Mechanics of Materials 166 (2022) 104193

Contents lists available at ScienceDirect

Mechanics of Materials

journal homepage: www.elsevier.com/locate/mecmat

Coordinate-free compatibility conditions for deformations of material surfaces

Brian Seguin [a,*], Eliot Fried [b,*]

[a] Department of Mathematics and Statistics, Loyola University Chicago, 1032 W. Sheridan Road, Chicago IL 60660, USA
[b] Mechanics and Materials Unit, Okinawa Institute of Science and Technology Graduate University, 1919-1 Tancha, 904-0495, Okinawa, Japan

ARTICLE INFO

MSC:
53A05
74K25
74K20
74K35

Keywords:
Covariant gradient
Deformable surfaces
Material interfaces
Shells
Plates
Kinematical constraints

ABSTRACT

The study of material surfaces uses notions from classical differential geometry, such as the covariant gradient, the mean and Gaussian curvatures, and the Peterson–Mainardi–Codazzi and Gauss equations. These notions are traditionally introduced relative to local surface coordinates and involve Christoffel symbols. We proceed instead without recourse to coordinates using direct notation. After developing the formula for the covariant gradient relative to a surface metric, we derive versions of the Peterson–Mainardi–Codazzi and Gauss equations and Gauss' Theorema Egregium relevant to a deformed material surface. We then apply our framework to kinematically constrained material surfaces. For material surfaces that can sustain only deformations that preserve either angles or lengths, we obtain explicit representations for the covariant gradient relative to the surface metric in terms of the surface gradient. We show also that a deformation of a material surface that preserves angles and areas must be length preserving and vice versa. Finally, we present an alternative derivation of the Peterson–Mainardi–Codazzi and Gauss equations for a deformed material surface subject to the provision that the surface metric derives from the metric for the ambient Euclidean space within which the surface is embedded. An Appendix involving coordinates is included to ease comparisons between our approach to covariant differentiation and associated derivations of the Peterson–Mainardi–Codazzi and Gauss equations and standard coordinate-based approaches.

1. Introduction

Material surfaces are fundamental to all theories of thin-walled plates and shells, whether established asymptotically with reference to three-dimensional theory or formulated directly, and have also been used extensively to model adhesive interphases in composites, surface coatings, and thin films. The kinematical framework needed to describe the deformation of a material surface relies intimately on tools and results from classical differential geometry. Orthodox treatments of this subject, as exemplified by the comprehensive treatise of Ciarlet (2005), typically begin by introducing coordinate charts for the reference and deformed configurations of the material surface and hinge on representing fundamental kinematical objects through their components relative to the associated coordinates.

The primary aim of the present work is to supply a nonstandard treatment that, by avoiding such ingredients, is resolutely coordinate free. In so doing, we seek to extend and complement the pioneering contributions of Gurtin and Murdoch (1975), Murdoch (1978), and Murdoch and Cohen (1979). Specifically, we emulate the perspective and approach synopsized in the following passage due to Murdoch (1978):

> [T]he essential simplicity and elegance of a treatment free of co-ordinate considerations allows for greater insight into and emphasis upon the fundamental geometric and algebraic concepts involved. In this respect it is to be observed that direct notation does not merely mean that the results are presented free of indicial notation but rather that, *inter alia*, precise definitions of surface, position and motion are made without recourse to any co-ordinate system. Thus by the employment of direct notation is implied a philosophy in which inessentials and non-physical considerations involved in the modeling of natural phenomena are carefully excised, the more clearly to perceive the relationship between the model and its subject.

To begin, we develop coordinate-free representations for the covariant gradient and curl operators on a surface. We then derive coordinate-free versions of the Peterson–Mainardi–Codazzi and Gauss equations and present an invariant proof of Gauss' Theorema Egregium, all for a deformed material surface. Our Peterson–Mainardi–Codazzi and Gauss equations involve the pullbacks, to the reference surface, of the metric

* Corresponding authors.
E-mail addresses: bseguin@luc.edu (B. Seguin), eliot.fried@oist.jp (E. Fried).

https://doi.org/10.1016/j.mechmat.2021.104193
Received 17 September 2021; Received in revised form 12 December 2021; Accepted 12 December 2021
Available online 3 January 2022

and curvature tensors of the deformed surface that must be met to ensure the existence of a deformation and, thus, constitute compatibility conditions. We find that a deformation serves effectively as a chart that describes the deformed surface in terms of the reference surface. However, there are three significant differences between a deformation and a coordinate chart. First, whereas a coordinate chart involves a bijection between a set of parameter pairs that can be identified with an open subset of $\mathbb{R}^2$ and a possibly curved surface, a deformation is a bijection between reference and deformed surfaces that may both be curved. Second, whereas the introduction of a chart entails the provision of a basis and, thus, dependence upon the associated coordinates, a deformation can be considered independent of any choice of coordinates, as we have done in the present work. Third, whereas there is no unique way of selecting a coordinate chart to describe a surface as a purely geometrical object and the parameter pairs of a coordinate chart need not be related to the points that describe a material surface, a deformation must preserve the identity of material points that comprise a material surface and, thus, supplies a natural correspondence between the reference and deformed configurations of a material surface.

As an initial application, we use our framework to derive representations for the covariant gradient operator on material surfaces that are subject to deformations which preserve angles and to deformations which preserve lengths. Building on those results, we next demonstrate that a deformation of a material surface that preserves both angles and areas must be length preserving and vice versa. In another application, we show how our framework can be applied to obtain an alternative derivation of the Peterson–Mainardi–Codazzi and Gauss equations that holds when the surface metric is derived from the metric for the three-dimensional Euclidean space in which the surface is embedded. This derivation exposes the unity of the Peterson–Mainardi–Codazzi and Gauss equations in that these two conditions are found to stem from the normal and tangential components of a single equation. Moreover, this alternative derivation yields the Gauss equation in a form commonly encountered in the literature. However, that condition is expressed in direct notation rather than relative to the associated coordinates.

To connect with traditional approaches to covariant differentiation, we include an appendix containing coordinate-based representations. Those representations hinge on the classical differential geometric notion of a coordinate chart.

2. Preliminaries from linear algebra

Given linear spaces $\mathcal{X}$ and $\mathcal{Y}$, let $\mathrm{Lin}(\mathcal{X},\mathcal{Y})$ denote the set of all linear mappings from $\mathcal{X}$ to $\mathcal{Y}$. If $\mathcal{X}$ and $\mathcal{Y}$ are inner-product spaces, then the transpose $\mathbf{M}^{\top}$ of $\mathbf{M}\in\mathrm{Lin}(\mathcal{X},\mathcal{Y})$ is an element of $\mathrm{Lin}(\mathcal{Y},\mathcal{X})$ and is determined by

$$\langle \mathbf{y},\mathbf{M}\mathbf{x}\rangle_{\mathcal{Y}}=\langle \mathbf{x},\mathbf{M}^{\top}\mathbf{y}\rangle_{\mathcal{X}},\qquad \mathbf{x}\in\mathcal{X},\ \mathbf{y}\in\mathcal{Y}. \tag{1}$$

Given $\mathbf{x}\in\mathcal{X}$ and $\mathbf{y}\in\mathcal{Y}$, the tensor product $\mathbf{y}\otimes\mathbf{x}\in\mathrm{Lin}(\mathcal{X},\mathcal{Y})$ is defined by

$$(\mathbf{y}\otimes\mathbf{x})\mathbf{z}=\langle \mathbf{x},\mathbf{z}\rangle_{\mathcal{X}}\mathbf{y},\qquad \mathbf{z}\in\mathcal{X}. \tag{2}$$

If $\mathcal{Y}=\mathcal{X}$ and $\mathbf{M}^{\top}=\mathbf{M}$, then $\mathbf{M}\in\mathrm{Sym}(\mathcal{X},\mathcal{X})\subseteq\mathrm{Lin}(\mathcal{X},\mathcal{X})$ and is said to be symmetric. The wedge product $\mathbf{x}_1\wedge\mathbf{x}_2\in\mathrm{Lin}(\mathcal{X},\mathcal{X})$ of $\mathbf{x}_1\in\mathcal{X}$ and $\mathbf{x}_2\in\mathcal{X}$ is defined by

$$\mathbf{x}_1\wedge\mathbf{x}_2=\mathbf{x}_1\otimes\mathbf{x}_2-\mathbf{x}_2\otimes\mathbf{x}_1. \tag{3}$$

If $\mathbb{A}\in\mathrm{Lin}(\mathcal{X},\mathrm{Lin}(\mathcal{X},\mathcal{Y}))$, then its right transpose $\mathbb{A}^{t}$ is a linear mapping of the same type defined by

$$(\mathbb{A}^{t}\mathbf{x}_1)\mathbf{x}_2=(\mathbb{A}\mathbf{x}_2)\mathbf{x}_1,\qquad \mathbf{x}_1,\mathbf{x}_2\in\mathcal{X}. \tag{4}$$

It is alternatively possible to interpret $\mathbb{A}$ as a bilinear mapping from $\mathcal{X}$ to $\mathcal{Y}$ through the relation

$$\mathbb{A}(\mathbf{x}_1,\mathbf{x}_2)=(\mathbb{A}\mathbf{x}_1)\mathbf{x}_2,\qquad \mathbf{x}_1,\mathbf{x}_2\in\mathcal{X}. \tag{5}$$

We use $\mathrm{Lin}_2(\mathcal{X},\mathcal{Y})$ to denote the set of all such bilinear mappings. Also, if $\mathbb{A}\in\mathrm{Lin}_2(\mathcal{X},\mathcal{Y})$ satisfies $\mathbb{A}^{t}=\mathbb{A}$, then $\mathbb{A}\in\mathrm{Sym}_2(\mathcal{X},\mathcal{Y})\subseteq\mathrm{Lin}_2(\mathcal{X},\mathcal{Y})$ and is called right symmetric.

Suppose that $\mathcal{X}$ is a two-dimensional inner-product space and that $\mathbf{M}\in\mathrm{Lin}(\mathcal{X},\mathcal{X})$. The determinant $\det\mathbf{M}$ of $\mathbf{M}$ is then characterized by the property:

$$\mathbf{M}\mathbf{x}_1\wedge\mathbf{M}\mathbf{x}_2=(\det\mathbf{M})(\mathbf{x}_1\wedge\mathbf{x}_2),\qquad \mathbf{x}_1,\mathbf{x}_2\in\mathcal{X}. \tag{6}$$

If, moreover, $\mathcal{Y}$ is a two-dimensional inner-product space and $\mathbf{N}\in\mathrm{Lin}(\mathcal{Y},\mathcal{X})$, then the wedge product on the left-hand side of (6) obeys

$$\begin{aligned}(\mathbf{N}^{\top}\mathbf{M}\mathbf{N})\mathbf{y}_1\wedge(\mathbf{N}^{\top}\mathbf{M}\mathbf{N})\mathbf{y}_2&=\mathbf{N}^{\top}(\mathbf{M}\mathbf{N}\mathbf{y}_1\wedge\mathbf{M}\mathbf{N}\mathbf{y}_2)\mathbf{N}\\&=(\det\mathbf{M})\mathbf{N}^{\top}(\mathbf{N}\mathbf{y}_1\wedge\mathbf{N}\mathbf{y}_2)\mathbf{N}\\&=(\det\mathbf{M})(\mathbf{N}^{\top}\mathbf{N})\mathbf{y}_1\wedge(\mathbf{N}^{\top}\mathbf{N})\mathbf{y}_2\\&=(\det\mathbf{M})\det(\mathbf{N}^{\top}\mathbf{N})\mathbf{y}_1\wedge\mathbf{y}_2\end{aligned} \tag{7}$$

for all choices of $\mathbf{y}_1\in\mathcal{Y}$ and $\mathbf{y}_2\in\mathcal{Y}$, with the consequence that the determinant of $\mathbf{N}^{\top}\mathbf{M}\mathbf{N}$ is given by the product of the determinants of $\mathbf{M}\in\mathrm{Lin}(\mathcal{X},\mathcal{X})$ and $\mathbf{N}^{\top}\mathbf{N}\in\mathrm{Lin}(\mathcal{X},\mathcal{X})$:

$$\det(\mathbf{N}^{\top}\mathbf{M}\mathbf{N})=(\det\mathbf{M})\det(\mathbf{N}^{\top}\mathbf{N}). \tag{8}$$

3. Surfaces in a Euclidean space

Consider a smooth surface $\mathcal{S}$ in a three-dimensional Euclidean point space $\mathcal{E}$ with associated vector space $\mathcal{V}$. Let the inner product of two vectors $\mathbf{a}\in\mathcal{V}$ and $\mathbf{b}\in\mathcal{V}$ be denoted by $\mathbf{a}\cdot\mathbf{b}$. For each $x\in\mathcal{S}$, the tangent space $T_x\mathcal{S}$ of $\mathcal{S}$ is a two-dimensional subspace of $\mathcal{V}$. We write $\mathbf{P}(x)\in\mathrm{Lin}(\mathcal{V},\mathcal{V})$ for the orthogonal projection of $\mathcal{V}$ onto $T_x\mathcal{S}$.

The tangent space $T_x\mathcal{S}$ can be used to parameterize the surface $\mathcal{S}$ local to $x\in\mathcal{S}$. More precisely, there is a neighborhood $\mathcal{N}_x$ of zero in $T_x\mathcal{S}$ and a smooth injective function $\pi_x:\mathcal{N}_x\to\mathcal{E}$ such that $\pi_x(\mathcal{N}_x)$ constitutes a neighborhood of x in $\mathcal{S}$. A function ϕ from $\mathcal{S}$ to a Euclidean space $\mathcal{F}$ with vector space $\mathcal{W}$ is said to be differentiable at x if $\phi\circ\pi_x$ is differentiable, and the surface gradient $\nabla^{\mathcal{S}}\phi$ of ϕ is defined by

$$\nabla^{\mathcal{S}}\phi(x)=\mathrm{grad}(\phi\circ\pi_x)(0)\in\mathrm{Lin}(T_x\mathcal{S},\mathcal{W}). \tag{9}$$

We consider only functions on $\mathcal{S}$ that are smooth in the sense that they can be differentiated as many times as needed.

It is sometimes convenient to view $\nabla^{\mathcal{S}}\phi(x)$ as a linear mapping on $\mathcal{V}$, rather than on the subspace $T_x\mathcal{S}$. This can be achieved by stipulating that $\nabla^{\mathcal{S}}\phi(x)$ vanish on the orthogonal complement of $T_x\mathcal{S}$. Put another way, we may identify $\nabla^{\mathcal{S}}\phi$ with the product $(\nabla^{\mathcal{S}}\phi)\mathbf{P}$. For the particular choice $\mathcal{F}=\mathbb{R}$, since $T_x\mathcal{S}$ is a subspace of $\mathcal{V}$, the inner product on $\mathcal{V}$ can be used to identify $\nabla^{\mathcal{S}}\phi(x)\in\mathrm{Lin}(T_x\mathcal{S},\mathbb{R})$ with an element of $T_x\mathcal{S}$ such that

$$(\nabla^{\mathcal{S}}\phi(x))\cdot\mathbf{a}=(\nabla^{\mathcal{S}}\phi(x))\mathbf{a},\qquad \mathbf{a}\in T_x\mathcal{S}. \tag{10}$$

A vector field on $\mathcal{S}$ is a function of the form $\mathbf{u}:\mathcal{S}\to\mathcal{V}$ and is called tangential if $\mathbf{u}(x)\in T_x\mathcal{S}$ for all $x\in\mathcal{S}$. Given a tangential vector field $\mathbf{u}$ and a field ϕ, the directional derivative $\nabla^{\mathcal{S}}_{\mathbf{u}}\phi$ of ϕ along $\mathbf{u}$ is the field defined such that

$$\nabla^{\mathcal{S}}_{\mathbf{u}}\phi=(\nabla^{\mathcal{S}}\phi)\mathbf{u}. \tag{11}$$

We will use the notation:

$\mathcal{D}$	scalar fields,
$\mathcal{D}(\mathcal{V})$	vector fields,
$\mathcal{D}(T\mathcal{S})$	tangential vector fields.

A two-tensor field $\mathbf{M}$ on $\mathcal{S}$ can be defined as a mapping from $\mathcal{D}(\mathcal{V})$ to itself that is $\mathcal{D}$-linear in the sense that

$$\mathbf{M}(g\mathbf{u}+h\mathbf{v})=g\mathbf{M}\mathbf{u}+h\mathbf{M}\mathbf{v},\qquad g,h\in\mathcal{D},\ \mathbf{u},\mathbf{v}\in\mathcal{D}(\mathcal{V}). \tag{12}$$

It follows that $\mathbf{M}$ can be viewed as a function on $\mathcal{S}$ defined such that for each $x\in\mathcal{S}$, $\mathbf{M}(x)\in\mathrm{Lin}(\mathcal{V},\mathcal{V})$. Any such function $\mathbf{M}$ is called tangential if it takes tangential vector fields to tangential vector fields. In this case, $\mathbf{M}(x) \in \mathrm{Lin}(T_x\mathcal{S}, T_x\mathcal{S})$. We use the following notation:

$\mathcal{D}_2(\mathcal{V})$ two-tensor fields,
$\mathcal{D}_2(T\mathcal{S})$ tangential two-tensor fields.

Lastly, we define a three-tensor $\mathbb{A}$ to be a $\mathcal{D}$-linear mapping from $\mathcal{D}(\mathcal{V})$ to $\mathcal{D}_2(\mathcal{V})$, call such a mapping tangential if it takes elements of $\mathcal{D}(T\mathcal{S})$ to elements of $\mathcal{D}_2(T\mathcal{S})$, and adopt the notation

$\mathcal{D}_3(\mathcal{V})$ three-tensor fields,
$\mathcal{D}_3(T\mathcal{S})$ tangential three-tensor fields.

Given the surface gradient $\nabla^{\mathcal{S}}\mathbf{M}$ of $\mathbf{M} \in \mathcal{D}_2(T\mathcal{S})$, we define its surface curl, $\mathrm{Curl}^{\mathcal{S}}\mathbf{M} \in \mathcal{D}_3(\mathcal{V})$, by

$$\mathrm{Curl}^{\mathcal{S}}\mathbf{M} = \nabla^{\mathcal{S}}\mathbf{M} - (\nabla^{\mathcal{S}}\mathbf{M})^t. \tag{13}$$

Suppose that $\mathcal{S}$ is orientable, so that there is a $\mathbf{n} \in \mathcal{D}(\mathcal{V})$ that is unit-vector valued and is orthogonal to $\mathcal{S}$. The orthogonal projection $\mathbf{P}$, which at each $x \in \mathcal{S}$ maps $\mathcal{V}$ onto $T_x\mathcal{S}$, can then be expressed in terms of $\mathbf{n}$ through

$$\mathbf{P} = \mathbf{1} - \mathbf{n} \otimes \mathbf{n}. \tag{14}$$

Moreover, the curvature tensor $\mathbf{L}$ of $\mathcal{S}$ is defined by

$$\mathbf{L} = -\nabla^{\mathcal{S}}\mathbf{n}. \tag{15}$$

Taking the surface gradient of the equation $\mathbf{n} \cdot \mathbf{n} = 1$ yields

$$\mathbf{L}^{\top}\mathbf{n} = \mathbf{0}, \tag{16}$$

from which we see that $\mathbf{L}$ is tangential. Until further notice, let $x \in \mathcal{S}$ be fixed, consider $\pi_x : \mathcal{N}_x \to \mathcal{E}$, and define $\boldsymbol{\Pi}_x = \mathrm{grad}\, \pi_x$. Since the range of π_x is contained in $\mathcal{S}$, it follows that $\boldsymbol{\Pi}_x\mathbf{a}$ is tangent to $\mathcal{S}$ for any $\mathbf{a} \in T_x\mathcal{S}$. Then, for any $\mathbf{a} \in T_x\mathcal{S}$,

$$\mathbf{n}(\pi_x(\mathbf{p})) \cdot \boldsymbol{\Pi}_x(\mathbf{p})\mathbf{a} = 0, \qquad \mathbf{p} \in \mathcal{N}_x. \tag{17}$$

Taking the gradient with respect to $\mathbf{p}$ in the direction $\mathbf{b} \in T_x\mathcal{S}$ of the previous equation produces the identity

$$\mathbf{a} \cdot \boldsymbol{\Pi}_x^{\top}(\mathbf{p})\mathbf{L}(\pi_x(\mathbf{p}))\boldsymbol{\Pi}_x(\mathbf{p})\mathbf{b} = \mathbf{n}(\pi_x(\mathbf{p})) \cdot [(\mathrm{grad}\, \boldsymbol{\Pi}_x)(\pi_x(\mathbf{p}))\mathbf{a}]. \tag{18}$$

Since $\mathrm{grad}\, \boldsymbol{\Pi}_x$ is the second gradient of π_x, it is symmetric. Thus, it follows from (18) that $\mathbf{L}$ is symmetric. The mean and Gaussian curvatures H and K of $\mathcal{S}$ are defined through the curvature tensor $\mathbf{L}$ of $\mathcal{S}$ by

$$H = \frac{1}{2}\mathrm{tr}\,\mathbf{L} \qquad \text{and} \qquad K = \det \mathbf{L}. \tag{19}$$

Notice that $\mathcal{D}(T\mathcal{S}) \subseteq \mathcal{D}(\mathcal{V})$ and thus that the surface gradient of a tangential vector $\mathbf{u}$ must satisfy

$$\nabla^{\mathcal{S}}\mathbf{u}(x) \in \mathrm{Lin}(T_x\mathcal{S}, \mathcal{V}), \qquad x \in \mathcal{S}. \tag{20}$$

From (20), we see that the surface gradient $\nabla^{\mathcal{S}}\mathbf{u}$ of a tangential vector field $\mathbf{u}$ need not be tangential. To elaborate on this, taking the surface gradient of the equation $\mathbf{u} \cdot \mathbf{n} = 0$ yields

$$(\nabla^{\mathcal{S}}\mathbf{u})^{\top}\mathbf{n} = \mathbf{L}\mathbf{u}, \tag{21}$$

from which it follows that

$$\nabla^{\mathcal{S}}\mathbf{u} = \mathbf{P}\nabla^{\mathcal{S}}\mathbf{u} + \mathbf{n} \otimes \mathbf{L}\mathbf{u}. \tag{22}$$

While this shows that generally the surface gradient of a tangential vector fields is not tangential, it is possible to combine two tangential vector fields with the surface gradient to obtain a tangential vector field. To verify this assertion, we observe from the product rule and the symmetry of $\mathbf{L}$ that for any $\mathbf{u} \in \mathcal{D}(T\mathcal{S})$ and $\mathbf{v} \in \mathcal{D}(T\mathcal{S})$,

$$\begin{aligned}\mathbf{n} \cdot (\nabla^{\mathcal{S}}_{\mathbf{u}}\mathbf{v} - \nabla^{\mathcal{S}}_{\mathbf{v}}\mathbf{u}) &= \nabla^{\mathcal{S}}_{\mathbf{u}}(\mathbf{n} \cdot \mathbf{v}) - \mathbf{v} \cdot \nabla^{\mathcal{S}}_{\mathbf{u}}\mathbf{n} - \nabla^{\mathcal{S}}_{\mathbf{v}}(\mathbf{n} \cdot \mathbf{u}) + \mathbf{u} \cdot \nabla^{\mathcal{S}}_{\mathbf{v}}\mathbf{n} \\ &= \mathbf{v} \cdot \mathbf{L}\mathbf{u} - \mathbf{u} \cdot \mathbf{L}\mathbf{v} \\ &= 0. \end{aligned} \tag{23}$$

Thus, the vector field $[\mathbf{u}, \mathbf{v}]$ defined by

$$[\mathbf{u}, \mathbf{v}] = \nabla^{\mathcal{S}}_{\mathbf{u}}\mathbf{v} - \nabla^{\mathcal{S}}_{\mathbf{v}}\mathbf{u}, \tag{24}$$

and known as the Lie bracket of $\mathbf{u}$ and $\mathbf{v}$, is tangential.

Given a tangential two-tensor field $\mathbf{M}$ defined on $\mathcal{S}$, it is possible to identify $\mathbf{M}$ with an element of $\mathbf{M} \in \mathcal{D}_2(\mathcal{V})$. This identification is accomplished by first extending $\mathbf{M}(x)$ to act on any element of $\mathcal{V}$ with the stipulation that it annihilate any vector orthogonal to $T_x\mathcal{S}$ and by recalling $T_x\mathcal{S}$ as a subspace of $\mathcal{V}$. In accord with this convention,

$$\mathbf{M} = \mathbf{M}\mathbf{P}. \tag{25}$$

Moreover, the surface gradient $\nabla^{\mathcal{S}}\mathbf{M}$ of $\mathbf{M}$ is the three-tensor field defined such that

$$\nabla^{\mathcal{S}}\mathbf{M}(x) \in \mathrm{Lin}(T_x\mathcal{S}, \mathrm{Lin}(\mathcal{V}, \mathcal{V})), \qquad x \in \mathcal{S}. \tag{26}$$

Taking the surface gradient in the direction $\mathbf{u} \in \mathcal{D}(T\mathcal{S})$ of (25) and utilizing the product rule, the representation (14) for the orthogonal projection $\mathbf{P}$ onto $\mathcal{S}$, and the definition (15) of the curvature tensor $\mathbf{L}$ of $\mathcal{S}$ produces the relation

$$\nabla^{\mathcal{S}}_{\mathbf{u}}\mathbf{M} = (\nabla^{\mathcal{S}}_{\mathbf{u}}\mathbf{M})\mathbf{P} + \mathbf{M}\mathbf{L}\mathbf{u} \otimes \mathbf{n}. \tag{27}$$

Thus, as with the surface gradient of a tangential vector field defined on $\mathcal{S}$, the surface gradient $\nabla^{\mathcal{S}}\mathbf{M}$ of a tangential two-tensor field $\mathbf{M}$ defined on $\mathcal{S}$ need not be tangential.

Remark 1. The identifications that allow us to compute the surface gradient of a tangential two-tensor are not always used. Specifically, given $x \in \mathcal{S}$, Gurtin and Murdoch (1975) introduce the inclusion map $\mathbf{I}(x) \in \mathrm{Lin}(T_x\mathcal{S}, \mathcal{V})$,[1] as defined by

$$\mathbf{I}(x)\mathbf{a} = \mathbf{a}, \qquad \mathbf{a} \in T_x\mathcal{S}, \tag{28}$$

and a projection map $\mathbf{P}(x) \in \mathrm{Lin}(\mathcal{V}, T_x\mathcal{S})$ that resembles the projection used in this paper except that its codomain is $T_x\mathcal{S}$ rather than $\mathcal{V}$. Given a tangential two-tensor field $\mathbf{M} \in \mathcal{D}_2(T\mathcal{S})$, the fields $\mathbf{I}$ and $\mathbf{P}$ so defined can be used to define $\hat{\mathbf{M}} = \mathbf{IMP} \in \mathcal{D}_2(\mathcal{V})$. With these provisions, the surface gradient $\nabla^{\mathcal{S}}\mathbf{M}$ of $\mathbf{M}$ is the three-tensor field defined such that

$$\nabla^{\mathcal{S}}\mathbf{M} = \nabla^{\mathcal{S}}\hat{\mathbf{M}}. \tag{29}$$

Although the approach adopted in this paper is less precise than that relying on the inclusion and projection mappings, it is less cumbersome from a notational perspective. We thus opt for simplicity of presentation over mathematical precision. As an outcome of this compromise, it is, however, essential to apply (27) when taking the surface gradient of a tangential two-tensor field.

4. Metrics

A metric on $\mathcal{S}$ is a rule that smoothly assigns to each $x \in \mathcal{S}$ an inner product $\langle \cdot, \cdot \rangle_{T_x\mathcal{S}}$ on the tangent space $T_x\mathcal{S}$ of $\mathcal{S}$ at x. Here, the notion of smoothness is embodied by the requirement that the scalar field $x \mapsto \langle \mathbf{u}, \mathbf{v} \rangle_{T_x\mathcal{S}}$ be smooth for all $\mathbf{u} \in \mathcal{D}(T\mathcal{S})$ and $\mathbf{v} \in \mathcal{D}(T\mathcal{S})$. Since $T_x\mathcal{S}$ is a subspace of $\mathcal{V}$, the inner product on $\mathcal{V}$ can be used to define a metric on $\mathcal{S}$. A metric so defined is referred to as "induced". Intuitively, this induced surface metric can be considered as a metric that derives from the metric for the space $\mathcal{E}$ in which the surface is embedded. It follows that any metric on $\mathcal{S}$ has associated with it a tangential two-tensor field $\mathbf{G}$ that satisfies

$$\langle \mathbf{a}, \mathbf{b} \rangle_{T_x\mathcal{S}} = \mathbf{a} \cdot \mathbf{G}(x)\mathbf{b}, \qquad \mathbf{a}, \mathbf{b} \in T_x\mathcal{S}, \tag{30}$$

and is called a metric tensor. As a consequence of (30), $\mathbf{G}(x) \in \mathrm{Sym}(T_x\mathcal{S}, T_x\mathcal{S})$ and must satisfy the inequality $\mathbf{a} \cdot \mathbf{G}(x)\mathbf{a} > 0$ for every

[1] See also Murdoch (1978), Murdoch and Cohen (1979), and Murdoch (1990).

nonzero $\mathbf{a} \in T_x\mathcal{S}$. These conditions imply that $\mathbf{G}(x)$ is invertible. If, in particular, the metric is the induced metric, then $\mathbf{G} = \mathbf{P}$.

Since the values of $\mathbf{G}$ are symmetric, the identity

$$\mathbf{u} \cdot \mathbf{G}\mathbf{v} = \mathbf{v} \cdot \mathbf{G}\mathbf{u} \tag{31}$$

must hold for all $\mathbf{u} \in \mathcal{D}(T\mathcal{S})$ and $\mathbf{v} \in \mathcal{D}(T\mathcal{S})$. Applying the surface gradient in the direction $\mathbf{w} \in \mathcal{D}(T\mathcal{S})$ to both sides of (31), setting

$$\mathbb{G} = \nabla^{\mathcal{S}}\mathbf{G}, \tag{32}$$

and invoking the symmetry of $\mathbf{G}$, we see that $\mathbb{G}$ must satisfy

$$\mathbf{u} \cdot (\mathbb{G}\mathbf{w})\mathbf{v} = \mathbf{v} \cdot (\mathbb{G}\mathbf{w})\mathbf{u} \tag{33}$$

for all $\mathbf{u} \in \mathcal{D}(T\mathcal{S})$, $\mathbf{v} \in \mathcal{D}(T\mathcal{S})$, and $\mathbf{w} \in \mathcal{D}(T\mathcal{S})$.

Given a metric on $\mathcal{S}$ with metric tensor $\mathbf{G}$, the inner product of $\mathbf{a} \in T_x\mathcal{S}$ and $\mathbf{b} \in T_x\mathcal{S}$ can be computed in two different ways: $\mathbf{a} \cdot \mathbf{b}$ and $\langle \mathbf{a}, \mathbf{b} \rangle_{T_x\mathcal{S}} = \mathbf{a} \cdot \mathbf{G}(x)\mathbf{b}$. Because of this, the meaning of the transpose of a two-tensor $\mathbf{M} \in \mathrm{Lin}(T_x\mathcal{S}, T_x\mathcal{S})$ suffers from ambiguity. Specifically, there exist linear mappings $\mathbf{M}_1$ and $\mathbf{M}_2$ on $T_x\mathcal{S}$ that satisfy

$$\mathbf{a} \cdot \mathbf{M}\mathbf{b} = \mathbf{b} \cdot \mathbf{M}_1\mathbf{a} \qquad \text{and} \qquad \langle \mathbf{a}, \mathbf{M}\mathbf{b} \rangle_{T_x\mathcal{S}} = \langle \mathbf{b}, \mathbf{M}_2\mathbf{a} \rangle_{T_x\mathcal{S}}, \tag{34}$$

for all $\mathbf{a} \in T_x\mathcal{S}$ and $\mathbf{b} \in T_x\mathcal{S}$. While $\mathbf{M}_1$ is the transpose of $\mathbf{M}$ relative to the induced inner product on $T_x\mathcal{S}$, $\mathbf{M}_2$ is the transpose of $\mathbf{M}$ relative to the inner product $\langle \cdot, \cdot \rangle_{T_x\mathcal{S}}$. Using (30), $(34)_2$ can be written as

$$\mathbf{a} \cdot \mathbf{G}(x)\mathbf{M}\mathbf{b} = \mathbf{b} \cdot \mathbf{G}(x)\mathbf{M}_2\mathbf{a}, \qquad \mathbf{a}, \mathbf{b} \in T_x\mathcal{S}. \tag{35}$$

Since $\mathbf{G}(x)$ is invertible and symmetric, (35) is equivalent to

$$\mathbf{a} \cdot \mathbf{M}\mathbf{b} = \mathbf{b} \cdot \mathbf{G}(x)\mathbf{M}_2\mathbf{G}(x)^{-1}\mathbf{a}, \qquad \mathbf{a}, \mathbf{b} \in T_x\mathcal{S}. \tag{36}$$

Comparing the previous equation with $(34)_1$ we see that $\mathbf{M}_1$ and $\mathbf{M}_2$ are related through

$$\mathbf{M}_1 = \mathbf{G}(x)\mathbf{M}_2\mathbf{G}(x)^{-1}. \tag{37}$$

Hereafter, we exclusively use the transpose relative to the induced inner product on $T_x\mathcal{S}$ and employ the notation $\mathbf{M}^{\top} = \mathbf{M}_1$ to denote said transpose. The notion of the magnitude of a tangent vector also suffers from an ambiguity. For $\mathbf{a} \in T_x\mathcal{S}$, we adopt the notation

$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} \qquad \text{and} \qquad |\mathbf{a}|_{\mathbf{G}} = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle_{T_x\mathcal{S}}}. \tag{38}$$

5. Covariant gradient and covariant curl

Although the surface gradient $\nabla^{\mathcal{S}}$ of a tangential vector field need not be tangential, there is, given a metric on $\mathcal{S}$ with associated metric tensor $\mathbf{G}$, another object that delivers a tangential two-tensor field when applied to a tangential vector field. Termed the covariant gradient relative to a metric $\mathbf{G}$ and denoted by $\nabla^{\mathbf{G}}$, that object is the unique mapping from $\mathcal{D}(T\mathcal{S})$ to $\mathcal{D}_2(T\mathcal{S})$ which, given $g \in \mathcal{D}$, $\mathbf{u} \in \mathcal{D}(T\mathcal{S})$, and $\mathbf{v} \in \mathcal{D}(T\mathcal{S})$, has the following properties:

(P1) $\nabla^{\mathbf{G}}(\mathbf{u} + \mathbf{v}) = \nabla^{\mathbf{G}}\mathbf{u} + \nabla^{\mathbf{G}}\mathbf{v}$,
(P2) $\nabla^{\mathbf{G}}(g\mathbf{u}) = g\nabla^{\mathbf{G}}\mathbf{u} + \mathbf{u} \otimes \nabla^{\mathcal{S}}g$,
(P3) $\nabla^{\mathbf{G}}_{\mathbf{v}}\mathbf{u} - \nabla^{\mathbf{G}}_{\mathbf{u}}\mathbf{v} = [\mathbf{v}, \mathbf{u}]$,
(P4) $\nabla^{\mathcal{S}}_{\mathbf{w}}\langle \mathbf{u}, \mathbf{v} \rangle = \langle \nabla^{\mathbf{G}}_{\mathbf{w}}\mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \nabla^{\mathbf{G}}_{\mathbf{w}}\mathbf{v} \rangle$.

While property P1 ensures that $\nabla^{\mathbf{G}}$ is additive, property P2 is a product rule for $\nabla^{\mathbf{G}}$. Together, P1 and P2 imply that $\nabla^{\mathbf{G}}$ is linear, though it should be noted that $\nabla^{\mathbf{G}}$ is not $\mathcal{D}$-linear and, thus, that $\nabla^{\mathbf{G}}$ should not be misconstrued as a three-tensor. It will become evident that P3 encompasses a notion of symmetry for $\nabla^{\mathbf{G}}$. Property P4 ensures that $\nabla^{\mathbf{G}}$ preserves the metric $\mathbf{G}$ on $\mathcal{S}$.

We next obtain an expression for $\nabla^{\mathbf{G}}$ in terms of the metric tensor $\mathbf{G}$. Given $\mathbf{u} \in \mathcal{D}(T\mathcal{S})$, $\mathbf{v} \in \mathcal{D}(T\mathcal{S})$, and $\mathbf{w} \in \mathcal{D}(T\mathcal{S})$, we may apply property P4 of $\nabla^{\mathbf{G}}$ to give

$$\left.\begin{aligned} \nabla^{\mathcal{S}}_{\mathbf{w}}\langle \mathbf{u}, \mathbf{v} \rangle &= \langle \nabla^{\mathbf{G}}_{\mathbf{w}}\mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \nabla^{\mathbf{G}}_{\mathbf{w}}\mathbf{v} \rangle, \\ \nabla^{\mathcal{S}}_{\mathbf{v}}\langle \mathbf{w}, \mathbf{u} \rangle &= \langle \nabla^{\mathbf{G}}_{\mathbf{v}}\mathbf{w}, \mathbf{u} \rangle + \langle \mathbf{w}, \nabla^{\mathbf{G}}_{\mathbf{v}}\mathbf{u} \rangle, \\ \nabla^{\mathcal{S}}_{\mathbf{u}}\langle \mathbf{v}, \mathbf{w} \rangle &= \langle \nabla^{\mathbf{G}}_{\mathbf{u}}\mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{v}, \nabla^{\mathbf{G}}_{\mathbf{u}}\mathbf{w} \rangle. \end{aligned}\right\} \tag{39}$$

Subtracting $(39)_1$ from the sum of $(39)_2$ and $(39)_3$, recalling from (32) that $\nabla^{\mathcal{S}}\mathbf{G} = \mathbb{G}$, and using the consequences

$$\left.\begin{aligned} \nabla^{\mathcal{S}}_{\mathbf{w}}\langle \mathbf{u}, \mathbf{v} \rangle &= \nabla^{\mathcal{S}}_{\mathbf{w}}(\mathbf{u} \cdot \mathbf{G}\mathbf{v}) \\ &= \nabla^{\mathcal{S}}_{\mathbf{w}}\mathbf{u} \cdot \mathbf{G}\mathbf{v} + \mathbf{u} \cdot (\mathbb{G}\mathbf{w})\mathbf{v} + \mathbf{u} \cdot \mathbf{G}\nabla^{\mathcal{S}}_{\mathbf{w}}\mathbf{v} \\ &= \langle \mathbf{P}\nabla^{\mathcal{S}}_{\mathbf{w}}\mathbf{u}, \mathbf{v} \rangle + \mathbf{u} \cdot (\mathbb{G}\mathbf{w})\mathbf{v} + \langle \mathbf{u}, \mathbf{P}\nabla^{\mathcal{S}}_{\mathbf{w}}\mathbf{v} \rangle, \\ \nabla^{\mathcal{S}}_{\mathbf{v}}\langle \mathbf{w}, \mathbf{u} \rangle &= \langle \mathbf{P}\nabla^{\mathcal{S}}_{\mathbf{v}}\mathbf{w}, \mathbf{u} \rangle + \mathbf{w} \cdot (\mathbb{G}\mathbf{v})\mathbf{u} + \langle \mathbf{w}, \mathbf{P}\nabla^{\mathcal{S}}_{\mathbf{v}}\mathbf{u} \rangle, \\ \nabla^{\mathcal{S}}_{\mathbf{u}}\langle \mathbf{v}, \mathbf{w} \rangle &= \langle \mathbf{P}\nabla^{\mathcal{S}}_{\mathbf{u}}\mathbf{v}, \mathbf{w} \rangle + \mathbf{v} \cdot (\mathbb{G}\mathbf{u})\mathbf{w} + \langle \mathbf{v}, \mathbf{P}\nabla^{\mathcal{S}}_{\mathbf{u}}\mathbf{w} \rangle \end{aligned}\right\} \tag{40}$$

of the product rule for $\nabla^{\mathcal{S}}$, we then find that

$$\begin{aligned} &\langle \nabla^{\mathbf{G}}_{\mathbf{v}}\mathbf{w} - \nabla^{\mathbf{G}}_{\mathbf{w}}\mathbf{v}, \mathbf{u} \rangle + \langle \nabla^{\mathbf{G}}_{\mathbf{u}}\mathbf{w} - \nabla^{\mathbf{G}}_{\mathbf{w}}\mathbf{u}, \mathbf{v} \rangle + \langle \nabla^{\mathbf{G}}_{\mathbf{v}}\mathbf{u} + \nabla^{\mathbf{G}}_{\mathbf{u}}\mathbf{v}, \mathbf{w} \rangle \\ &\quad = \langle [\mathbf{v}, \mathbf{w}], \mathbf{u} \rangle + \langle [\mathbf{u}, \mathbf{w}], \mathbf{v} \rangle + \langle \mathbf{P}(\nabla^{\mathcal{S}}_{\mathbf{v}}\mathbf{u} + \nabla^{\mathcal{S}}_{\mathbf{u}}\mathbf{v}), \mathbf{w} \rangle \\ &\qquad - \mathbf{u} \cdot (\mathbb{G}\mathbf{w})\mathbf{v} + \mathbf{w} \cdot (\mathbb{G}\mathbf{v})\mathbf{u} + \mathbf{v} \cdot (\mathbb{G}\mathbf{u})\mathbf{w}. \end{aligned} \tag{41}$$

From property P3 of $\nabla^{\mathbf{G}}$, we find that (41) reduces to

$$\langle \mathbf{w}, \nabla^{\mathbf{G}}_{\mathbf{v}}\mathbf{u} \rangle = \langle \mathbf{w}, \mathbf{P}\nabla^{\mathcal{S}}_{\mathbf{v}}\mathbf{u} \rangle + \frac{1}{2}(\mathbf{v} \cdot (\mathbb{G}\mathbf{u})\mathbf{w} + \mathbf{w} \cdot (\mathbb{G}\mathbf{v})\mathbf{u} - \mathbf{u} \cdot (\mathbb{G}\mathbf{w})). \tag{42}$$

Using (4) and (33), we notice that (42) can be written as

$$\langle \mathbf{w}, \nabla^{\mathbf{G}}_{\mathbf{v}}\mathbf{u} \rangle = \langle \mathbf{w}, \mathbf{P}\nabla^{\mathcal{S}}_{\mathbf{v}}\mathbf{u} \rangle + \frac{1}{2}\mathbf{w} \cdot [(\mathbb{G}\mathbf{u})\mathbf{v} + (\mathbb{G}^{t}\mathbf{u})\mathbf{v} - (\mathbb{G}^{t}\mathbf{u})^{\top}\mathbf{v}] \tag{43}$$

or, equivalently, as

$$\langle \mathbf{w}, \nabla^{\mathbf{G}}_{\mathbf{v}}\mathbf{u} \rangle = \langle \mathbf{w}, \mathbf{P}\nabla^{\mathcal{S}}_{\mathbf{v}}\mathbf{u} \rangle + \frac{1}{2}\langle \mathbf{w}, \mathbf{G}^{-1}[(\mathbb{G}\mathbf{u})\mathbf{v} + (\mathbb{G}^{t}\mathbf{u})\mathbf{v} - (\mathbb{G}^{t}\mathbf{u})^{\top}\mathbf{v}] \rangle. \tag{44}$$

Since (44) holds for all $\mathbf{v} \in \mathcal{D}(T\mathcal{S})$ and $\mathbf{w} \in \mathcal{D}(T\mathcal{S})$, we conclude that the covariant gradient $\nabla^{\mathbf{G}}\mathbf{u}$ of $\mathbf{u}$ admits a representation of the form

$$\nabla^{\mathbf{G}}\mathbf{u} = \mathbf{P}\nabla^{\mathcal{S}}\mathbf{u} + \frac{1}{2}\mathbf{G}^{-1}(\mathbb{G}\mathbf{u} + \mathbb{G}^{t}\mathbf{u} - (\mathbb{G}^{t}\mathbf{u})^{\top}). \tag{45}$$

We next define the tensor $\mathbb{A} \in \mathcal{D}_3(T\mathcal{S})$ by

$$(\mathbb{A}\mathbf{u})\mathbf{v} = \frac{1}{2}\mathbf{G}^{-1}(\mathbb{G}\mathbf{u} + \mathbb{G}^{t}\mathbf{u} - (\mathbb{G}^{t}\mathbf{u})^{\top})\mathbf{v}, \qquad \mathbf{u}, \mathbf{v} \in \mathcal{D}(T\mathcal{S}). \tag{46}$$

From (46) and property P3 of $\nabla^{\mathbf{G}}$, we observe that $\mathbb{A}(x) \in \mathrm{Sym}_2(T_x\mathcal{S}, T_x\mathcal{S})$, which justifies identifying property P3 as a symmetry condition. Finally, using the definition (46), we express (45) in the abbreviated form

$$\nabla^{\mathbf{G}}\mathbf{u} = (\mathbf{P}\nabla^{\mathcal{S}} + \mathbb{A})\mathbf{u}. \tag{47}$$

The covariant gradient $\nabla^{\mathbf{G}}\mathbf{M}$ of $\mathbf{M} \in \mathcal{D}_2(T\mathcal{S})$ is the tangential three-tensor defined such that

$$\nabla^{\mathbf{G}}_{\mathbf{u}}\mathbf{M} = \mathbf{P}(\nabla^{\mathcal{S}}_{\mathbf{u}}\mathbf{M})\mathbf{P} - (\mathbb{A}\mathbf{u})^{\top}\mathbf{M} - \mathbf{M}\mathbb{A}\mathbf{u}, \qquad \mathbf{u} \in \mathcal{D}(T\mathcal{S}). \tag{48}$$

The definition (48) ensures the satisfaction of the product rule

$$\nabla^{\mathcal{S}}_{\mathbf{w}}(\mathbf{u} \cdot \mathbf{M}\mathbf{v}) = \nabla^{\mathbf{G}}_{\mathbf{w}}\mathbf{u} \cdot \mathbf{M}\mathbf{v} + \mathbf{u} \cdot (\nabla^{\mathbf{G}}_{\mathbf{w}}\mathbf{M})\mathbf{v} + \mathbf{u} \cdot \mathbf{M}\nabla^{\mathbf{G}}_{\mathbf{w}}\mathbf{v}, \tag{49}$$

which generalizes property P4 of $\nabla^{\mathbf{G}}$. A simple but important consequence of the definition (48) is that

$$\nabla^{\mathbf{G}}\mathbf{G} = \mathbf{0}. \tag{50}$$

The covariant curl, $\mathrm{Curl}^{\mathbf{G}}\mathbf{M} \in \mathcal{D}_3(T\mathcal{S})$, of $\mathbf{M} \in \mathcal{D}_2(T\mathcal{S})$ is defined by

$$\mathrm{Curl}^{\mathbf{G}}\mathbf{M} = \nabla^{\mathbf{G}}\mathbf{M} - (\nabla^{\mathbf{G}}\mathbf{M})^{t} \tag{51}$$

and, thus, mimics the definition (13) of the surface curl of $\mathbf{M} \in \mathcal{D}_2(T\mathcal{S})$.

Consider the special case where the metric is the induced metric, so that $\mathbf{G}=\mathbf{P}$. Since

$$(\nabla^{\mathcal{S}}_{\mathbf{u}}\mathbf{P})\mathbf{v}=\mathbf{0},\qquad \mathbf{u},\mathbf{v}\in\mathcal{D}(T\mathcal{S}), \tag{52}$$

it follows from (45) and (46) that

$$\nabla^{\mathbf{P}}=\mathbf{P}\nabla^{\mathcal{S}}. \tag{53}$$

Moreover, from (13) and (51) we find that for any $\mathbf{M}\in\mathcal{D}_2(T\mathcal{S})$

$$(\mathrm{Curl}^{\mathbf{P}}\mathbf{M})(\mathbf{u},\mathbf{v})=\mathbf{P}(\mathrm{Curl}^{\mathcal{S}}\mathbf{M})(\mathbf{u},\mathbf{v}),\qquad \mathbf{u},\mathbf{v}\in\mathcal{D}(T\mathcal{S}). \tag{54}$$

Remark 2. Instead of defining the covariant gradient of $\mathbf{M}\in\mathcal{D}_2(T\mathcal{S})$ to ensure satisfaction of (49), it is possible to motivate a definition of $\nabla^{\mathbf{G}}\mathbf{M}$ that ensures satisfaction of the alternative product rule

$$\nabla^{\mathcal{S}}_{\mathbf{w}}\langle\mathbf{u},\mathbf{M}\mathbf{v}\rangle=\langle\nabla^{\mathbf{G}}_{\mathbf{w}}\mathbf{u},\mathbf{M}\mathbf{v}\rangle+\langle\mathbf{u},(\nabla^{\mathbf{G}}_{\mathbf{w}}\mathbf{M})\mathbf{v}\rangle+\langle\mathbf{u},\mathbf{M}\nabla^{\mathbf{G}}_{\mathbf{w}}\mathbf{v}\rangle. \tag{55}$$

This is the approach taken in standard treatments of Riemannian geometry — as expounded by do Carmo (1992), among others. Since we utilize the induced inner product, rather than the metric tensor, to identify the dual of each tangent space with itself (see (10)), we find it more natural to define $\nabla^{\mathbf{G}}\mathbf{M}$ so that (49), rather than (55), holds.

6. Deformation of a material surface

Viewing $\mathcal{S}$ as a material surface, we consider a deformation $f:\mathcal{S}\to\mathcal{E}$ of $\mathcal{S}$ to another surface, namely the set $\bar{\mathcal{S}}=f(\mathcal{S})$. For each $x\in\mathcal{S}$ the deformation gradient is defined through the surface gradient:

$$\mathbf{F}(x)=\nabla^{\mathcal{S}}f(x)\in\mathrm{Lin}(T_x\mathcal{S},\mathcal{V}). \tag{56}$$

Since the range of $\mathbf{F}(x)$ is $T_{f(x)}\bar{\mathcal{S}}$, it is possible to consider $\mathbf{F}(x)$ as a linear mapping from $T_x\mathcal{S}$ to $T_{f(x)}\bar{\mathcal{S}}$. In this context, $\mathbf{F}(x)$ is invertible. To take advantage of this property, we will consider $\mathbf{F}(x)$ as a mapping from one tangent space to another.

This deformation gradient can be used to induce a metric on $\mathcal{S}$ through

$$\langle\mathbf{a},\mathbf{b}\rangle_{T_x\mathcal{S}}=\mathbf{F}(x)\mathbf{a}\cdot\mathbf{F}(x)\mathbf{b}=\mathbf{a}\cdot\mathbf{F}^{\top}(x)\mathbf{F}(x)\mathbf{b},\qquad \mathbf{a},\mathbf{b}\in T_x\mathcal{S}. \tag{57}$$

The right Cauchy–Green tensor $\mathbf{C}=\mathbf{F}^{\top}\mathbf{F}\in\mathcal{D}_2(T\mathcal{S})$ is therefore the metric tensor that measures any length and angle changes caused by deforming $\mathcal{S}$ to $\bar{\mathcal{S}}$. This tensor can also be recognized as the pullback to $\mathcal{S}$ of the induced metric on $\bar{\mathcal{S}}$.

Now consider the surface gradient

$$\mathbb{F}=\nabla^{\mathcal{S}}\mathbf{F} \tag{58}$$

of $\mathbf{F}$. Since $\mathbb{F}$ is the second surface gradient of f, its restriction to tangent vectors is symmetric:

$$\mathbb{F}(x)\in\mathrm{Sym}_2(T_x\mathcal{S},\mathcal{V}),\qquad x\in\mathcal{S}. \tag{59}$$

Referring to (46), we find that the three-tensor $\mathbb{A}$ associated with the metric tensor $\mathbf{C}=\mathbf{F}^{\top}\mathbf{F}$ is given in terms of $\mathbf{F}$ and $\mathbb{F}$ such that

$$(\mathbb{A}\mathbf{u})\mathbf{v}=\mathbf{F}^{-1}(\mathbb{F}\mathbf{u})\mathbf{v},\qquad \mathbf{u},\mathbf{v}\in\mathcal{D}(T\mathcal{S}). \tag{60}$$

Suppose that $\mathcal{S}$ and $\bar{\mathcal{S}}$ are orientable and let $\bar{\mathbf{n}}\in\mathcal{D}(\mathcal{V})$ be a unit-normal vector field on $\bar{\mathcal{S}}$. Since $\mathbf{F}$ takes tangential vector fields on $\mathcal{S}$ to tangential vector fields on $\bar{\mathcal{S}}$, we have for any $\mathbf{u}\in\mathcal{D}(T\mathcal{S})$ that

$$\bar{\mathbf{n}}\cdot\mathbf{F}\mathbf{u}=0. \tag{61}$$

Applying the surface gradient in the direction $\mathbf{v}\in\mathcal{D}(T\mathcal{S})$ to both sides of (61) and using (61) to simplify the resulting identity yields

$$-\bar{\mathbf{L}}\mathbf{F}\mathbf{v}\cdot\mathbf{F}\mathbf{u}+\bar{\mathbf{n}}\cdot(\mathbb{F}\mathbf{v})\mathbf{u}=0, \tag{62}$$

where $\bar{\mathbf{L}}$ is the curvature tensor for $\bar{\mathcal{S}}$. It then follows from (59) that the tangential tensor field $\mathbf{H}\in\mathcal{D}_2(T\mathcal{S})$ defined by

$$\mathbf{H}(x)=\mathbf{F}^{\top}(x)\bar{\mathbf{L}}(f(x))\mathbf{F}(x),\qquad x\in\mathcal{S} \tag{63}$$

is related to $\mathbb{F}$ through

$$\mathbf{v}\cdot\mathbf{H}\mathbf{u}=\bar{\mathbf{n}}\cdot(\mathbb{F}\mathbf{v})\mathbf{u},\qquad \mathbf{u},\mathbf{v}\in\mathcal{D}(T\mathcal{S}). \tag{64}$$

As defined through (63), $\mathbf{H}$ is the pull-back, to $\mathcal{S}$, of the curvature tensor $\bar{\mathbf{L}}$ of $\bar{\mathcal{S}}$.

7. Versions of the Peterson–Mainardi–Codazzi and Gauss equations for a deformed material surface

In classical differential geometry, the induced metric $\mathbf{P}$ and curvature tensor $\mathbf{L}$ of a surface $\mathcal{S}$ can be represented as matrices relative to a basis induced by a coordinate chart.[2] The coefficients of those matrices must necessarily satisfy the Peterson–Mainardi–Codazzi and Gauss equations.[3] Conversely, if the coefficients of a pair of given matrices satisfy the Peterson–Mainardi–Codazzi and Gauss equations, then there is a surface and a coordinate chart for that surface such that the matrix representations of $\mathbf{P}$ and $\mathbf{L}$ relative to that coordinate chart are identical to the given matrices. Moreover, the surface so determined is unique up to an orientation preserving isometry of the space $\mathcal{E}$ in which it is embedded. We next derive counterparts of the Peterson–Mainardi–Codazzi and Gauss equations that, given a material surface with reference configuration $\mathcal{S}$, the referential surface metric $\mathbf{C}$ of $\mathcal{S}$ and the pullback $\mathbf{H}$, to $\mathcal{S}$, of the curvature tensor $\bar{\mathbf{L}}$ of a surface $\bar{\mathcal{S}}$ must satisfy to ensure the existence of deformation f from $\mathcal{S}$ to $\bar{\mathcal{S}}$.

Proposition 1. *The tensor* $\mathbf{H}$ *satisfies*

$$\mathrm{Curl}^{\mathbf{C}}\mathbf{H}=\mathbf{0}. \tag{65}$$

Proof. We begin by fixing $\mathbf{u}\in\mathcal{D}(T\mathcal{S})$, $\mathbf{v}\in\mathcal{D}(T\mathcal{S})$, and $\mathbf{w}\in\mathcal{D}(T\mathcal{S})$. Combining (48) and (60), we next obtain the relation

$$\mathbf{u}\cdot(\nabla^{\mathbf{C}}_{\mathbf{w}}\mathbf{H})\mathbf{v}=\mathbf{u}\cdot(\nabla^{\mathcal{S}}_{\mathbf{w}}\mathbf{H})\mathbf{v}-\mathbf{F}^{-1}(\mathbb{F}\mathbf{u})\mathbf{w}\cdot\mathbf{H}\mathbf{v}-\mathbf{u}\cdot\mathbf{H}\mathbf{F}^{-1}(\mathbb{F}\mathbf{v})\mathbf{w}. \tag{66}$$

Using (63), the chain rule, and the product rule, we find further that

$$(\nabla^{\mathcal{S}}_{\mathbf{w}}\mathbf{H})\mathbf{v}=(\mathbb{F}\mathbf{w})^{\top}\bar{\mathbf{L}}\mathbf{F}\mathbf{v}+\mathbf{F}^{\top}(\nabla^{\bar{\mathcal{S}}}_{\mathbf{F}\mathbf{w}}\bar{\mathbf{L}})\mathbf{F}\mathbf{v}+\mathbf{F}^{\top}\bar{\mathbf{L}}\mathbb{F}\mathbf{v}. \tag{67}$$

Since $\nabla^{\bar{\mathcal{S}}}\bar{\mathbf{L}}=-\nabla^{\bar{\mathcal{S}}}\nabla^{\bar{\mathcal{S}}}\bar{\mathbf{n}}$ is a second gradient, we notice that its values are symmetric in the sense that

$$(\nabla^{\bar{\mathcal{S}}}_{\mathbf{F}\mathbf{w}}\bar{\mathbf{L}})\mathbf{F}\mathbf{v}=(\nabla^{\bar{\mathcal{S}}}_{\mathbf{F}\mathbf{v}}\bar{\mathbf{L}})\mathbf{F}\mathbf{w}. \tag{68}$$

In view of (59) and the symmetry of $\mathbb{A}$, we infer from the previous equation that

$$\begin{aligned}&\mathbf{u}\cdot[(\nabla^{\mathbf{C}}_{\mathbf{w}}\mathbf{H})\mathbf{v}-(\nabla^{\mathbf{C}}_{\mathbf{v}}\mathbf{H})\mathbf{w}]\\&\quad=(\mathbb{F}\mathbf{w})\mathbf{u}\cdot\bar{\mathbf{L}}\mathbf{F}\mathbf{v}-(\mathbb{F}\mathbf{v})\mathbf{u}\cdot\bar{\mathbf{L}}\mathbf{F}\mathbf{w}-\mathbf{F}^{-1}(\mathbb{F}\mathbf{u})\mathbf{w}\cdot\mathbf{H}\mathbf{v}+\mathbf{F}^{-1}(\mathbb{F}\mathbf{u})\mathbf{v}\cdot\mathbf{H}\mathbf{w}.\end{aligned} \tag{69}$$

Finally, using the identities

$$\begin{aligned}\mathbf{F}^{-1}(\mathbb{F}\mathbf{u})\mathbf{w}\cdot\mathbf{H}\mathbf{v}&=\mathbf{F}^{-1}(\mathbb{F}\mathbf{u})\mathbf{w}\cdot\mathbf{F}^{\top}\bar{\mathbf{L}}\mathbf{F}\mathbf{v}\\&=(\mathbb{F}\mathbf{w})\mathbf{u}\cdot\bar{\mathbf{L}}\mathbf{F}\mathbf{v}\end{aligned} \tag{70}$$

and

$$\mathbf{F}^{-1}(\mathbb{F}\mathbf{u})\mathbf{v}\cdot\mathbf{H}\mathbf{w}=(\mathbb{F}\mathbf{v})\mathbf{u}\cdot\bar{\mathbf{L}}\mathbf{F}\mathbf{w} \tag{71}$$

to simplify (69), we obtain (65), which completes the proof. □

The mean $\bar{H}$ and Gaussian $\bar{K}$ curvatures of $\bar{\mathcal{S}}$ can be written in terms of $\mathbf{H}$ and $\mathbf{C}$ since

$$\bar{H}=\frac{1}{2}\mathrm{tr}\,\bar{\mathbf{L}}=\mathbf{H}\cdot\mathbf{C}^{-1} \tag{72}$$

[2] An overview of this approach is provided in the Appendix.

[3] Mainardi (1856) and Codazzi (1868–1869) obtained (65) independently while being unaware of an earlier derivation due to Peterson (1853); (73) was one of many results in Gauss' (1828) groundbreaking contribution to the differential geometry of curves and surfaces.

and, by (8),

$$\bar{K} = \frac{\det \mathbf{H}}{\det \mathbf{C}}. \tag{73}$$

In writing (72) and (73), it is important to note that $\mathbf{H}$ and $\mathbf{C}$ are considered as linear mappings from each tangent space to itself rather than as linear mappings on the ambient space $\mathcal{V}$. If $\mathbf{H}$ and $\mathbf{C}$ are viewed instead as linear mappings on $\mathcal{V}$, then $\mathbf{C}$ is no longer invertible and Eqs. (72) and (73) become problematic. In this case, it is possible to obtain $\bar{H}$ and $\bar{K}$ determined directly from $\bar{\mathbf{L}}$ through $\bar{H} = \frac{1}{2}\operatorname{tr}\bar{\mathbf{L}}$ and $\bar{K} = \frac{1}{2}[(\operatorname{tr}\bar{\mathbf{L}})^2 - \operatorname{tr}(\bar{\mathbf{L}}^2)]$.

If $\mathcal{S}$ is a smooth surface $\mathcal{S}$ with metric $\mathbf{G}$ and there is a symmetric $\mathbf{H} \in \mathcal{D}_2(T\mathcal{S})$ such that

$$\det \mathbf{H} = (\det \mathbf{G})\bar{K} \qquad \text{and} \qquad \operatorname{Curl}^{\mathbf{G}}\mathbf{H} = \mathbf{0}, \tag{74}$$

then there exists a deformation $f : \mathcal{S} \to \bar{\mathcal{S}}$ with surface gradient $\mathbf{F} = \nabla^{\mathcal{S}} f$ such that the right Cauchy–Green tensor $\mathbf{C} = \mathbf{F}^{\top}\mathbf{F}$ equals $\mathbf{G}$ and the deformed surface $\bar{\mathcal{S}}$ has curvature tensor $\bar{\mathbf{L}}$ with referential pull-back $\mathbf{H} = \mathbf{F}^{\top}\bar{\mathbf{L}}\mathbf{F}$. Thus, when solving problems involving material surfaces in which $\mathbf{G}$ and $\mathbf{H}$ are unknown quantities that are needed to determine the deformed surface $\bar{\mathcal{S}}$, it is necessary to ensure that (74) hold. In this sense, the versions (65) and (73) of the Peterson–Mainardi–Codazzi equations derived here constitute compatibility conditions that any deformation $f : \mathcal{S} \to \bar{\mathcal{S}}$ must satisfy.

8. Gauss' Theorema Egregium

In the present context, the Theorema Egregium of Gauss (1828) takes the following form:

Proposition 2. *The Gaussian curvature $\bar{K}$ of the surface $\bar{\mathcal{S}}$ obtained by a deformation $f : \mathcal{S} \to \mathcal{E}$ with gradient $\mathbf{F} = \nabla^{\mathcal{S}} f$ only depends on the associated metric $\mathbf{C} = \mathbf{F}^{\top}\mathbf{F}$ induced on $\mathcal{S}$.*

Proof. For $\mathbf{u}, \mathbf{v} \in \mathcal{D}(T\mathcal{S})$, we use (60) to give

$$(\mathbb{F}\mathbf{u})\mathbf{v} = \mathbf{F}(\mathbb{A}\mathbf{u})\mathbf{v} + (\bar{\mathbf{n}} \cdot (\mathbb{F}\mathbf{u})\mathbf{v})\bar{\mathbf{n}}. \tag{75}$$

With reference to (64), we notice that the previous equation can be recast as

$$(\mathbb{F}\mathbf{u})\mathbf{v} = \mathbf{F}(\mathbb{A}\mathbf{u})\mathbf{v} + (\bar{\mathbf{n}} \otimes \mathbf{H}\mathbf{u})\mathbf{v}. \tag{76}$$

Taking into account (27), we find from (76) that

$$\mathbb{F}\mathbf{u} = \mathbf{F}(\mathbb{A}\mathbf{u} + \mathbf{L}\mathbf{u} \otimes \mathbf{n}) + \bar{\mathbf{n}} \otimes \mathbf{H}\mathbf{u}. \tag{77}$$

Applying the surface gradient in the direction $\mathbf{v}$ to both sides of the previous equation and simultaneously multiplying each term of the resulting identity from the left by $\mathbf{F}^{\top}$ and from the right by $\mathbf{P}$, we see that

$$\mathbf{F}^{\top}(\nabla^{\mathcal{S}}_{\mathbf{v}}\mathbb{F}\mathbf{u})\mathbf{P} = \mathbf{C}[(\mathbb{A}\mathbf{v})\mathbb{A}\mathbf{u} + \nabla^{\mathcal{S}}_{\mathbf{v}}\mathbb{A}\mathbf{u} - \mathbf{L}\mathbf{u} \otimes \mathbf{L}\mathbf{v}] - \mathbf{H}\mathbf{v} \otimes \mathbf{H}\mathbf{u}. \tag{78}$$

We thus find that

$$\begin{aligned}&\mathbf{F}^{\top}(\nabla^{\mathcal{S}}_{\mathbf{u}}\mathbb{F}\mathbf{v} - \nabla^{\mathcal{S}}_{\mathbf{v}}\mathbb{F}\mathbf{u})\mathbf{P}\\ &\quad = \mathbf{C}[(\mathbb{A}\mathbf{u})\mathbb{A}\mathbf{v} - (\mathbb{A}\mathbf{v})\mathbb{A}\mathbf{u} + \nabla^{\mathcal{S}}_{\mathbf{u}}\mathbb{A}\mathbf{v} - \nabla^{\mathcal{S}}_{\mathbf{v}}\mathbb{A}\mathbf{u} + \mathbf{L}\mathbf{u} \wedge \mathbf{L}\mathbf{u}] - \mathbf{H}\mathbf{u} \wedge \mathbf{H}\mathbf{v}.\end{aligned} \tag{79}$$

Since $\mathbb{F}$ is the surface gradient of $\mathbf{F}$, we recognize that $\nabla^{\mathcal{S}}\mathbb{F}$ must be symmetric in the sense that

$$\nabla^{\mathcal{S}}_{\mathbf{u}}\mathbb{F}\mathbf{v} = \nabla^{\mathcal{S}}_{\mathbf{v}}\mathbb{F}\mathbf{u}. \tag{80}$$

Combining the previous two equations, we find that

$$\mathbf{H}\mathbf{u} \wedge \mathbf{H}\mathbf{v} = \mathbf{C}[(\mathbb{A}\mathbf{u})\mathbb{A}\mathbf{v} - (\mathbb{A}\mathbf{v})\mathbb{A}\mathbf{u} + (\nabla^{\mathcal{S}}_{\mathbf{u}}\mathbb{A}\mathbf{v})\mathbf{P} - (\nabla^{\mathcal{S}}_{\mathbf{v}}\mathbb{A}\mathbf{u})\mathbf{P} + \mathbf{L}\mathbf{u} \wedge \mathbf{L}\mathbf{u}]. \tag{81}$$

Using (73), we infer that the left-hand side of (81) can be expressed in the form

$$\mathbf{H}\mathbf{u} \wedge \mathbf{H}\mathbf{v} = \bar{K}(\det \mathbf{C})\mathbf{u} \wedge \mathbf{v}. \tag{82}$$

Finally, since the right-hand side of (81) only depends on the surface $\bar{\mathcal{S}}$ through $\mathbf{C}$, and since $\mathbb{A}$ is determined by $\mathbf{C}$, we confirm that the Gaussian curvature $\bar{K}$ depends on $\bar{\mathcal{S}}$ only through $\mathbf{C}$, which completes the proof. □

Consider the case where f is the identity mapping on the surface $\mathcal{S}$, so that no deformation occurs. Then, $\mathbf{C} = \mathbf{P}$ is the induced metric on $\mathcal{S}$ and (53) holds. Moreover, a calculation shows that

$$\nabla^{\mathbf{P}}_{\mathbf{u}}\nabla^{\mathbf{P}}_{\mathbf{v}}\mathbf{w} - \nabla^{\mathbf{P}}_{\mathbf{v}}\nabla^{\mathbf{P}}_{\mathbf{u}}\mathbf{w} - \nabla^{\mathbf{P}}_{[\mathbf{u},\mathbf{v}]}\mathbf{w} = (\mathbf{L}\mathbf{u} \wedge \mathbf{L}\mathbf{v})\mathbf{w}. \tag{83}$$

Using (6) in the foregoing equality yields

$$\nabla^{\mathbf{P}}_{\mathbf{u}}\nabla^{\mathbf{P}}_{\mathbf{v}}\mathbf{w} - \nabla^{\mathbf{P}}_{\mathbf{v}}\nabla^{\mathbf{P}}_{\mathbf{u}}\mathbf{w} - \nabla^{\mathbf{P}}_{[\mathbf{u},\mathbf{v}]}\mathbf{w} = (\det \mathbf{L})(\mathbf{u} \wedge \mathbf{v})\mathbf{w}. \tag{84}$$

Some authors refer to (84), rather than (73), as the Gauss equation when the surface metric is the induced metric $\mathbf{P}$.

Remark 3. In Riemannian geometry, the Riemannian curvature tensor $\mathbf{R}_{\mathbf{C}}$ for $\mathcal{S}$ relative to the metric $\mathbf{C}$ is a mapping that associates to each pair of vector fields $\mathbf{u} \in \mathcal{D}(T\mathcal{S})$ and $\mathbf{v} \in \mathcal{D}(T\mathcal{S})$ a tangential two-tensor $\mathbf{R}(\mathbf{u}, \mathbf{v})$ defined by

$$\mathbf{R}_{\mathbf{C}}(\mathbf{u}, \mathbf{v})\mathbf{w} = (\nabla^{\mathbf{C}}_{\mathbf{u}}\nabla^{\mathbf{C}}_{\mathbf{v}} - \nabla^{\mathbf{C}}_{\mathbf{v}}\nabla^{\mathbf{C}}_{\mathbf{u}})\mathbf{w} - \nabla^{\mathbf{C}}_{[\mathbf{u},\mathbf{v}]}\mathbf{w}. \tag{85}$$

A calculation using (47) shows that

$$\mathbf{R}_{\mathbf{C}}(\mathbf{u}, \mathbf{v}) = \mathbf{R}_{\mathbf{P}}(\mathbf{u}, \mathbf{v}) + (\mathbb{A}\mathbf{u})\mathbb{A}\mathbf{v} - (\mathbb{A}\mathbf{v})\mathbb{A}\mathbf{u} + \mathbf{P}\nabla^{\mathcal{S}}_{\mathbf{u}}\mathbb{A}\mathbf{v} - \mathbf{P}\nabla^{\mathcal{S}}_{\mathbf{v}}\mathbb{A}\mathbf{u}, \tag{86}$$

where $\mathbf{R}_{\mathbf{P}}$ is the Riemannian curvature for the induced metric $\mathbf{P}$. A further calculation shows that

$$\mathbf{R}_{\mathbf{P}}(\mathbf{u}, \mathbf{v}) = \mathbf{L}\mathbf{u} \wedge \mathbf{L}\mathbf{v}. \tag{87}$$

Inserting (86) and (87) into (81) yields

$$\mathbf{H}\mathbf{u} \wedge \mathbf{H}\mathbf{v} = \mathbf{C}\mathbf{R}_{\mathbf{C}}(\mathbf{u}, \mathbf{v}). \tag{88}$$

Finally, using (6) and (73), the previous equation can be written as

$$\bar{K}\mathbf{C}\mathbf{u} \wedge \mathbf{C}\mathbf{v} = \mathbf{C}\mathbf{R}_{\mathbf{C}}(\mathbf{u}, \mathbf{v}), \tag{89}$$

which shows that the Riemann curvature tensor relative to $\mathbf{C}$ is completely determined by the Gaussian curvature $\bar{K}$ of $\bar{\mathcal{S}}$.

9. Kinematical constraints

We next specialize the results of the preceding section to certain special classes of deformations. For each class of deformations considered, the kinematics are constrained by a scalar- or tensor-valued condition on the surface metric $\mathbf{C}$. We refer to such constraints as kinematical since they impose restrictions on the kinds of deformations a material surface can sustain. Common alternatives to this choice of terms include "internal constraints" and "material constraints".

9.1. Angle preserving deformations

Consider a deformation $f : \mathcal{S} \to \mathcal{E}$, with gradient $\mathbf{F} = \nabla^{\mathcal{S}} f$, that preserves angles, in the sense that for each $x \in \mathcal{S}$, given $\mathbf{a} \in T_x\mathcal{S}$ and $\mathbf{b} \in T_x\mathcal{S}$, the angle between these vectors is the same as the angle between $\mathbf{F}(x)\mathbf{a}$ and $\mathbf{F}(x)\mathbf{b}$. Hence,

$$\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{\mathbf{F}(x)\mathbf{a} \cdot \mathbf{F}(x)\mathbf{b}}{|\mathbf{F}(x)\mathbf{a}||\mathbf{F}(x)\mathbf{b}|} = \frac{\langle \mathbf{a}, \mathbf{b}\rangle_{T_x\mathcal{S}}}{|\mathbf{a}|_{\mathbf{C}}|\mathbf{b}|_{\mathbf{C}}}, \qquad \mathbf{a}, \mathbf{b} \in T_x\mathcal{S},\ x \in \mathcal{S}. \tag{90}$$

Such a deformation $f : \mathcal{S} \to \mathcal{E}$ is often called conformal.

Proposition 3. *A deformation preserves angles if and only if there is a positive $\lambda \in \mathcal{D}$ such that*

$$\lambda\mathbf{u} \cdot \mathbf{v} = \mathbf{F}\mathbf{u} \cdot \mathbf{F}\mathbf{v}, \qquad \mathbf{u}, \mathbf{v} \in \mathcal{D}(T\mathcal{S}). \tag{91}$$

Proof. If (91) holds for some positive $\lambda \in \mathcal{D}$, it is then evident that (90) holds. Suppose that (90) holds. Fix $x \in \mathcal{S}$. Define λ by

$$\lambda(x, \mathbf{a}) = \frac{|\mathbf{F}(x)\mathbf{a}|}{|\mathbf{a}|}, \qquad \mathbf{a} \in T_x\mathcal{S},\ \mathbf{a} \neq \mathbf{0}. \tag{92}$$

From this definition it follows that λ is positive, smooth, and that, for any $\varepsilon > 0$ and $\mathbf{a} \in T_x\mathcal{S}$,

$$\lambda(x, \mathbf{a}) = \lambda(x, \varepsilon\mathbf{a}). \tag{93}$$

Differentiating the previous equation with respect to ε and evaluating the resulting expression at $\varepsilon = 1$ yields

$$(\operatorname{grad} \lambda(x, \mathbf{a}))\mathbf{a} = 0, \tag{94}$$

where grad denotes the derivative of λ with respect to its second variable. Moreover, setting $\varepsilon = 1/|\mathbf{a}|$ in (93) yields

$$\lambda(x, \mathbf{a}) = \lambda\Big(x, \frac{\mathbf{a}}{|\mathbf{a}|}\Big). \tag{95}$$

Referring to the definition (92) of λ, we see that, for any $\mathbf{a}, \mathbf{b} \in T_x\mathcal{S}$ and $\varepsilon > 0$,

$$|\mathbf{F}(x)(\mathbf{a} + \varepsilon\mathbf{b})|^2 = \lambda(x, \mathbf{u} + \varepsilon\mathbf{v})^2|\mathbf{a} + \varepsilon\mathbf{b}|^2. \tag{96}$$

Granted that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, so that $\mathbf{a} \cdot \mathbf{b} = 0$, it is evident from (90) that $\mathbf{F}(x)\mathbf{a} \cdot \mathbf{F}(x)\mathbf{b} = 0$. Thus, (96) can be recast as

$$\lambda(x, \mathbf{a})^2|\mathbf{u}|^2 + \varepsilon^2\lambda(\mathbf{v})^2 = \lambda\Big(x, \frac{\mathbf{a} + \varepsilon\mathbf{b}}{|\mathbf{a} + \varepsilon\mathbf{b}|}\Big)(|\mathbf{u}|^2 + \varepsilon^2|\mathbf{v}|^2). \tag{97}$$

Differentiating the previous equation with respect to ε and then evaluating it at $\varepsilon = 0$ yields

$$(\operatorname{grad} \lambda(x, \mathbf{a}))\mathbf{b} = \mathbf{0}. \tag{98}$$

In combination, (94) and (98) lead to the conclusion that $\operatorname{grad} \lambda(x, \mathbf{a}) = \mathbf{0}$ and, hence, that λ is independent of $\mathbf{a}$. Thus, $\lambda \in \mathcal{D}$ as claimed. □

Notice that if the deformation preserves angles, then it follows from (91) that

$$|\mathbf{F}\mathbf{u} \times \mathbf{F}\mathbf{v}| = \lambda|\mathbf{u} \times \mathbf{v}|, \qquad \mathbf{u}, \mathbf{v} \in \mathcal{D}(T\mathcal{S}). \tag{99}$$

Since the magnitude of the cross product of two vectors gives the area of the parallelogram spanned by those vectors, (99) shows that λ quantifies how infinitesimal areas scale under a deformation $f : \mathcal{S} \to \mathcal{E}$ which preserves angles as indicated in (90).

It follows from Proposition 3 that a deformation $f : \mathcal{S} \to \mathcal{E}$ with gradient $\mathbf{F} = \nabla^{\mathcal{S}} f$ preserves angles if and only if the associated metric tensor $\mathbf{C} = \mathbf{F}^{\mathsf{T}}\mathbf{F}$ satisfies

$$\mathbf{C} = \lambda\mathbf{P} \tag{100}$$

for some positive $\lambda \in \mathcal{D}$. If specialized in accord with the above result, (46) simplifies to

$$\mathbb{A}\mathbf{u} = \frac{1}{2\lambda}((\nabla^{\mathcal{S}}_{\mathbf{u}}\lambda)\mathbf{P} + \mathbf{u} \wedge \nabla^{\mathcal{S}}\lambda). \tag{101}$$

Thus, from (47), the covariant gradient $\nabla^{\mathbf{C}}$ relative to $\mathbf{C} = \mathbf{F}^{\mathsf{T}}\mathbf{F}$ is given by

$$\begin{aligned}\nabla^{\mathbf{C}}\mathbf{u} &= \nabla^{\lambda\mathbf{P}}\mathbf{u} \\ &= \mathbf{P}\nabla^{\mathcal{S}}\mathbf{u} + \frac{1}{2\lambda}((\nabla^{\mathcal{S}}_{\mathbf{u}}\lambda)\mathbf{P} + \mathbf{u} \wedge \nabla^{\mathcal{S}}\lambda).\end{aligned} \tag{102}$$

9.2. Locally area preserving deformations

Consider a deformation $f : \mathcal{S} \to \mathcal{E}$, with gradient $\mathbf{F} = \nabla^{\mathcal{S}} f$, that preserves areas in the sense that

$$\operatorname{Area}(\mathcal{A}) = \operatorname{Area}(f(\mathcal{A})), \qquad \mathcal{A} \subseteq \mathcal{S}. \tag{103}$$

This condition can be written using integrals as

$$\int_{\mathcal{A}} d\mathrm{A} = \int_{f(\mathcal{A})} d\mathrm{A}, \qquad \mathcal{A} \subseteq \mathcal{S}. \tag{104}$$

By the area formula $\int_{f(\mathcal{A})} d\mathrm{A} = \int_{\mathcal{A}} \det \mathbf{C}\, d\mathrm{A}$, so the previous equation is equivalent to

$$\int_{\mathcal{A}} (1 - \det \mathbf{C})\, d\mathrm{A} = 0, \qquad \mathcal{A} \subseteq \mathcal{S}. \tag{105}$$

It follows that the deformation $f : \mathcal{S} \to \mathcal{E}$ is area preserving if and only if

$$\det \mathbf{C} = 1. \tag{106}$$

In contrast to what occurs for angle preserving deformations, the foregoing restriction on $\mathbf{C}$ does not lead to a simplification of the representation (46) for the corresponding covariant gradient $\nabla^{\mathbf{C}}$.

9.3. Local length preserving deformations

Consider a deformation $f : \mathcal{S} \to \mathcal{E}$, with gradient $\mathbf{F} = \nabla^{\mathcal{S}} f$, that preserves the lengths of tangent vectors in the sense that

$$|\mathbf{u}| = |\mathbf{F}\mathbf{u}| = |\mathbf{u}|_{\mathbf{C}}, \qquad \mathbf{u} \in \mathcal{D}(T\mathcal{S}). \tag{107}$$

Such a deformation is called a local isometry. It follows from (107) that, for all $\mathbf{u} \in \mathcal{D}(T\mathcal{S})$ and $\mathbf{v} \in \mathcal{D}(T\mathcal{S})$,

$$\begin{aligned}|\mathbf{u}|^2 + 2\mathbf{u} \cdot \mathbf{v} + |\mathbf{v}|^2 &= |\mathbf{u} + \mathbf{v}|^2 \\ &= |\mathbf{F}\mathbf{u}|^2 + 2\mathbf{F}\mathbf{u} \cdot \mathbf{F}\mathbf{v} + |\mathbf{F}\mathbf{v}|^2.\end{aligned} \tag{108}$$

Referring again to (107) simplifies the previous relation to

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{F}\mathbf{u} \cdot \mathbf{F}\mathbf{v}, \qquad \mathbf{u}, \mathbf{v} \in \mathcal{D}(T\mathcal{S}). \tag{109}$$

It follows that the deformation $f : \mathcal{S} \to \mathcal{E}$ preserves angles and, moreover, (91) holds with $\lambda = 1$. Thus, the metric tensor $\mathbf{C}$ of a locally length preserving deformation $f : \mathcal{S} \to \mathcal{E}$ is equal to the orthogonal projection $\mathbf{P}$ onto $\mathcal{S}$:

$$\mathbf{C} = \mathbf{P}. \tag{110}$$

It then follows from (53) that

$$\nabla^{\mathbf{C}} = \nabla^{\mathbf{P}} = \mathbf{P}\nabla^{\mathcal{S}}. \tag{111}$$

9.4. Locally length and area preserving deformations

We next show that, consistent with intuitive expectations, a deformation $f : \mathcal{S} \to \mathcal{E}$ that is both angle and locally area preserving must also preserve lengths, and vice versa.

Proposition 4. *A deformation $f : \mathcal{S} \to \mathcal{E}$ is locally length preserving if and only if it is both angle preserving and locally area preserving.*

Proof. If a deformation locally preserves lengths, then $\mathbf{C} = \mathbf{P}$. It follows that $\det \mathbf{C} = 1$, so it is locally area preserving, and also (91) holds with $\lambda = 1$, so it is angle preserving.

On the other hand, suppose that a deformation is both angle and locally area preserving. We can then combine (100) and (106) to find that $\mathbf{C} = \mathbf{P}$. It immediately follows from (107) that the deformation is locally length preserving. □

This last result affords an alternative argument leading to (111). Specifically, any deformation that is locally length preserving must, by Proposition 4, also be angle and locally area preserving. Thus, (100) and (102) must hold for some λ. It then follows from (106) that $\lambda = 1$ and, hence, that (102) reduces to (111).

10. Derivation of the Peterson–Mainardi–Codazzi and Gauss equations subject to the provision that the surface metric is the induced metric

Wempner (1967) and Steele (1971) derive the Peterson–Mainardi–Codazzi and Gauss equations under the provision that the surface metric derives from the metric for the ambient three-dimensional Euclidean space $\mathcal{E}$ within which the surface is embedded. In our setting, that assumption amounts to requiring that the metric $\mathbf{C}$ of $\mathcal{S}$ be equal to the induced metric $\mathbf{P}$. Intuitively, this amounts to selecting the deformed configuration $\bar{\mathcal{S}}$ of a material surface as the reference configuration $\mathcal{S}$ for that surface. We next show how our framework specializes if this is so.

Apart from the choice

$$\mathbf{C} = \mathbf{P} \tag{112}$$

inspired by the works of Wempner (1967) and Steele (1971), our alternative derivation the Peterson–Mainardi–Codazzi and Gauss equations begins with the understanding that the surface curl of the surface gradient of any tangent vector field $\mathbf{w} \in \mathcal{D}(T\mathcal{S})$ must vanish:

$$\mathrm{Curl}^{\mathcal{S}}(\nabla^{\mathcal{S}}\mathbf{w}) = \mathbf{0}. \tag{113}$$

This fact is a consequence of the symmetry of the second surface gradient $\nabla^{\mathcal{S}}(\nabla^{\mathcal{S}}\mathbf{w})$ of $\mathbf{w}$. The basic idea is to decompose this equation into parts tangent and normal to the surface. Towards this end, we first use (22) to split $\nabla^{\mathcal{S}}\mathbf{w}$ into the sum of a tangential two-tensor $\mathbf{P}\nabla^{\mathcal{S}}\mathbf{w} = \nabla^{\mathbf{P}}\mathbf{w}$ and its complement $\mathbf{n} \otimes \mathbf{L}\mathbf{w}$. We then use (15) and the product rule to find that for all $\mathbf{u} \in \mathcal{D}(T\mathcal{S})$ and $\mathbf{v} \in \mathcal{D}(T\mathcal{S})$

$$\begin{aligned}\mathrm{Curl}^{\mathcal{S}}(\nabla^{\mathcal{S}}\mathbf{w})(\mathbf{u},\mathbf{v}) \;=\; &\nabla^{\mathcal{S}}_{\mathbf{u}}(\nabla^{\mathbf{P}}\mathbf{w})\mathbf{v} - \nabla^{\mathcal{S}}_{\mathbf{v}}(\nabla^{\mathbf{P}}\mathbf{w})\mathbf{u}\\ &-(\mathbf{L}\mathbf{u}\otimes\mathbf{L}\mathbf{w})\mathbf{v} + (\mathbf{L}\mathbf{v}\otimes\mathbf{L}\mathbf{w})\mathbf{u}\\ &+(\nabla^{\mathcal{S}}_{\mathbf{u}}(\mathbf{L}\mathbf{w})\cdot\mathbf{v} - \nabla^{\mathcal{S}}_{\mathbf{v}}(\mathbf{L}\mathbf{w})\cdot\mathbf{u})\mathbf{n}.\end{aligned} \tag{114}$$

Since $\nabla^{\mathbf{P}}\mathbf{w} = \mathbf{P}\nabla^{\mathbf{P}}\mathbf{w}$, we see from the product rule that

$$\nabla^{\mathcal{S}}_{\mathbf{v}}(\nabla^{\mathbf{P}}\mathbf{u})\mathbf{w} = (\nabla^{\mathbf{P}}_{\mathbf{w}}\mathbf{u}\cdot\mathbf{L}\mathbf{v})\mathbf{n} + \mathbf{P}\nabla^{\mathcal{S}}_{\mathbf{v}}(\nabla^{\mathbf{P}}\mathbf{u})\mathbf{w}. \tag{115}$$

On augmenting (114) with the previous identity and applying the product rule again, it follows that

$$\begin{aligned}\mathrm{Curl}^{\mathcal{S}}(\nabla^{\mathcal{S}}\mathbf{w})(\mathbf{u},\mathbf{v}) \;=\; &\mathrm{Curl}^{\mathbf{P}}(\nabla^{\mathbf{P}}\mathbf{u})(\mathbf{a},\mathbf{b}) - (\mathbf{L}\mathbf{u}\otimes\mathbf{L}\mathbf{w})\mathbf{v} + (\mathbf{L}\mathbf{v}\otimes\mathbf{L}\mathbf{w})\mathbf{u}\\ &+[(\nabla^{\mathcal{S}}_{\mathbf{u}}\mathbf{L})\mathbf{w}\cdot\mathbf{v} - (\nabla^{\mathcal{S}}_{\mathbf{v}}\mathbf{L})\mathbf{w}\cdot\mathbf{u}]\mathbf{n}.\end{aligned} \tag{116}$$

To obtain the Peterson–Mainardi–Codazzi equation, we take the inner-product of (113) with $\mathbf{n}$ and use (116). This gives

$$0 = \mathbf{w}\cdot[(\nabla^{\mathcal{S}}_{\mathbf{u}}\mathbf{L})^{\top}\mathbf{v} - (\nabla^{\mathcal{S}}_{\mathbf{v}}\mathbf{L})^{\top}\mathbf{u}]. \tag{117}$$

Since $\mathbf{L} = -\nabla^{\mathcal{S}}\mathbf{n}$ and $\nabla^{\mathcal{S}}(\nabla^{\mathcal{S}}\mathbf{n})$ is symmetric, we see that (117) can be expressed as

$$0 = \mathbf{w}\cdot\mathrm{Curl}^{\mathcal{S}}\mathbf{L}(\mathbf{u},\mathbf{v}). \tag{118}$$

Thus, since the foregoing condition must hold for any $\mathbf{w} \in \mathcal{D}(T\mathcal{S})$, we infer that

$$\mathbf{0} = \mathbf{P}\mathrm{Curl}^{\mathcal{S}}\mathbf{L}(\mathbf{u},\mathbf{v}), \tag{119}$$

which, by (54), is (65) for $\mathbf{G} = \mathbf{P}$.

To obtain the Gauss equation, we first notice that since $\mathbf{L}$ is symmetric, (6) can be used to find that

$$\begin{aligned}(\mathbf{L}\mathbf{u}\otimes\mathbf{L}\mathbf{w})\mathbf{v} - (\mathbf{L}\mathbf{v}\otimes\mathbf{L}\mathbf{w})\mathbf{u} &= (\mathbf{L}\mathbf{u}\wedge\mathbf{L}\mathbf{v})\mathbf{w}\\ &= K(\mathbf{u}\wedge\mathbf{v})\mathbf{w}.\end{aligned} \tag{120}$$

Thus, if we apply $\mathbf{P}$ to (113) and invoke (116), we obtain

$$K(\mathbf{u}\wedge\mathbf{w})\mathbf{w} = \mathrm{Curl}^{\mathbf{P}}(\nabla^{\mathbf{P}}\mathbf{w})(\mathbf{u},\mathbf{v}). \tag{121}$$

A final calculation then gives

$$\mathrm{Curl}^{\mathbf{P}}(\nabla^{\mathbf{P}}\mathbf{w})(\mathbf{u},\mathbf{v}) = \nabla^{\mathbf{P}}_{\mathbf{u}}\nabla^{\mathbf{P}}_{\mathbf{v}}\mathbf{w} - \nabla^{\mathbf{P}}_{\mathbf{v}}\nabla^{\mathbf{P}}_{\mathbf{u}}\mathbf{w} - \nabla^{\mathbf{P}}_{[\mathbf{u},\mathbf{v}]}\mathbf{w} \tag{122}$$

and, hence, (121) is (84), which some authors refer to as the Gauss equation granted, of course, that the surface metric is given by $\mathbf{P}$.

Remark 4. Rather than using the surface curl operator defined in (13), many authors use the surface curl $\mathrm{curl}^{\mathcal{S}}\mathbf{M} \in \mathcal{D}_2(\mathcal{V})$ of $\mathbf{M} \in \mathcal{D}_2(T\mathcal{S})$ defined such that

$$((\mathrm{curl}^{\mathcal{S}}\mathbf{M})^{\top}\mathbf{a})\times\mathbf{b} = (\nabla^{\mathcal{S}}(\mathbf{M}^{\top}\mathbf{a}) - \nabla^{\mathcal{S}}(\mathbf{M}^{\top}\mathbf{a})^{\top})\times\mathbf{b} \tag{123}$$

for all $\mathbf{a},\mathbf{b} \in \mathcal{V}$. It should be stressed that the object $\mathrm{curl}^{\mathcal{S}}\mathbf{M}$ so defined is a two-tensor but that $\mathrm{Curl}^{\mathcal{S}}\mathbf{M}$ is a three-tensor. This allows for a derivation that is essentially identical to the one presented above. In particular, the decomposition (22) can be used and the vector equation $\mathrm{curl}^{\mathcal{S}}(\nabla^{\mathcal{S}}\mathbf{w})\mathbf{n} = \mathbf{0}$ can be decomposed into a tangential and normal components. While the normal component,

$$\mathbf{P}(\mathrm{curl}^{\mathcal{S}}\mathbf{L})\mathbf{n} = \mathbf{0}, \tag{124}$$

can be seen to be equivalent to (65) with $\mathbf{C} = \mathbf{P}$, the tangential component is

$$|\mathbf{w}|^2K = (\mathbf{w}\times\mathbf{n})\cdot(\mathrm{curl}^{\mathcal{S}}(\nabla^{\mathbf{P}}\mathbf{w}))\mathbf{n}. \tag{125}$$

A direct connection between (125) and (121) is, however, elusive because (125) is scalar valued and involves a single arbitrary tangential vector field $\mathbf{w}$ while (84) is vector valued and involves three tangential vector fields $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. For this reason, we prefer the derivation based on the definition (13) of the surface curl operator.

Remark 5. Instead of the local condition (113), Wempner (1967) and Steele (1971) begin by stipulating that the integral of the differential increment $d\mathbf{u}$ of a tangential vector field $\mathbf{u}$ about all simple closed contours on the surface must vanish. The condition (113) imposed here follows from that global stipulation on applying Stokes' theorem and localizing the resulting surface integral at an arbitrary point on the portion of the surface enclosed by the original contour.

Remark 6. The condition (112) embodying the assumption that the surface metric be derived from the metric for the space $\mathcal{E}$ in which the surface is embedded should not be misconstrued with the constraint (110) that applies if the surface can sustain only length preserving deformations. However, the argument leading to the versions (119) and (121) of the Peterson–Mainardi–Codazzi and Gauss equations does apply if (112) is interpreted as a constraint on the surface metric instead of as a choice. Hence, (119) and (121) are the versions of the Peterson–Mainardi–Codazzi and Gauss equations that apply for a surface that can sustain only length preserving deformations.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The authors thank an anonymous referee for constructive feedback and suggestions. E.F. gratefully acknowledges support from the Okinawa Institute of Science and Technology Graduate University with subsidy funding from the Cabinet Office, Government of Japan.

Appendix. Coordinate based representations

Let $\mathbf{r} : \mathcal{U} \to \mathcal{E}$ be a coordinate chart for the surface $\mathcal{S}$, meaning that:

- $\mathcal{U}$ is an open subset of $\mathbb{R}^2$;
- $\mathbf{r}$ is smooth;
- $(\operatorname{grad}\mathbf{r})(\mathbf{p})$ is injective for every $\mathbf{p} \in \mathcal{U}$;
- the range $\operatorname{Rng}\mathbf{r}$ of $\mathbf{r}$ is contained in $\mathcal{S}$; and
- the inverse $\mathbf{r}^{-1} : \operatorname{Rng}\mathbf{r} \to \mathcal{U}$ of $\mathbf{r}$ exists.

Since $\mathbf{r}$ is injective, it follows that if $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis for $\mathbb{R}^2$, then $\mathbf{s}_1$ and $\mathbf{s}_2$ defined for each $\mathbf{p} \in \mathcal{U}$ by[4]

$$\mathbf{s}_i(\mathbf{r}(\mathbf{p})) = \mathbf{r}_{,i}(\mathbf{p}) = (\operatorname{grad}\mathbf{r})(\mathbf{p})\mathbf{e}_i, \tag{A.1}$$

form a basis for $T_{\mathbf{r}(\mathbf{p})}\mathcal{S}$. Notice from (A.1) that $\mathbf{s}_1$ and $\mathbf{s}_2$ are viewed as functions that are defined on $\operatorname{Rng}\mathbf{r}$. The corresponding dual basis $\{\mathbf{s}^1(x), \mathbf{s}^2(x)\}$ on $T_x\mathcal{S}$ is determined by the criteria

$$\mathbf{s}^i(x) \cdot \mathbf{s}_j(x) = \delta^i{}_j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{A.2}$$

Given the field $\mathbf{n} = \mathbf{s}_1 \times \mathbf{s}_2$, which is defined on $\operatorname{Rng}\mathbf{r}$, is orthogonal to $\mathcal{S}$, and has unit magnitude, $\{\mathbf{s}_1(x), \mathbf{s}_2(x), \mathbf{n}(x)\}$ and $\{\mathbf{s}^1(x), \mathbf{s}^2(x), \mathbf{n}(x)\}$ are bases for $\mathcal{V}$.

We can represent a tangential vector field $\mathbf{u} \in \mathcal{D}(T\mathcal{S})$ on $\operatorname{Rng}\mathbf{r}$ using either the basis $\{\mathbf{s}_1, \mathbf{s}_2\}$ or the dual basis $\{\mathbf{s}^1, \mathbf{s}^2\}$:

$$\mathbf{u} = u^i\mathbf{s}_i = u_i\mathbf{s}^i, \tag{A.3}$$

where $u^i = \mathbf{u} \cdot \mathbf{s}^i$ and $u_i = \mathbf{u} \cdot \mathbf{s}_i$. Likewise, a tangential two-tensor field $\mathbf{M} \in \mathcal{D}_2(T\mathcal{S})$ can be represented using either basis:

$$\mathbf{M} = M^{ij}\mathbf{s}_i \otimes \mathbf{s}_j = M_{ij}\mathbf{s}^i \otimes \mathbf{s}^j, \tag{A.4}$$

where $M^{ij} = \mathbf{s}^i \cdot \mathbf{M}\mathbf{s}^j$ and $M_{ij} = \mathbf{s}_i \cdot \mathbf{M}\mathbf{s}_j$. However, it is also possible to represent $\mathbf{M}$ using both bases simultaneously:

$$\mathbf{M} = M^i{}_j\mathbf{s}_i \otimes \mathbf{s}^j = M_i{}^j\mathbf{s}^i \otimes \mathbf{s}_j, \tag{A.5}$$

where $M^i{}_j = \mathbf{s}^i \cdot \mathbf{M}\mathbf{s}_j$ and $M_i{}^j = \mathbf{s}_i \cdot \mathbf{M}\mathbf{s}^j$. Finally, a tangential three-tensor field $\mathbb{A} \in \mathcal{D}_3(T\mathcal{S})$ can be represented in eight different ways involving the bases. One example of such a representation is

$$\mathbb{A} = A^i{}_{jk}\mathbf{s}_i \otimes \mathbf{s}^j \otimes \mathbf{s}^k, \tag{A.6}$$

where $A^i{}_{jk} = \mathbf{s}^i \cdot (\mathbb{A}\mathbf{s}_k)\mathbf{s}_j$. The remaining alternatives take similar forms.

We can also define the symbols g_{ij} and g^{ij} by

$$g_{ij} = \mathbf{s}_i \cdot \mathbf{s}_j \qquad \text{and} \qquad g^{ij} = \mathbf{s}^i \cdot \mathbf{s}^j. \tag{A.7}$$

It follows from the definitions in (A.7) and (A.3) that

$$g_{ij}u^j = u_i \qquad \text{and} \qquad g^{ij}u_j = u^i. \tag{A.8}$$

Thus, we see that the symbols g_{ij} and g^{ij} can be used to raise and lower the index used in the components of a vector. Similar computations can be done involving the components of second and third order tensors. Just as an example, we have

$$M_{ij}g^{jk} = M_i{}^k \qquad \text{and} \qquad A^k{}_{lm}g_{ki}g^{ln} = A_i{}^n{}_m. \tag{A.9}$$

Combining the two equations in (A.8) shows that

$$g_{ij}g^{jk} = \delta_i{}^k \qquad \text{and} \qquad g^{ij}g_{jk} = \delta^i{}_k. \tag{A.10}$$

Taking the gradient of (A.1) in the direction $\mathbf{e}_j$ and invoking the chain rule gives

$$\nabla^{\mathcal{S}}_{\mathbf{s}_j(\mathbf{r}(\mathbf{p}))}\mathbf{s}_i(\mathbf{r}(\mathbf{p})) = \mathbf{r}_{,ij}(\mathbf{p}). \tag{A.11}$$

The Christoffel symbols Γ^k_{ij} are defined through

$$\Gamma^k_{ij} = \mathbf{s}^k \cdot \nabla^{\mathcal{S}}_{\mathbf{s}_j}\mathbf{s}_i. \tag{A.12}$$

From (A.12) and the identity $\mathbf{r}_{,ij} = \mathbf{r}_{,ji}$, it follows that $\Gamma^k_{ij} = \Gamma^k_{ji}$. Since $\mathbf{s}_i$ is tangent to $\mathcal{S}$, $\mathbf{s}_i \cdot \mathbf{n} = 0$. Applying the surface gradient $\nabla^{\mathcal{S}}$ in the direction $\mathbf{s}_j$ to $\mathbf{s}_i \cdot \mathbf{n} = 0$ yields

$$\nabla^{\mathcal{S}}_{\mathbf{s}_j}\mathbf{s}_i \cdot \mathbf{n} = -\mathbf{s}_i \cdot \nabla^{\mathcal{S}}_{\mathbf{s}_j}\mathbf{n} = \mathbf{s}_i \cdot \mathbf{L}\mathbf{s}_j = L_{ij}. \tag{A.13}$$

Combining (A.12) and (A.13) results in

$$\nabla^{\mathcal{S}}_{\mathbf{s}_j}\mathbf{s}_i = \Gamma^k_{ij}\mathbf{s}_k + L_{ij}\mathbf{n}. \tag{A.14}$$

Applying $\nabla^{\mathcal{S}}$ in the direction $\mathbf{s}_k$ of both sides of (A.2) gives

$$\mathbf{s}_j \cdot \nabla^{\mathcal{S}}_{\mathbf{s}_k}\mathbf{s}^i = -\mathbf{s}^i \cdot \nabla^{\mathcal{S}}_{\mathbf{s}_k}\mathbf{s}_j = -\Gamma^i{}_{jk}. \tag{A.15}$$

Calculations analogous to (A.13) and (A.14) deliver

$$\nabla^{\mathcal{S}}_{\mathbf{s}_j}\mathbf{s}^i \cdot \mathbf{n} = \mathbf{s}^i \cdot \mathbf{L}\mathbf{s}_j = L^i{}_j \tag{A.16}$$

and

$$\nabla^{\mathcal{S}}_{\mathbf{s}_j}\mathbf{s}^i = -\Gamma^i{}_{jk}\mathbf{s}^k + L^i{}_j\mathbf{n}. \tag{A.17}$$

Given $\phi \in \mathcal{D}$, it follows that $\nabla^{\mathcal{S}}\phi \in \mathcal{D}(T\mathcal{S})$ and, thus, that we can write

$$\nabla^{\mathcal{S}}\phi = \phi_{,i}\mathbf{s}^i, \tag{A.18}$$

where $\phi_{,i} = \nabla^{\mathcal{S}}\phi \cdot \mathbf{s}_i$. By the product rule, $(\text{A.3})_1$, (A.14), and (A.18), it follows that, for all $\mathbf{u} \in \mathcal{D}(T\mathcal{S})$ and $\mathbf{v} \in \mathcal{D}(T\mathcal{S})$,

$$\nabla^{\mathcal{S}}_{\mathbf{v}}\mathbf{u} = (u^k_{,i}v^i + \Gamma^k_{ij}u^jv^i)\mathbf{s}_k + L_{ij}u^jv^i\mathbf{n}. \tag{A.19}$$

Given a metric on $\mathcal{S}$ with metric tensor $\mathbf{G}$, it then follows from (47) and (A.19) that

$$\nabla^{\mathbf{G}}_{\mathbf{v}}\mathbf{u} = [u^k_{,i}v^i + (\Gamma^k_{ij} + \Lambda^k_{ij})u^jv^i]\mathbf{s}_k, \tag{A.20}$$

which is a coordinate representation of (47). While Γ^k_{ij} are the Christoffel symbols for the covariant gradient $\mathbf{P}\nabla^{\mathcal{S}}$, we see from (A.20) that the sum $\Gamma^k_{ij} + \Lambda^k_{ij}$ are the Christoffel symbols for $\nabla^{\mathbf{G}}$.

Similarly, for $\mathbf{M} \in \mathcal{D}_2(T\mathcal{S})$, the product rule, (A.4), and (A.17) can be used to show that

$$\begin{aligned}\nabla^{\mathcal{S}}_{\mathbf{u}}\mathbf{M} = {}& M_{ij,k}u^k\mathbf{s}_i \otimes \mathbf{s}_j \\ & -M_{ij}u^k(\Gamma^i{}_{kl}\mathbf{s}^l \otimes \mathbf{s}^j + \Gamma^j{}_{kl}\mathbf{s}^i \otimes \mathbf{s}^l - L^i{}_k\mathbf{s}_i \otimes \mathbf{n} - L^j{}_k\mathbf{n} \otimes \mathbf{s}_j).\end{aligned} \tag{A.21}$$

A coordinate representation of (48) follows from (A.21):

$$\begin{aligned}\nabla^{\mathbf{G}}_{\mathbf{u}}\mathbf{M} = {}& M_{ij,k}u^k\mathbf{s}^i \otimes \mathbf{s}^j \\ & -M_{ij}u^k[(\Gamma^i{}_{kl} + \Lambda^i{}_{kl})\mathbf{s}^l \otimes \mathbf{s}^j + (\Gamma^j{}_{kl} + \Lambda^j{}_{kl})\mathbf{s}^i \otimes \mathbf{s}^l].\end{aligned} \tag{A.22}$$

Comparing (A.20) to (A.19) and (A.22) to (A.21) shows that the components of the three-tensor $\mathbb{A}$ that determines the relationship between the covariant and surface gradients through (47) enter the component-based representations for $\nabla^{\mathbf{G}}\mathbf{u}$ and $\nabla^{\mathbf{G}}\mathbf{M}$ in a fashion completely analogous to the Christoffel symbols. Whereas the Christoffel symbols are needed to account for the particular features of the chosen coordinates and, thus, are not the components of a three-tensor, the components

[4] Throughout this section subscripts and superscripts using i, j, k etc. are assumed to equal either 1 or 2. Moreover, if there are repeated indices with one index a superscript and the other a subscript, then summation over that index from 1 to 2 is implied.

of $\mathbb{A}$ reflect the process of converting the surface gradient operator to an object that operates on tangential vector fields to produce tangential two-tensor fields and, thus, is independent of any choice of coordinates.

It follows from (A.21) that the Peterson–Mainardi–Codazzi (65), when written using components, becomes

$$H_{ij,k} - H_{ik,j} - H_{lj}(\Gamma^l{}_{ki} + \Lambda^l{}_{ki}) + H_{lk}(\Gamma^l{}_{ji} + \Lambda^l{}_{ji}) = 0. \tag{A.23}$$

If the reference and deformed surfaces coincide, so that the metric $\mathbf{C}$ is the induced metric $\mathbf{P}$ and $\mathbf{H}$ is the curvature tensor $\mathbf{L}$ on $\mathcal{S}$, rather than the pull-back of the curvature tensor of the deformed surface, then (A.23) becomes

$$L_{ij,k} - L_{ik,j} - L_{lj}\Gamma^l{}_{ki} + H_{lk}\Gamma^l{}_{ji} = 0. \tag{A.24}$$

Next we obtain a component version of the alternate form of the Gauss Eq. (84) for the particular case in which the metric is the induced metric $\mathbf{P}$ and $\mathbb{A} = \mathbf{0}$. Then, it follows from (A.19) that, for any tangent vector fields $\mathbf{u}$ and $\mathbf{v}$,

$$\nabla^{\mathbf{P}}_{\mathbf{v}}\mathbf{u} = \mathbf{P}\nabla^{\mathcal{S}}\mathbf{u} = (u^k_{,i}v^i + \Gamma^k{}_{ij}u^jv^i)\mathbf{s}_k. \tag{A.25}$$

Using the foregoing identity in (84) with the particular choices $\mathbf{u} = \mathbf{s}_i$, $\mathbf{v} = \mathbf{s}_j$, and $\mathbf{w} = \mathbf{s}_k$ and taking the inner product with $\mathbf{s}^l$ on both sides of the resulting identity gives

$$L^l{}_iL_{kj} - L^l{}_jL_{ik} = \Gamma^l{}_{mi}\Gamma^m_{kj} - \Gamma^l{}_{mj}\Gamma^m_{ki} + \Gamma^l_{kj,i} - \Gamma^l_{ki,j}, \tag{A.26}$$

which is a commonly encountered form of the Gauss equation.

Remark 7. In traditional approaches that rely on coordinate charts to describe the reference and deformed surfaces $\mathcal{S}$ and $\bar{\mathcal{S}}$, the Peterson–Mainardi–Codazzi (A.24) and Gauss (A.26) equations must be satisfied on both surfaces and, hence, constitute conditions of compatibility, as Koiter and Simmonds (1972) explain in their incisive and comprehensive survey of the foundations of shell theory.

Remark 8. It is possible to express the covariant gradient $\nabla^{\mathbf{G}}\mathbf{M}$ of a tangential two-tensor $\mathbf{M}$ using any of the four component representations for $\mathbf{M}$ in (A.4) and (A.5). For example, on using

$$\mathbf{M} = M^i{}_j\mathbf{s}_i \otimes \mathbf{s}^j \tag{A.27}$$

it follows from (A.7)–(A.10) that

$$\begin{aligned}\nabla^{\mathbf{G}}_{\mathbf{u}}\mathbf{M} = {} & M^i{}_{j,k}u^k\mathbf{s}_i \otimes \mathbf{s}^j \\ & + M^i{}_ju^k[(\Gamma^l{}_{ik} - \Lambda_i{}^l{}_k)\mathbf{s}_l \otimes \mathbf{s}^j - (\Gamma^j{}_{kl} + \Lambda^j{}_{kl})\mathbf{s}_i \otimes \mathbf{s}^l].\end{aligned} \tag{A.28}$$

Since $\Gamma^k{}_{ij}$ and $\Lambda^k{}_{ij}$ appear in the same way in (A.22), it can be argued that the expression (A.22) is less ragged than and, thus preferable to, (A.28). This difference stems from our chosen definition of $\nabla^{\mathbf{G}}\mathbf{M}$ through (49). Within that formula, it is most natural to view $\mathbf{M}$ as a bilinear mapping that takes in two vector fields $\mathbf{u} \in \mathcal{D}(T\mathcal{S})$ and $\mathbf{v} \in \mathcal{D}(T\mathcal{S})$ and outputs the scalar field $\mathbf{u}\cdot\mathbf{M}\mathbf{v}$. Such a bilinear mapping is most naturally expressed using the dual basis $\{\mathbf{s}^1, \mathbf{s}^2\}$ and, hence, through the representation $(A.4)_2$.

References

do Carmo, M.P., 1992. Riemannian Geometry. Birkhäuser, Basel.

Ciarlet, P.G., 2005. An introduction to differential geometry with applications to elasticity. J. Elasticity 78 (1), 1–215.

Codazzi, D., 1868–1869. Sulle coordinate curvilnee d'una sperficie e dello spazio (memoria terza). Ann. Mat. Pura Appl. 2, 269–287.

Gauss, K.F., 1828. Disquisitiones generales circa superficies curvas. Commentationes societatis regiæscientiarum Gottingensis recentiores. Comment. Cl. Math. 6, 99–146.

Gurtin, M.E., Murdoch, A.I., 1975. A continuum theory of elastic material surfaces. Arch. Ration. Mech. Anal. 57, 291–323.

Koiter, W.T., Simmonds, J.G., 1972. Foundations of shell theory. In: Proceedings of the Thirteenth International Congress of Theoretical and Applied Mechanics. Springer-Verlag, Berlin, Heidelberg, New York.

Mainardi, G., 1856. Su la Teoria Generale delle Superficie, Vol. 9. Giornale dell'Imperial Regio Istituto Lombardo di Scienze Lettere ed Arti e Biblioteca Italiana, pp. 385–398.

Murdoch, A.I., 1978. Direct Notation for Surfaces with Application to the Thermodynamics of Elastic Material Surfaces of Second Grade. Research Report ES 78-134, University of Cincinnati, Department of Engineering Science.

Murdoch, A.I., 1990. A coordinate-free approach to surface kinematics. Glasg. Math. J. 32 (3), 299–307.

Murdoch, A.I., Cohen, H., 1979. Symmetry considerations for material surfaces. Arch. Ration. Mech. Anal. 72, 61–98.

Peterson, K.M., 1853. Über die Biegung der Flächen. In: Kandidatenschrift, Kaiserliche Universität zu Dorpat.

Steele, C.R., 1971. A geometric optics solution for the thin shell equation. Internat. J. Engrg. Sci. 9, 681–704.

Wempner, G.A., 1967. The deformation of thin shells. In: Shaw, W.H. (Ed.), Developments in Theoretical and Applied Mechanics. Pergamon, Oxford, pp. 245–254.