (21)$$

where $\vec{p} = \{p_1, p_2, \dots, p_N\}$ is an element of the set S_N of $N!$ permutations of N elements. The term $s_{\vec{p}} = \pm 1$ is the sign of the permutation for fermions, while $s_{\vec{p}} = 1$ is for bosons. A global normalization constant has been omitted because it will be irrelevant. In particular, we consider that the single-particle wave functions $\psi_i(\vec{r}, 0)$ and $\psi_j(\vec{r}, 0)$ are either identical or without spatial overlapping. For any \vec{r} and $\psi_f(\vec{r}, 0)$, we have:

$$\begin{aligned} \psi_f(\vec{r}, 0) &= \psi_i(\vec{r}, 0) & \forall f \in N_i, \\ \psi_f(\vec{r}, 0) \psi_i(\vec{r}, 0) &\simeq 0 & \forall f \notin N_i, \end{aligned} \quad (22)$$

where N_i is the subset of wave functions identical to $\psi_i(\vec{r}, 0)$. We now check if the quantum state defined by equations (21) and (22) is a quantum state full of identical particles. The initial modulus squared of the wave

function in (21) can be written as

$$|\Psi|^2 = \sum_{\vec{p}, \vec{p}' \in S_N} \prod_{i=1}^N \psi_{p_i}(\vec{r}_i, 0) \psi_{p'_i}^*(\vec{r}_i, 0) s_{\vec{p}} s_{\vec{p}'}, \quad (23)$$

and the marginal distribution for each particle is then given from (12) as

$$D(\vec{r}, 0) = \sum_{\vec{p}, \vec{p}' \in S_N} \psi_{p_1}(\vec{r}, 0) \psi_{p'_1}^*(\vec{r}, 0) \prod_{i=2}^N d_{p_i, p'_i} s_{\vec{p}} s_{\vec{p}'}, \quad (24)$$

with the matrix element $d_{i,f}$ defined as

$$d_{i,f} = \int \psi_i(\vec{r}, 0) \psi_f^*(\vec{r}, 0) d^3\vec{r}. \quad (25)$$

Because of (22), $d_{i,f} = 1$ for all $f \in N_i$ and $d_{i,f} \simeq 0$ for $f \notin N_i$. Then, only the summands in (24) with all the terms $d_{i,f} = 1$ are different from zero, and we can rewrite $D(\vec{r}, 0)$ as

$$D(\vec{r}, 0) = \alpha \sum_{i=1}^N |\psi_i(\vec{r}, 0)|^2. \quad (26)$$

where α is the product of the number of permutations of each N_i to provide a properly normalized distribution in (14).

On the other hand, the selection of the N positions in a single experiment $\{\vec{r}_i^j(0)\}$ has to satisfy (9). The selection of the first particle $\vec{r}_1^j(0)$ (independently of all other particles) is given by (26). To select the second particle $\vec{r}_2^j(0)$, one needs to take into account the already selected $\vec{r}_1^j(0)$. In general, according to definitions (15) and (16) and using (23)–(25), the selection of the position $\vec{r}_m^j(0)$ as a function of the previous $m - 1$ positions $\vec{r}_1^j(0), \dots, \vec{r}_{m-1}^j(0)$ is given by the distribution

$$D^{j,m}(\vec{r}, 0) = \sum_{\vec{p}, \vec{p}' \in S_N} \left(\prod_{k=1}^{m-1} w_{k,p_k,p'_k}^j \right) \psi_{p_m}(\vec{r}, 0) \psi_{p'_m}^*(\vec{r}, 0) \left(\prod_{i=m+1}^N d_{p_i, p'_i} \right) s_{\vec{p}} s_{\vec{p}'}, \quad (27)$$

with the matrix element w_{k,p_k,p'_k}^j defined as

$$w_{k,p_k,p'_k}^j = \psi_{p_k}(\vec{r}_k^j(0), 0) \psi_{p'_k}^*(\vec{r}_k^j(0), 0). \quad (28)$$

For each position $\vec{r}_k^j(0)$, because of (22), there is a N_i set of wave functions whose value is $w_{k,i,f}^j = |\psi_i(\vec{r}_k^j(0), 0)|^2$ for any $f \in N_i$, and $w_{k,i,f}^j \simeq 0$ for any $f \notin N_i$. Again, we can assume that only the summands with the products $w_{k,i,f}^j = |\psi_i(\vec{r}_k^j(0), 0)|^2$ and $d_{i,f} = 1$ will remain different from zero in (27) giving $\psi_i(\vec{r}, 0) \psi_f^*(\vec{r}, 0) = |\psi_i(\vec{r}, 0)|^2$. We can then rewrite $D^{j,m}(\vec{r}, 0)$ as

$$D^{j,m}(\vec{r}, 0) = \beta_m \left(\sum_{i=1}^N |\psi_i(\vec{r}, 0)|^2 \right), \quad (29)$$

$$\beta_m = \alpha \sum_{\vec{p} \in S_{M-1}} \prod_{k=1}^{m-1} |\psi_{p_k}(\vec{r}_k^j(0), 0)|^2. \quad (30)$$

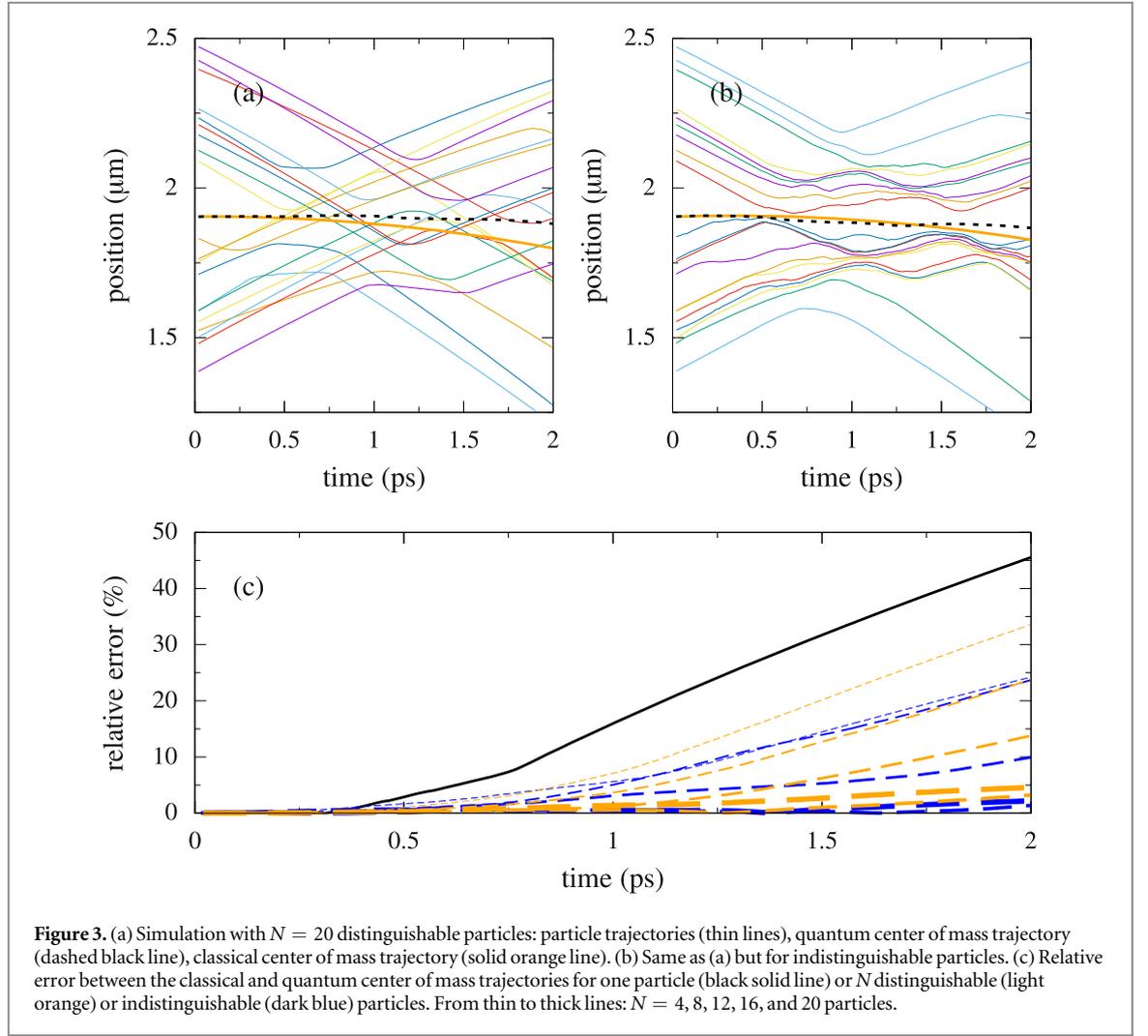
Again, the parameter β_m is irrelevant because the selection of the particles can be done through an expression of $D^{j,m}(\vec{r}, 0)$ properly normalized to unity, where only the dependence on \vec{r} matters.

In summary, for the quantum state defined by equations (21) and (22) plus a set of trajectories $\{\vec{r}_i^j(0)\}$, we conclude that the (normalized versions of the) distributions $D(\vec{r}, 0)$ in (26) and $D^{j,m}(\vec{r}, 0)$ in (29) for any m are identical. Therefore we are dealing with a quantum state full of identical particles whose center of mass follows a classical trajectory.

As we have demonstrated in section 3.1, whether $\Psi(\vec{r}_1, \dots, \vec{r}_N, t)$ fulfills the condition in (14) or not has to be tested at a single time. Since we have shown that (21) is a quantum state full of identical particles at $t = 0$, we conclude that any quantum state with the wave function $\Psi(\vec{r}_1, \dots, \vec{r}_N, t)$ solution of the many-particle Schrödinger equation in (1), with or without external V_{ext} or inter-particle V_{int} potentials, and with the initial state defined by equations (21) and (22), is a quantum state full of identical particles when $N \rightarrow \infty$.

Numerical example

In what follows, we investigate numerically this system. We will show that the center of mass of the quantum state effectively tends to a classical result, even for quite a small number of particles. The evolution of the initial wave function in (21) in the limit of $N \rightarrow \infty$ is numerically intractable. We will consider here a finite number of non-interacting bosons in a 1D space and test if the center of mass tends to a classical trajectory when N increases. Each single-particle wave function $\psi_i(x_i, t)$ is a solution of a single-particle Schrödinger equation



under the potential V_{ext} . Therefore, the bosonic many-particle wave function can be written at any time t as

$$\Psi(x_1, \dots, x_N, t) = \sum_{\vec{p} \in S_N} \prod_{i=1}^N \psi_{p_i}(x_i, t). \quad (31)$$

For comparison, we also consider the same state in (31), but without exchange interaction

$$\Psi(x_1, \dots, x_N, t) = \prod_{i=1}^N \psi_i(x_i, t). \quad (32)$$

In particular, we will consider each of the ψ_i in equations (31) and (32) as a sum of two initially separated Gaussian wave packets, but with opposite central momenta to ensure that they impinge at a later time

$$\psi_i(x_j, 0) = \frac{\exp(ik_{iL}x_j)}{2(\pi\sigma^2)^{1/4}} \exp\left(-\frac{(x_j - x_{iL})^2}{2\sigma^2}\right) + \frac{\exp(ik_{iR}x_j)}{2(\pi\sigma^2)^{1/4}} \exp\left(-\frac{(x_j - x_{iR})^2}{2\sigma^2}\right), \quad (33)$$

The x_{iL} and x_{iR} are the centers of two (non-overlapping) Gaussian wave packets, with respective momenta k_{iL} and k_{iR} , and spatial dispersion $\sigma = 15$ nm. Each of the wave functions have different random values for x_{iL} , x_{iR} , k_{iL} , and k_{iR} . These wave functions evolve using the Schrödinger equation with an external potential V_{ext} implying a constant electric field of 3.3×10^5 Vm $^{-1}$.

We show in figures 3(a) and (b), for the cases with and without exchange interaction, the evolution of the quantum trajectories (thin lines). We plot their quantum center of mass (dashed black line) computed from (10) for $N = 20$. We also plot the classical center of mass (solid orange line), computed from a Newtonian trajectory with the same initial position and velocity as the previous quantum center of mass. We notice that the Bohmian trajectories for states with exchange interaction do not cross in the physical space. This is a well-known property [37] that obviously remains valid, even if the center of mass becomes classical.

Moreover, in figure 3(c) we show the difference between the quantum and classical centers of mass for different values of N , with and without exchange interaction. (See appendix A for a discussion of the error of a quantum state full of identical particles when a large, but finite, number of particles is considered.) We see that the quantum center of mass $x_{\text{cm}}(t)$ becomes more and more classical as N grows, and the indistinguishable case reduces the quantum non-classical effects faster than the case without exchange interaction. These results can be interpreted in a simple way: a single experiment with N distinguishable particles represents effectively only one experiment, while a single experiment with N indistinguishable particles represents, in fact, $N!$ different experiments, each one with the initial (Bohmian) positions interchanged. This explains why the latter center of mass become more similar to that given by the Ehrenfest theorem which involves an infinite number of experiments.

3.4. Wave equation for the center of mass

While the description of a classical state requires only a trajectory, a complete Bohmian quantum state requires a wave function plus trajectories. Moreover, because of its exponential complexity, solutions to the Schrödinger equation in the whole many-particle configuration space are not accessible. However, an equation describing the evolution of a wave function associated with the center of mass of a quantum state full of identical particles will help to certify that a classical center of mass behavior is fully compatible with a *pure* quantum state. In addition, such an equation will provide an accessible numerical framework to analyze practical quantum systems under decoherence. One route towards this equation could be obtained from the reduced density matrix of the center of mass, and assuming some kind of collapse. Alternatively, as mentioned throughout the paper, we will follow a Bohmian procedure which allows the construction of such a wave equation for the center of mass through the use of the (Bohmian) conditional wave function [39–41].

To simplify the derivations, in the following we restrict ourselves to a 1D physical space. We define the center of mass of our N -particle state, x_{cm} , and a set of relative coordinates, $\vec{y} = \{y_2, \dots, y_N\}$, as

$$x_{\text{cm}} = \frac{1}{N} \sum_{i=1}^N x_i, \quad (34)$$

$$y_j = x_j - \frac{\sqrt{N}x_{\text{cm}} + x_1}{\sqrt{N} + 1}. \quad (35)$$

With these substitutions, the 1D version of the Schrödinger equation (see equation (1)) can be rewritten as

$$i\hbar \frac{\partial \Psi}{\partial t} = \left(-\frac{\hbar^2}{2M_{\text{cm}}} \frac{\partial^2}{\partial x_{\text{cm}}^2} - \frac{\hbar^2}{2m} \sum_{i=2}^N \frac{\partial^2}{\partial y_i^2} + V \right) \Psi, \quad (36)$$

where $M_{\text{cm}} \equiv Nm$ and $\Psi \equiv \Psi(x_{\text{cm}}, \vec{y}, t)$ is the many-particle wave function with the new coordinates. The coordinates \vec{y} in (34) are chosen such that no crossed terms appear in the Laplacian of (36) (see appendix B). Notice that the many-particle Schrödinger equation in (36) is, in general, non separable because of the potential V defined in (2), see also (B.11).

Hereafter, we derive the wave equation associated with the conditional wave function for the center of mass [39–41] defined as $\psi_{\text{cd}}(x_{\text{cm}}, t) \equiv \Psi(x_{\text{cm}}, \vec{y}^j(t), t)$ associated with the j -experiment. By construction, the velocity (and therefore the trajectory) of the center of mass only depends on the spatial derivatives along x_{cm} [39, 41]. Therefore, $x_{\text{cm}}^j(t)$ can be equivalently computed from either ψ_{cd} or Ψ . Following [40], (36) can be written in the conditional form as

$$i\hbar \frac{\partial \psi_{\text{cd}}}{\partial t} = -\frac{\hbar^2}{2M_{\text{cm}}} \frac{\partial^2 \psi_{\text{cd}}}{\partial x_{\text{cm}}^2} - \frac{\hbar^2}{2m} \sum_{i=2}^N \frac{\partial^2 \Psi(x_{\text{cm}}, \vec{y}, t)}{\partial y_i^2} \Bigg|_{\vec{y}^j(t)} - i\hbar \sum_{i=2}^N v_i^j(t) \frac{\partial \Psi(x_{\text{cm}}, \vec{y}, t)}{\partial y_i} \Bigg|_{\vec{y}^j(t)} + V_{\text{cm}}(x_{\text{cm}}) \psi_{\text{cd}}, \quad (37)$$

where $V_{\text{cm}}(x_{\text{cm}}) = NV_{\text{ext}}(x_{\text{cm}})$. See appendix B to see how the term V in the many-particle Schrödinger equation (36) is translated into the term V_{cm} in the equation of motion of the conditional wave function (37). By inserting the polar decomposition of the full and conditional wave functions, $\Psi \equiv R \exp(iS/\hbar)$ and $\psi_{\text{cd}} \equiv R_{\text{cd}} \exp(iS_{\text{cd}}/\hbar)$, into (37), one can then derive a continuity-like equation,

$$0 = \frac{\partial R_{\text{cd}}^2}{\partial t} + \frac{\partial}{\partial x_{\text{cm}}} \left(R_{\text{cd}}^2 \frac{\partial S_{\text{cd}}}{\partial x_{\text{cm}}} \frac{1}{M_{\text{cm}}} \right) + J|_{\vec{y}^j(t)}, \quad (38)$$

$$J = \hbar \sum_{i=2}^N \left[\frac{\partial R^2}{\partial y_i} v_i^j(t) - \frac{\partial}{\partial y_i} \left(\frac{1}{m} r^2 \frac{\partial S}{\partial y_i} \right) \right], \quad (39)$$

plus a quantum Hamilton–Jacobi-like equation

$$0 = \frac{\partial S_{\text{cd}}}{\partial t} + \frac{1}{2M_{\text{cm}}} \left(\frac{\partial S_{\text{cd}}}{\partial x_{\text{cm}}} \right)^2 + V_{\text{cm}} + G|_{\vec{y}=\vec{y}^j(t)}, \quad (40)$$

$$G = Q_{\text{cm}} + \sum_{i=2}^N \left(\frac{1}{2m} \left(\frac{\partial S}{\partial y_i} \right)^2 + Q_i - v_i^j(t) \frac{\partial S}{\partial y_i} \right). \quad (41)$$

They include the definition of the quantum potentials

$$Q_{\text{cm}} = Q_{\text{cm}}(x_{\text{cm}}, \vec{y}, t) = -\frac{\hbar^2}{2M_{\text{cm}}R} \frac{\partial^2 R}{\partial x_{\text{cm}}^2}, \quad (42)$$

$$Q_i = Q_i(x_{\text{cm}}, \vec{y}, t) = -\frac{\hbar^2}{2mR} \frac{\partial^2 R}{\partial y_i^2}, \quad (43)$$

and the (non-local) velocity fields

$$v_{\text{cm}} = v_{\text{cm}}(x_{\text{cm}}, \vec{y}, t) = \frac{1}{M_{\text{cm}}} \frac{\partial S}{\partial x_{\text{cm}}}, \quad (44)$$

$$v_i = v_i(x_{\text{cm}}, \vec{y}, t) = \frac{1}{m} \frac{\partial S}{\partial y_i}. \quad (45)$$

The behavior of the quantum Hamilton–Jacobi equation (40) would be classical if the effect of the ‘potential’ G could be ignored. Therefore, the key point in our demonstration is to show that G in (41) fulfills

$$\left. \frac{\partial G}{\partial x_{\text{cm}}} \right|_{\vec{y}=\vec{y}^j(t)} = 0, \quad (46)$$

for a quantum state full of identical particles. The first part of this proof shows that

$$\left. \frac{\partial}{\partial x_{\text{cm}}} \sum_{i=2}^N \left(\frac{1}{2m} \left(\frac{\partial S}{\partial y_i} \right)^2 - v_i^j(t) \frac{\partial S}{\partial y_i} \right) \right|_{\vec{y}^j(t)} = \left(\frac{1}{m} \frac{\partial S}{\partial y_i} \frac{\partial^2 S}{\partial x_{\text{cm}} \partial y_i} - v_i^j(t) \frac{\partial^2 S}{\partial x_{\text{cm}} \partial y_i} \right) \Big|_{\vec{y}^j(t)} = 0, \quad (47)$$

where we have used $\partial S/\partial y_i$ that depends on x_{cm} , but $v_i^j(t)$ does not. The second part of the proof shows that

$$\left[\frac{\partial}{\partial x_{\text{cm}}} \left(Q_{\text{cm}} + \sum_{i=2}^N Q_i \right) \right]_{\vec{y}^j(t)} = 0. \quad (48)$$

Up to here all equations have involved only the j -experiment. Since we know from section 2.2 that any other trajectory of the center of mass associated with the k -experiment will satisfy $x_{\text{cm}}^k(t) = x_{\text{cm}}^j(t) \equiv x_{\text{cm}}(t)$, the shape of the potential term in (48) for the j -experiment must also be equal to that of any other k -experiment. Therefore, we substitute (48) by an average over an ensemble of experiments,

$$\left[\frac{\partial}{\partial x_{\text{cm}}} \left(Q_{\text{cm}} + \sum_{i=2}^N Q_i \right) \right]_{\vec{y}^j(t)} = \frac{1}{M} \sum_{k=1}^M \left[\frac{\partial}{\partial x_{\text{cm}}} \left(Q_{\text{cm}} + \sum_{i=2}^N Q_i \right) \right]_{x_{\text{cm}}^k(t), \vec{y}^k(t)}. \quad (49)$$

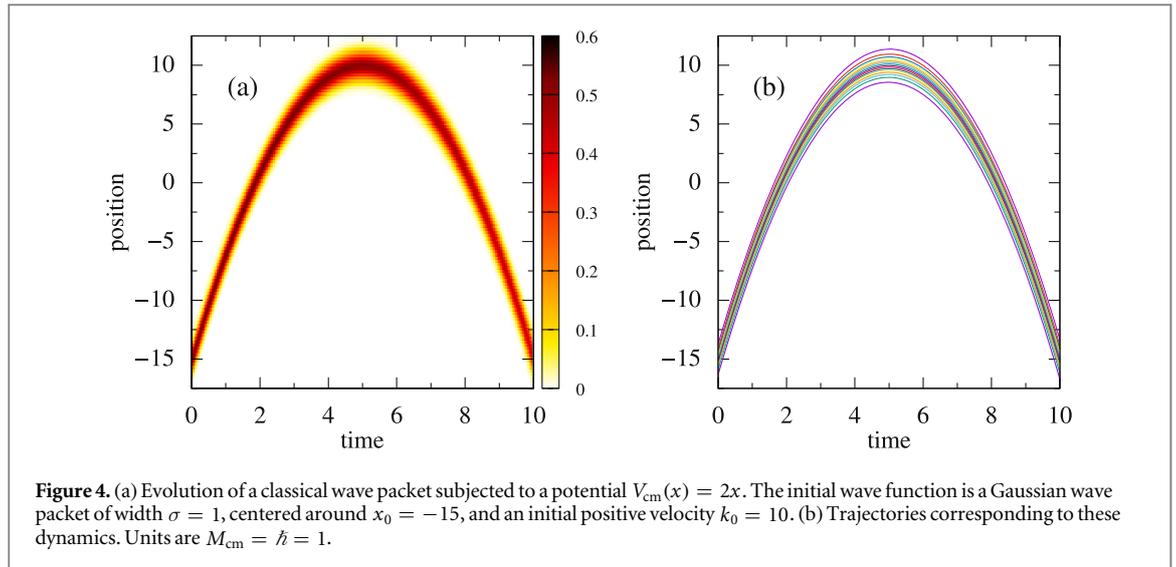
Since the trajectories $x_{\text{cm}}^k(t)$ and $\vec{y}^k(t)$ in the rhs are selected according to (9), we can substitute the sum in (49) by an integral weighted by R^2 ,

$$\frac{1}{M} \sum_{k=1}^M \left[\frac{\partial}{\partial x_{\text{cm}}} \left(Q_{\text{cm}} + \sum_{i=2}^N Q_i \right) \right]_{x_{\text{cm}}^k(t), \vec{y}^k(t)} = \int_{x_{\text{cm}}} \int_{y_2} \dots \int_{y_N} R^2 \frac{\partial}{\partial x_{\text{cm}}} \left(Q_{\text{cm}} + \sum_{i=2}^N Q_i \right) dx_{\text{cm}} dy_2 \dots dy_N. \quad (50)$$

For each term Q_i we have that

$$\int_{x_{\text{cm}}} R^2(x_{\text{cm}}, \vec{y}) \frac{\partial Q_i(x_{\text{cm}}, \vec{y})}{\partial x_{\text{cm}}} dx_{\text{cm}} = \frac{\hbar^2}{2m} \left[\int_{x_{\text{cm}}} \frac{\partial R}{\partial x_{\text{cm}}} \frac{\partial^2 R}{\partial y_i^2} dx_{\text{cm}} - \int_{x_{\text{cm}}} R \frac{\partial^3 R}{\partial x_{\text{cm}} \partial y_i^2} dx_{\text{cm}} \right]. \quad (51)$$

It can be easily seen that these two terms are equal (but with opposite signs) by integrating by parts the first term (assuming that R is zero for $x \rightarrow \pm\infty$). Therefore (51) is equal to 0. A similar argument can be made to show that the term with Q_{cm} in (50) is also zero. The fact that (50) vanishes can be anticipated by knowing that this type of integral on the whole configuration space also appears (and is zero) in the derivation of Ehrenfest’s theorem if the polar form of the wave function is used.



We have just demonstrated that the (conditional) wave equation of a center of mass associated with a quantum state full of identical particles implies (46). In this case, the Hamilton–Jacobi equation in (40) has no dependence on R_{cd} , and only on S_{cd} . Therefore, the velocity of the center of mass,

$$v_{\text{cm}} = \frac{1}{M_{\text{cm}}} \frac{\partial S_{\text{cd}}}{\partial x_{\text{cm}}}, \quad (52)$$

and its trajectory can be computed from (40) independently of (38). Moreover, (40), ignoring the ‘potential’ G , is analogous to the (classical) Hamilton–Jacobi equation, from which one can derive a Schrödinger-like equation

$$i\hbar \frac{\partial \psi_{\text{cd}}}{\partial t} = \left(-\frac{\hbar^2}{2M_{\text{cm}}} \frac{\partial^2}{\partial x_{\text{cm}}^2} + V_{\text{cm}} - Q_{\text{cm}} \right) \psi_{\text{cd}}. \quad (53)$$

In the derivation of this wave equation, we have also used (38). The exact shape of the term J in (38) is irrelevant for computing the velocity of the center of mass (which only depends on (40)), and we have assumed the term $J = 0$ to deal with a conditional wave function with norm equal to one. This equation is also known as the (non-linear) classical Schrödinger wave equation [11, 26, 42]. A study of the dynamics associated with this equation can be found in [43]. We emphasize that the correlations among x_{cm} and the rest of y_i present in (36) are included through the non-linear term $-Q_{\text{cm}}$ in the conditional equation of motion (53).

Numerical examples

In order to illustrate the previous derivation, in what follows we will solve the (non-linear) classical Schrödinger wave equation in (53). We show in figure 4 the case of the evolution of a wave packet under a potential $V_{\text{cm}}(x) = 2x$. One can see that the classical wave packet preserves its shape, and its corresponding trajectories are the expected classical parabolic ones. This contrasts with the simulation of the same initial quantum wave packet in figure 1, which expanded over time. Another simulation is shown in figure 5, in this case for a harmonic potential with a narrow initial wave packet displaced from the origin. As expected from the classical behavior, the trajectories oscillate around the origin, while the wave packet maintains its narrow shape. We emphasize that the initial wave packet has to reflect that the probability distribution of the center of mass is very sharp [43].

4. Quantum states without a classical center of mass

There are certainly many examples of quantum states whose center of mass do not behave classically [5, 7, 44, 45]. In the following we discuss two paradigmatic examples.

4.1. Single-particle states

For a single particle state, the center of mass in a single experiment is the Bohmian position of the particle itself. Moreover, it cannot satisfy condition 1 because different experiments will provide different results. Therefore, the center of mass of a quantum system with one particle (or few particles) cannot follow our classical intuition.

Let us analyze the problems that appear when Bohmian mechanics is used to study the quantum-to-classical transition for a single-particle state. By inserting $\psi = R \exp(iS/\hbar)$ into the single-particle Schrödinger equation one arrives at a quantum continuity equation

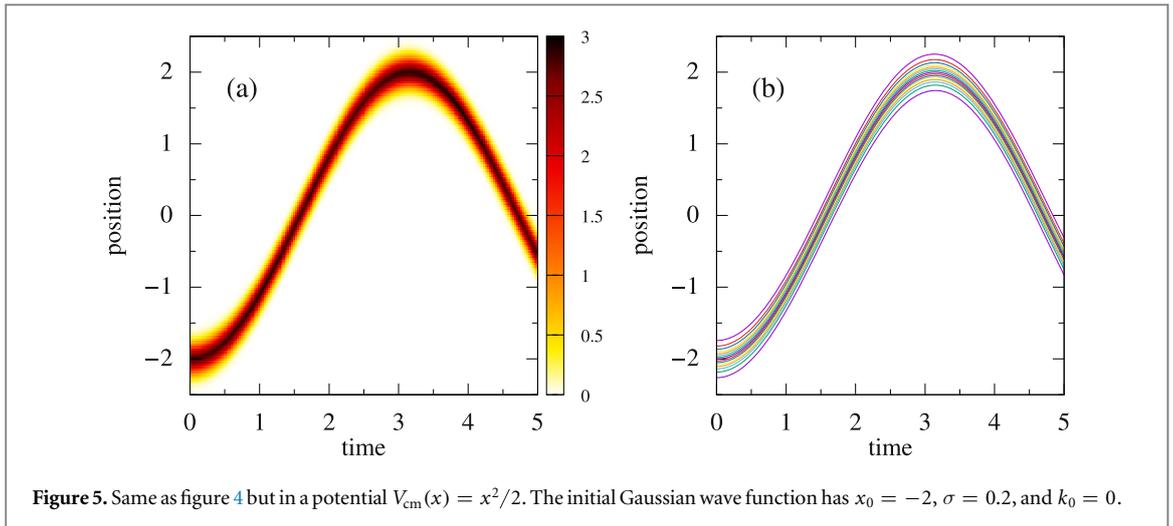


Figure 5. Same as figure 4 but in a potential $V_{\text{cm}}(x) = x^2/2$. The initial Gaussian wave function has $x_0 = -2$, $\sigma = 0.2$, and $k_0 = 0$.

$$\frac{\partial R^2}{\partial t} + \frac{\partial}{\partial x} \left(\frac{R^2}{m} \frac{\partial S}{\partial x} \right) = 0, \quad (54)$$

plus a quantum Hamilton–Jacobi equation [23] given by

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \left(\frac{\partial S}{\partial x} \right)^2 + V_{\text{ext}} + Q = 0. \quad (55)$$

It can be easily demonstrated that (54) and (55) give a Newton-like equation for the (Bohmian) trajectories [23, 26]

$$m \frac{dv(x^j(t), t)}{dt} = \left[-\frac{\partial}{\partial x} (V_{\text{ext}} + Q) \right]_{x=x^j(t)}. \quad (56)$$

It has been argued [46] that a classical (Newtonian) trajectory could be obtained from (56) by just adding a *new* condition

$$\frac{\partial Q}{\partial x} = 0. \quad (57)$$

The problem with this statement is that the classical state given by $x^j(t)$ is not compatible with a quantum state given by the same trajectory $x^j(t)$ and a wave function ψ . The reason for such incompatibility is that ψ does not exist in general. The wave function ψ would have to satisfy, in each position, three equations, equations (54), (55) and (57), but with only two unknowns, R and S .

Another single-particle approach to reach classical dynamics is to interpret the potential V_{ext} as an additional unknown that allows us to define some (exotic) systems where the trajectory and the wave function belong to a state which is simultaneously classical and quantum [47]. The simplest example is a plane wave with a constant $R = 1$, giving $Q = 0$. However, even these particular solutions have some unphysical features that disagree with our classical intuition. The initial positions of the Bohmian trajectories $x^j(t)$ associated with these systems obviously have to be selected according to the distribution $|\psi|^2$ obtained from (9). This means that different initial positions are obtained in different experiments. For the plane wave, the particle can depart from anywhere at an initial time, contradicting our classical intuition of having well defined initial positions.

On the contrary, we have shown in section 3 that a quantum state full of identical particles is compatible with a center of mass following a classical trajectory. The reason why both classical and quantum states are compatible in our case is because the condition in (46) is satisfied in a *natural* way by a quantum state full of identical particles (without imposing very restrictive conditions on V_{ext}). In addition, the classical trajectory of the center of mass of such states directly implies that its initial position and velocity do not change when the experiment is repeated.

4.2. Many-particle states

Our definition of a quantum state full of identical particles discussed in section 3.1 is quite *natural* when the number of particles tends to be very large. However, we define here a quantum state with a large number N of particles with strong correlations that do not satisfy our requirements for a quantum state full of identical particles.

One can think of wave functions of identical particles which make it impossible for a single experiment to fill the whole support of the marginal distribution. Macroscopic quantum many-particle superpositions [44, 45, 48] will not satisfy the condition in (14) and therefore we do not expect a classical behavior for their center of mass,

even when $N \rightarrow \infty$. An extreme example would be the superposition of two separated wave packets (a Schrödinger cat-like state) such as

$$\Psi(x_1, \dots, x_N) = \frac{1}{\sqrt{2}} \left(\prod_{i=1}^N \phi(x_i - x_L) + \prod_{i=1}^N \phi(x_i - x_R) \right). \quad (58)$$

We assume that $\phi(x)$ is a (properly normalized) wave packet centered around $x = 0$, whose support is much smaller than the distance between the two wave packets ($x_R - x_L$) so that the overlap between $\phi(x_i - x_L)$ and $\phi(x_i - x_R)$ is zero. The wave function in (58) only allows for two kinds of quantum states. The first one corresponds to the wave function above plus all particles around x_L . The second one corresponds to the same wave function plus all particles around x_R .

In order to see these two types of quantum states from the point of view of the probability distributions, we calculate the marginal probability distribution of this state, using (12),

$$D(x, 0) = \frac{1}{2} (|\phi(x_i - x_L)|^2 + |\phi(x_i - x_R)|^2). \quad (59)$$

Therefore, the first particle position in the j -experiment has equal probability to be in either $x_1^j(0) \approx x_L$ or $x_1^j(0) \approx x_R$. If, for instance, it is $x_1^j(0) \approx x_L$, then, using (15) and (16), the second particle is selected according to $D^{j,2}(x, 0) = |\phi(x_i - x_L)|^2$, and it will also be $x_2^j(0) \approx x_L$. In fact, all subsequent particles are located around x_L because (15) and (16) show that $D^{j,i}(x_i, 0) = |\phi(x_i - x_L)|^2$ for $i > 1$. Similarly, if in another experiment the first particle is $x_1^j(0) \approx x_R$, then, all particles will be around $x_i^j(0) \approx x_R$. It is obvious then that in this case $D(x, 0) \neq D^{j,i}(x, 0)$ in all experiments. This is because the marginal distribution for this state has a non-zero support around both x_L and x_R , while the quantum state in any experiment involves only particles on the left or only particles on the right, but never particles at both sides.

We discuss here why the center of mass of a quantum state like the one in (58) can show quantum interference. Although the marginal distribution has support on both sides of the physical space, in a particular experiment, the Bohmian trajectories associated with this state will be present on only one side, say the left support. Thus, the dynamics of the center of mass is associated only with the particles on the left support of the wave function. However, (classically unexpected) interferences could appear later if the left wave function overlaps and interferes with the right one (empty of particles), thus modifying the velocities of the particles. On the contrary, in the numerical example of section 3.3 where the marginal distribution also has two separated supports, such (classically unexpected) interferences will not appear because it is a quantum state full of identical particles. Bohmian trajectories will always fill up both left and right supports and the center of mass will always be an average over all (left and right) particles. If the left and right supports are large enough to be macroscopically distinguishable, we will see two classical particles, described by the center of mass of the left and right Bohmian particles, respectively. The trajectories of these centers of mass will correspond to the elastic collision between classical particles. We conclude that quantum states whose supports are partially empty of particles are required to observe effects against our classical intuition.

5. Conclusions

In summary, by using the peculiar properties of the center of mass of a large set of a Bohmian particles, we have provided a *natural* route to explain the quantum-to-classical transition. We have defined a quantum state full of identical particles as the state whose distribution of the Bohmian positions in a single experiment is always equal to the marginal distribution. The center of mass of such states satisfies our classical intuition in the sense that, first, its initial position and velocity are perfectly fixed when experiments are repeated (prepared with the same wave function) and, second, it follows a classical trajectory. We emphasize that only the center of mass behaves classically, while the rest of the microscopic degrees of freedom can and will show quantum dynamics. In this sense, the quantum-to-classical transition appears due to the *natural* coarse-graining description of the center of mass.

Due to the compatibility between Bohmian and orthodox results [23, 25–27], the Bohmian explanation of the quantum-to-classical transition in this paper can be equivalently derived using orthodox arguments. The Bohmian route explored here avoids dealing with the reduced density matrix and the collapse law. There is a commonly accepted wisdom in the orthodox attempts that decoherence plays a relevant role in the quantum-to-classical transition, and this work does not contradict this. One can see that the center of mass (our open system) is strongly entangled with the rest of the degrees of freedom of the macroscopic object (the environment). Notice, from the definition of the potential in (B.12), that the many-particle Schrödinger equation in (36) is, in general, non separable. Without this entanglement, we will not arrive at the classical (dispersionless) wave equation in section 3.4, but to a single-particle Schrödinger equation with the typical spreading of wave packets. Notice that the original Schrödinger equation is linear, while the classical version is non-linear, breaking the

superposition principle. A paradigmatic example of the role of decoherence in destroying superposition (and avoiding wave packet spreading) was initially presented by Zurek using the example of Hyperion, a chaotically tumbling moon of Saturn [49–52]. He estimated that, without decoherence, within 20 years the quantum state of Hyperion would evolve into a highly nonlocal coherent superposition of macroscopically distinguishable orientations. It is important to emphasize that, in our work, the environment of the center of mass of Hyperion would consist of $N \approx 10^{44}$ particles, which would be responsible for the decoherence of the center of mass.

The conclusions in this paper for a quantum state full of identical particles, derived for an infinite number of particles, can be translated into a macroscopic system with a very large but finite number of particles when the error defined in appendix A remains smaller than some predetermined measuring accuracy. In particular, for the two numerical examples of this paper, the central limit theorem [38] ensures that the center of mass of a quantum state full of identical particles with a finite number of particles tends to the exact classical value as N grows. However, further research on quantum states full of identical particles where the selection of the initial positions are dependent on each other is still needed.

Finally, an explanation as to why we have ignored the measurement apparatus in this article is in order. It is well-known that the Bohmian formalism does not include any collapse law but, instead, one has to include the interaction between the system and a measuring apparatus. We have ignored this interaction because we are only dealing with a classical object measured by a classical apparatus. Both the classical object and the classical measuring apparatus are in a quantum state full of identical particles whose centers of mass follow a classical trajectory $\vec{r}_{s,cm}(t)$ and $\vec{r}_{a,cm}(t)$, respectively. Then, the interaction between the system and the apparatus, i.e. between $\vec{r}_{s,cm}(t)$ and $\vec{r}_{a,cm}(t)$, is unproblematic and it can be ignored if the type of classical measurement is assumed to not perturb the classical macroscopic object. On the contrary, the present work cannot be directly applied to the measurement of a quantum system in general. Obviously, quantum systems cannot be directly described by a quantum state full of identical particles when different experiments (with identical wave function preparation) provide different measured results. Nevertheless, a straightforward generalization of the present work can explain why the measuring apparatus (entangled with the quantum system) presents a classical behavior with its macroscopic pointer (in fact, its center of mass) following a classical trajectory.

Acknowledgments

We would like to thank David Tena for fruitful discussions. This work has been partially supported by the Fondo Europeo de Desarrollo Regional (FEDER) and the ‘Ministerio de Ciencia e Innovación’ through the Spanish Project TEC2015-67462-C2-1-R, the Generalitat de Catalunya (2014 SGR-384), the European Union’s Horizon 2020 research and innovation program under grant agreement No 696656, and the Okinawa Institute of Science and Technology Graduate University.

Appendix A. Evolution of the error of the center of mass for a quantum state full of identical particles with a finite number of particles

A definition of a quantum state full of identical particles in (14) of the text, in principle, requires $N \rightarrow \infty$. Let us now study the properties of a quantum state with a finite number, N_F , of particles that becomes a quantum state full of identical particles when $N_F \rightarrow \infty$. We use the subscript F in N_F as a reminder that the number of particles is finite. In particular, the selection of the initial position of the trajectories associated with these new quantum states with only N_F particles also follows (15) and (16). Once the N_F particles are selected, we can distribute them following

$$C^{j_0,F}(\vec{r}, t) = \frac{1}{N_F} \sum_{i=1}^{N_F} \delta(\vec{r} - \vec{r}_i^{j_0}(t)), \quad (\text{A.1})$$

and define their center of mass as

$$\vec{r}_{cm}^{j_0,F}(t) = \int d\vec{r} \vec{r} C^{j_0,F}(\vec{r}, t) = \frac{1}{N_F} \sum_{i=1}^{N_F} \vec{r}_i^{j_0}(t). \quad (\text{A.2})$$

Notice again that $\vec{r}_{cm}^{j_0,F}(t) \neq \vec{r}_{cm}^{j_0}(t)$ because we are dealing here with a finite number of particles N_F , while we know that $\vec{r}_{cm}^{j_0}(t) = \langle \vec{r}_{cm} \rangle(t)$. The error resulting from comparing this center of mass $\vec{r}_{cm}^{j_0,F}(t)$ with the one obtained for $N_F \rightarrow \infty$, can be estimated as

$$\text{Err}(t) = |\langle \vec{r}_{cm} \rangle(t) - \vec{r}_{cm}^{j_0,F}(t)|. \quad (\text{A.3})$$

As indicated in (17), $\langle \vec{r}_{cm} \rangle(t)$ is independent of the experiment, but $\vec{r}_{cm}^{j_0,F}(t)$ in (A.2) varies between experiments due to quantum randomness.

To further develop expression (A.3), let us assume now that the selections of all $\vec{r}_i^{j_0}(t)$ are independent, i.e. we select each $\vec{r}_i^{j_0}(t)$ according to $D(\vec{r}, t)$. This is exactly the case in the two numerical examples explained in sections 3.2 and 3.3. The center of mass in (A.2) corresponds to a sequence of independent and identically distributed random variables \vec{r}_i drawn from a distribution $D(\vec{r}, t)$ with a mean value given by $\langle \vec{r}_{\text{cm}} \rangle(t) = \int \vec{r} D(\vec{r}, t) d\vec{r}$ and with a finite variance given by

$$\sigma^2(t) = \int (\vec{r} - \langle \vec{r}_{\text{cm}} \rangle(t))^2 D(\vec{r}, t) d\vec{r}. \quad (\text{A.4})$$

We know from the central limit theorem [38] that the distribution of $\vec{r}_{\text{cm}}^{j_0, F}(t)$ in different experiments given by (A.2) follows a normal distribution when N_F grows with mean value and variance

$$\vec{r}_{\text{cm}}^{j_0, F}(t) = \int d\vec{r} \vec{r} C^{j_0, F}(\vec{r}, t) \approx \langle \vec{r}_{\text{cm}} \rangle(t), \quad (\text{A.5})$$

$$\int d\vec{r} (\vec{r} - \langle \vec{r}_{\text{cm}} \rangle(t))^2 C^{j_0, F}(\vec{r}, t) \approx \frac{\sigma(t)^2}{N_F}. \quad (\text{A.6})$$

These results are valid for any initial distribution $D(\vec{r}, t)$ as far as N_F is large enough.

The error in expression (A.3) can now be rewritten in terms of the probability of obtaining a difference between $\langle \vec{r}_{\text{cm}} \rangle(t)$ and $\vec{r}_{\text{cm}}^{j_0, F}(t)$ smaller than a given error, Err,

$$\mathcal{P}(|\langle \vec{r}_{\text{cm}} \rangle - \vec{r}_{\text{cm}}^{j_0, F}| < \text{Err}) = 2\Phi_N\left(\frac{\sqrt{N_F}\text{Err}}{\sigma}\right) - 1 \quad (\text{A.7})$$

where $\Phi_N(x)$ is the cumulative distribution function of the standard normal distribution,

$$\Phi_N(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} \exp(-t^2/2) dt, \quad (\text{A.8})$$

and we have used its property $\Phi_N(x) + \Phi_N(-x) = 1$. If we require, for example, the difference $\langle \vec{r}_{\text{cm}} \rangle - \vec{r}_{\text{cm}}^{j_0, F}$ to be smaller than $\text{Err} = 0.005\sigma$ with a probability of $\mathcal{P}(|\langle \vec{r}_{\text{cm}} \rangle - \vec{r}_{\text{cm}}^{j_0, F}| < 0.005\sigma) = 0.98$, then we get that the number of particles N_F has to be equal or larger than:

$$N_F \geq \frac{(\Phi_N^{-1}(0.99))^2}{0.005^2} \simeq 2 \times 10^5. \quad (\text{A.9})$$

In summary, if we consider 0.005σ an acceptable error for $\vec{r}_{\text{cm}}^{j_0, F}$, then we are sure that 98% of the experiments with our quantum state with a number of particles $N_F \gtrsim 2 \times 10^5$ satisfy the fixed error.

As a more realistic example, let us consider a macroscopic system with the number of particles equal to a mol of the matter, i.e. $N_F = 6 \times 10^{23}$ particles. In addition, we require that the value of $\vec{r}_{\text{cm}}^{j_0, F}$ always gives the classical value, i.e. that only once in $M_F = 2 \times 10^{12}$ experiments, the value of $\vec{r}_{\text{cm}}^{j_0, F}$ overcomes a fixed value of the Err. Then, we can compute the required error by solving the relation $\mathcal{P} = 1 - 1/M_F$ in (A.7) as:

$$\frac{\text{Err}}{\sigma} = \frac{\Phi_N^{-1}(1 - 10^{-12})}{\sqrt{N_F}} \simeq 9 \times 10^{-12}. \quad (\text{A.10})$$

In summary, for a quantum state with a number of particles typical of a macroscopic system, i.e. $N_F = 6 \times 10^{23}$, the error of $\vec{r}_{\text{cm}}^{j_0, F}$ is always smaller than $\text{Err} \approx 10^{-11}\sigma$ (except in one experiment every $M_F = 2 \times 10^{12}$).

The time evolution of the error in (A.3) can be obtained once we know the particular time-dependence of the variance of $D(x, t)$. For example, in the case of $D(x, t)$ given by the modulus square of a Gaussian wave packet in free space, then the standard deviation is given (for larger times) by

$$\sigma(t) = \sigma_0 \sqrt{1 + \left(\frac{\hbar t}{2 m \sigma_0^2}\right)^2} \approx \frac{\hbar t}{2 m \sigma_0}. \quad (\text{A.11})$$

For example, assuming an initial spatial dispersion $\sigma_0 = 100$ nm, a mol of carbon atoms ($m = 2 \times 10^{-26}$ kg), after $t = 1$ year of classical evolution, the absolute error in (A.10) is given by $\text{Err}(t) \simeq 10^{-12}\sigma(t) \simeq 8 \mu\text{m}$. In summary, in the overwhelming majority of experiments (all $M_F = 2 \times 10^{12}$ experiments except one), the error in the center of mass after 1 year of evolution, between the exact value (with $N \rightarrow \infty$) and the approximate center of mass (with $N_F = 6 \times 10^{23}$) for the described quantum state is smaller than $10 \mu\text{m}$.

Certainly, in this example $\text{Err}(t)$ grows with time due to the intrinsic expansion of a free wave packet. However, we want to emphasize that our classical intuition is based on crystalline materials where particles have an ordered structure due to their attractive interactions. Thus, classical objects (i.e. their particles) will tend to remain much more localized than in the above example. These interactions will also introduce correlations among the different particles and, in principle, the assumption that the selection of all $\vec{r}_i^{j_0}(t)$ are independent might not seem fully rigorous. However, one can argue that in a realistic classical system, with $N_F \simeq 6 \times 10^{23}$ interacting particles, the accurate selection of the first, say $N_F/100$, particles with the procedure in (15) and (16)

will be roughly independent. This is due to the selection of points in a huge (and basically empty) configuration space of $3N_F \sim 10^{24}$ dimensions. Only the selection of the particles whose positions are very close to previous ones will be influenced by the non-negligible correlations.

Appendix B. Wave equation for the center of mass coordinates

Our aim here is to find a change of coordinates in the 1D many-particle Schrödinger equation (see the 1D version of equation (1)) with the usual definition of the center of mass,

$$x_{\text{cm}} = \frac{1}{N} \sum_{i=1}^N x_i. \quad (\text{B.1})$$

and without cross terms appearing in the Laplacian. The additional set of $N - 1$ coordinates can be written as

$$y_j = \sum_{i=1}^N \alpha_i^{(j)} x_i \quad \text{for } j = 2, \dots, N, \quad (\text{B.2})$$

and the $\alpha_i^{(j)}$ will be fixed by the condition that cross terms do not appear in the Laplacian

$$\sum_{i=1}^N \frac{\partial^2 \psi}{\partial x_i^2} = \frac{1}{N} \frac{\partial^2 \psi}{\partial x_{\text{cm}}^2} + \sum_{j=2}^N \frac{\partial^2 \psi}{\partial y_j^2}. \quad (\text{B.3})$$

Substituting equations (B.1) and (B.2) into the lhs. of (B.3), one obtains

$$\sum_{i=1}^N \frac{\partial^2 \psi}{\partial x_i^2} = \frac{1}{N} \frac{\partial^2 \psi}{\partial x_{\text{cm}}^2} + \frac{2}{N} \sum_{k=2}^N \left[\frac{\partial^2 \psi}{\partial x_{\text{cm}} \partial y_k} \sum_{i=1}^N \alpha_i^{(k)} \right] + \sum_{k=2}^N \sum_{j=2}^N \left[\frac{\partial^2 \psi}{\partial y_j \partial y_k} \sum_{i=1}^N \alpha_i^{(j)} \alpha_i^{(k)} \right]. \quad (\text{B.4})$$

Comparing this with (B.3) we see that the conditions for our change of variables are

$$0 = \sum_{i=1}^N \alpha_i^{(j)}, \quad 1 = \sum_{i=1}^N (\alpha_i^{(j)})^2, \quad 0 = \sum_{i=1}^N \alpha_i^{(j)} \alpha_i^{(k)} \quad \text{for } j \neq k. \quad (\text{B.5})$$

We propose a change of variables with the following structure (using x_1 separately as we only need $N - 1$ variables besides the center of mass):

$$y_j = ax_j + bx_{\text{cm}} + cx_1 = ax_j + \frac{b}{N} \sum_{i=1}^N x_i + cx_1 \quad \Rightarrow \quad \alpha_k^{(j)} = a \delta_{jk} + \frac{b}{N} + c \delta_{1k}. \quad (\text{B.6})$$

We impose conditions (B.5) in order to get the following system

$$0 = \sum_{i=1}^N \alpha_i^{(j)} = a + b + c, \quad (\text{B.7})$$

$$1 = \sum_{i=1}^N (\alpha_i^{(j)})^2 = \left(c + \frac{b}{N}\right)^2 + \left(a + \frac{b}{N}\right)^2 + (N-2) \left(\frac{b}{N}\right)^2, \quad (\text{B.8})$$

$$0 = \sum_{i=1}^N \alpha_i^{(j)} \alpha_i^{(k)} = \left(c + \frac{b}{N}\right)^2 + \frac{2b}{N} \left(a + \frac{b}{N}\right) + (N-3) \left(\frac{b}{N}\right)^2. \quad (\text{B.9})$$

This can be solved to yield the variable changes in equation (34) and the final many-particle Schrödinger equation in (36).

Now, in order to see how the term V in the many-particle Schrödinger equation (36) is translated into the term V_{cm} in the conditional wave function (37), we invert (34) to obtain

$$x_1 = x_{\text{cm}} - \frac{1}{\sqrt{N}} \sum_{i=2}^N y_i, \quad x_j = x_{\text{cm}} + y_j - \frac{1}{\sqrt{N} + N} \sum_{i=2}^N y_i. \quad (\text{B.10})$$

We can now rewrite the potential (2) as:

$$\begin{aligned} V(x_{\text{cm}}, \vec{y}) &= V_{\text{ext}} \left(x_{\text{cm}} - \frac{1}{\sqrt{N}} \sum_{i=2}^N y_i \right) + \sum_{j=2}^N V_{\text{ext}} \left(x_{\text{cm}} + y_j - \frac{1}{\sqrt{N} + N} \sum_{i=2}^N y_i \right) \\ &+ \frac{1}{2} \sum_{j=2}^N V_{\text{int}} \left(-\frac{1}{1 + \sqrt{N}} \sum_{i=2}^N y_i - y_j \right) + \frac{1}{2} \sum_{i=2}^N \sum_{j=2; i \neq j}^N V_{\text{int}}(y_i - y_j). \end{aligned} \quad (\text{B.11})$$

The terms V_{int} have no dependence on x_{cm} . Therefore, when considering the conditional wave function of the center of mass with $\vec{y} = \vec{y}(t)$ in (B.11), they will just become a purely time-dependent potential. Their only

effect will then be a pure time-dependent phase in the wave function, which can be neglected in the computation of the conditional equation of motion of the center of mass.

Each of the other two terms V_{ext} in (B.11) have a dependence on x_{cm} plus a dependence on $\sum_{i=2}^N y_i$. We provide a Taylor expansion around x_{cm}

$$V_{\text{ext}}(x_{\text{cm}} + \Delta x) = V_{\text{ext}}(x_{\text{cm}}) + \left. \frac{\partial V_{\text{ext}}(x)}{\partial x} \right|_{x=x_{\text{cm}}} \Delta x + \frac{1}{2} \left. \frac{\partial^2 V_{\text{ext}}(x)}{\partial x^2} \right|_{x=x_{\text{cm}}} \Delta x^2 + \dots \quad (\text{B.12})$$

We define, in order to simplify, the expressions,

$$\beta(x_{\text{cm}}) = \left. \frac{\partial V_{\text{ext}}(x)}{\partial x} \right|_{x=x_{\text{cm}}}, \quad \gamma(x_{\text{cm}}) = \frac{1}{2} \left. \frac{\partial^2 V_{\text{ext}}(x)}{\partial x^2} \right|_{x=x_{\text{cm}}}. \quad (\text{B.13})$$

This allows us to rewrite the part of the potential that depends on x_{cm} as

$$\begin{aligned} & V_{\text{ext}}\left(x_{\text{cm}} - \frac{1}{\sqrt{N}} \sum_{i=2}^N y_i\right) + \sum_{j=2}^N V_{\text{ext}}\left(x_{\text{cm}} + y_j - \frac{1}{\sqrt{N} + N} \sum_{i=2}^N y_i\right) \\ &= N V_{\text{ext}}(x_{\text{cm}}) + \beta(x_{\text{cm}}) \left(1 - \frac{1}{\sqrt{N}} - \frac{N-1}{\sqrt{N} + N}\right) \sum_{i=2}^N y_i \\ &+ \gamma(x_{\text{cm}}) \left[\sum_{j=2}^N y_j^2 + \left(\frac{1}{N} + \frac{N-1}{(\sqrt{N} + N)^2} - \frac{2}{\sqrt{N} + N}\right) \left(\sum_{j=2}^N y_j\right)^2 \right] + \dots \end{aligned} \quad (\text{B.14})$$

We see that the factor of $\beta(x_{\text{cm}})$ is zero, i.e. $1 - \frac{1}{\sqrt{N}} - \frac{N-1}{\sqrt{N} + N} = 0$, and the factor of $\gamma(x_{\text{cm}})$ can be simplified as $\frac{1}{N} + \frac{N-1}{(\sqrt{N} + N)^2} - \frac{2}{\sqrt{N} + N} = 0$, so we arrive at

$$V_{\text{ext}}\left(x_{\text{cm}} - \frac{1}{\sqrt{N}} \sum_{i=2}^N y_i\right) + \sum_{j=2}^N V_{\text{ext}}\left(x_{\text{cm}} + y_j - \frac{1}{\sqrt{N} + N} \sum_{i=2}^N y_i\right) = N V_{\text{ext}}(x_{\text{cm}}) + \gamma(x_{\text{cm}}) \sum_{i=2}^N y_i^2 + \dots \quad (\text{B.15})$$

The $\gamma(x_{\text{cm}})$ in the second term and higher orders still have, in principle, some x_{cm} spatial dependence. We now invoke condition 2 (see section 2.2) that assumes a quadratic approximation for the (long range) external potential, with a negligible dependence of γ on x_{cm} . This means that $\gamma(x_{\text{cm}}) = \gamma$ and the rest of the higher order derivatives of the Taylor expansion become zero. Under such conditions, when calculating the conditional wave function of the center of mass at $\vec{y}(t)$, the term $\gamma \sum_{i=2}^N y_i^2(t)$ can be neglected as a purely time-dependent term (as occurred for the previously discussed V_{int} terms). Therefore, we finally get the external potential of the equation of motion of the conditional wave function of the center of mass

$$\sum_{j=1}^N V_{\text{ext}}(x_j) |_{\vec{y}=\vec{y}(t)} = N V_{\text{ext}}(x_{\text{cm}}) \equiv V_{\text{cm}}(x_{\text{cm}}). \quad (\text{B.16})$$

The same simple potential can be exactly recovered for a quadratic external potential $V_{\text{ext}}(x) = \alpha + \beta x + \gamma x^2$ with constant α , β , and γ . Notice that our derivation above demands a more relaxed condition on V_{ext} , as it only requires that this shape (constant γ) occurs along the extension of the object in physical space.

ORCID

Xavier Oriols  <https://orcid.org/0000-0003-2181-4284>

Albert Benseny  <https://orcid.org/0000-0003-4785-0945>

References

- [1] Landau L D and Lifschitz E M 1981 *Quantum Mechanics, Non-Relativistic Theory* (Boston: Butterworth–Heinemann)
- [2] Ballentine L E 1990 *Quantum Mechanics* (Englewood Cliffs, NJ: Prentice-Hall)
- [3] Herbert N 1984 *Quantum Reality* (New York: Anchor Books)
- [4] Zurek W H 2003 Decoherence, einselection, and the quantum origins of the classical *Rev. Mod. Phys.* **75** 715
- [5] Giulini D, Joos E, Kiefer C, Kupsch J, Stamatescu I O and Zeh H D 1996 *Decoherence and the Appearance of a Classical World in Quantum Theory* (Berlin: Springer)
- [6] Diósi L 2003 *Progressive Decoherence and Total Environmental Disentanglement in Irreversible Quantum Dynamics (Lecture Notes in Physics)* (Berlin: Springer)
- [7] Schlosshauer M 2007 *The Quantum-to-Classical Transition and Decoherence* (Berlin: Springer)
- [8] Zeh H D 1970 On the interpretation of measurement in quantum theory *Found. Phys.* **1** 69
- [9] Maudlin T 1995 Three measurement problems *Topoi* **14** 7

- [10] Bohm D and Bub J 1966 A proposed solution of the measurement problem in quantum mechanics by a hidden variable theory *Rev. Mod. Phys.* **38** 453
- [11] Nikolić H 2006 Classical mechanics without determinism *Found. Phys. Lett.* **19** 553
- [12] Bassi A, Lochan K, Satin S, Singh T P and Ulbricht H 2013 Models of wave-function collapse, underlying theories, and experimental tests *Rev. Mod. Phys.* **85** 471
- [13] Ghirardi G C, Rimini A and Weber T 1986 Unified dynamics for microscopic and macroscopic system *Phys. Rev. D* **34** 470
- [14] Bohr N 1920 Über die serienspektren der elementen *Z. Phys.* **2** 423
- [15] Jeong H, Lim Y and Kim M S 2014 Coarsening measurement references and the quantum-to-classical transition *Phys. Rev. Lett.* **112** 010402
- [16] Kofler J and Brukner C 2007 Classical world arising out of quantum physics under the restriction of coarse-grained measurements *Phys. Rev. Lett.* **99** 180403
- [17] Yang H, Miao H, Lee D-S, Helou B and Chen Y 2013 Macroscopic quantum mechanics in a classical spacetime *Phys. Rev. Lett.* **110** 170401
- [18] Everett H 1957 Relative state formulation of quantum mechanics *Rev. Mod. Phys.* **29** 454
- [19] Wallace D 2012 *Chaos, Decoherence, and Branching in the Emergent Multiverse: Quantum Theory according to the Everett Interpretation* (Oxford: Oxford University Press)
- [20] Saunders S, Barrett J, Kent A and Wallace D (ed) 2010 *Many Worlds? Everett, Quantum Theory, and Reality* (Oxford: Oxford University Press)
- [21] Callender C 2007 The emergence and interpretation of probability in Bohmian mechanics *Stud. Hist. Phil. Sci. B* **38** 351
- [22] Towler M D, Russell N J and Valentini A 2011 Time scales for dynamical relaxation to the Born rule *Proc. R. Soc. A* **468** 990
- [23] Bohm D and Suggested A 1952 Interpretation of the quantum theory in terms of hidden variables *Phys. Rev.* **85** 166
- [24] Bell J S 1966 On the problem of hidden variables in quantum mechanics *Rev. Mod. Phys.* **38** 447
- [25] Holland P R 1993 *The Quantum Theory of Motion* (Cambridge: Cambridge University Press)
- [26] Oriols X and Mompert J (ed) 2012 *Applied Bohmian Mechanics: From Nanoscale Systems to Cosmology* (Singapore: Pan Stanford)
- [27] Benseny A, Albareda G, Sanz A S, Mompert J and Oriols X 2014 Applied Bohmian mechanics *Eur. Phys. J. D* **68** 286
- [28] Dürr D, Goldstein S and Zanghì N 2013 *Quantum Mechanics without Quantum Philosophy* (Berlin: Springer)
- [29] Allori V, Dürr D, Goldstein S and Zanghì N 2002 Seven steps towards the classical world *J. Opt. B* **4** 482
- [30] Nassar A B and Miret-Artés S 2013 Dividing line between quantum and classical trajectories in a measurement problem: bohmian time constant *Phys. Rev. Lett.* **111** 150401
- [31] Allori V and Zanghì N 2009 On the classical limit of quantum mechanics *Found. Phys.* **39** 20
- [32] Ehrenfest P 1927 Bemerkung über die angenäherte Gültigkeit der klassischen Mechanik innerhalb der Quantenmechanik *Z. Physik* **45** 455
- [33] Lusanna L and Pauri M 2014 On the transition from the quantum to the classical regime for massive scalar particles: a spatiotemporal approach *Eur. Phys. J. Plus* **129** 178
- [34] Traversa F L, Albareda G, Di Ventura M and Oriols X 2013 Robust weak-measurement protocol for Bohmian velocities *Phys. Rev. A* **87** 052124
- [35] Marian D, Zanghì N and Oriols X 2016 Weak values from displacement currents in multiterminal electron devices *Phys. Rev. Lett.* **116** 110404
- [36] Solé A, Oriols X, Marian D and Zanghì N 2016 How does quantum uncertainty emerge from deterministic bohmian mechanics? *Fluct. Noise Lett.* **15** 1640010
- [37] Alarcón A, Yaro S, Cartoixa X and Oriols X 2013 Computation of many-particle quantum trajectories with exchange interaction: application to the simulation of nanoelectronic devices *J. Phys.: Condens. Matter* **25** 325601
- [38] Rice J 1995 *Mathematical Statistics and Data Analysis* 2nd edn (Belmont, CA: Duxbury Press)
- [39] Dürr D, Goldstein S and Zanghì N 1992 Quantum equilibrium and the origin of absolute uncertainty *J. Stat. Phys.* **67** 843
- [40] Oriols X 2007 Quantum-trajectory approach to time-dependent transport in mesoscopic systems with electron-electron interactions *Phys. Rev. Lett.* **98** 066803
- [41] Norsen T, Marian D and Oriols X 2015 Can the wave function in configuration space be replaced by single-particle wave functions in physical space? *Synthese* **192** 3125
- [42] Richardson C D, Schlagheck P, Martin J, Vandewalle N and Bastin T 2014 Nonlinear Schrödinger wave equation with linear quantum behavior *Phys. Rev. A* **89** 032118
- [43] Benseny A, Tena D and Oriols X 2016 On the classical Schrödinger equation *Fluct. Noise Lett.* **15** 1640011
- [44] Romero-Isart O 2011 Quantum superposition of massive objects and collapse models *Phys. Rev. A* **84** 052121
- [45] Hornberger K, Uttenthaler S, Brezger B, Hackermüller L, Arndt M and Zeilinger A 2003 Collisional decoherence observed in matter wave interferometry *Phys. Rev. Lett.* **90** 160401
- [46] Rosen N 1964 Identical motion in quantum and classical mechanics *Am. J. Phys.* **32** 377
- [47] Makowski A J 2002 Exact classical limit of quantum mechanics: central potentials and specific states *Phys. Rev. A* **65** 032103
- [48] Romero-Isart O, Pflanzer A C, Blaser F, Kaltenbaek R, Kiesel N, Aspelmeyer M and Cirac J I 2011 Large quantum superpositions and interference of massive nanometer-sized objects *Phys. Rev. Lett.* **107** 020405
- [49] Zurek W H 1998 Decoherence, chaos, quantum-classical correspondence and the arrow of time *Phys. Scr. T* **76** 186
- [50] Wiebe N and Ballentine L E 2005 Quantum mechanics of Hyperion *Phys. Rev. A* **72** 022109
- [51] Schlosshauer M 2008 Classicality, the ensemble interpretation, and decoherence: resolving the hyperion dispute *Found. Phys.* **38** 796
- [52] Ballentine L E 2008 Classicality without decoherence: a reply to Schlosshauer *Found. Phys.* **38** 916