FUNCTIONS OF BOUNDED VARIATION ON COMPLETE AND CONNECTED ONE-DIMENSIONAL METRIC SPACES

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ABSTRACT. In this paper, we study functions of bounded variation on a complete and connected metric space with finite one-dimensional Hausdorff measure. The definition of BV functions on a compact interval based on pointwise variation is extended to this general setting. We show this definition of BV functions is equivalent to the BV functions introduced by Miranda [18]. Furthermore, we study the necessity of conditions on the underlying space in Federer's characterization of sets of finite perimeter on metric measure spaces. In particular, our examples show that the doubling and Poincaré inequality conditions are essential in showing that a set has finite perimeter if the codimension one Hausdorff measure of the measure-theoretic boundary is finite.

1. INTRODUCTION

Functions of bounded variation, also known as BV functions, have been extensively studied and widely applied in different areas including the calculus of variations, hyperbolic conservation laws, and minimal surfaces [3, 6, 9]. In the context of metric measure spaces, the notion of functions of bounded variation is introduced by Miranda [18] and it has attracted significant attention in recent years (e.g. [1, 2, 13, 16, 17]). Motivated by the observation that various function classes including Sobolev functions and BV functions defined on the real line $\mathbb{R}$ have simple characterizations, in this work we focus our study on BV functions in one-dimensional metric spaces. Our main result gives a simple alternative definition of BV functions in a general one-dimensional space based on pointwise variation.

Let Ω denote an open set in the Euclidean space $\mathbb{R}^n$. A function $u \in L^1_{\mathrm{loc}}(\Omega)$ is said to have bounded variation in Ω if

$$\|Du\|(\Omega) := \sup\left\{\int_\Omega u \operatorname{div}\varphi\, dx : \varphi \in C^1_c(\Omega;\mathbb{R}^n),\ |\varphi| \le 1\right\} < \infty.$$

By the Riesz representation theorem, the class of functions with bounded variation in Ω, denoted by $\mathrm{BV}(\Omega)$, is the collection of functions whose weak first partial derivatives are Radon measures. An equivalent characterization of BV functions is given as the L^1 limits of sequences of smooth functions with gradients bounded in L^1. By replacing smooth functions with locally Lipschitz functions and the absolute value of the gradient by a local

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Lipschitz constant, Miranda [18] introduced functions of bounded variation on a complete doubling metric measure space (X, d, μ) supporting a Poincaré inequality. Equivalent definitions of BV functions on complete and separable metric measure spaces are studied by Ambrosio and Di Marino [2]. They relax the locally Lipschitz functions in Miranda's definition to a more general class of functions, with the local Lipschitz constants replaced by upper gradients. We recall the definition of BV functions on general metric measure spaces using upper gradients.

Definition 1.1. Given an open set $\Omega \subset X$ and a function u on Ω, the total variation of u in Ω is defined by

$$\|Du\|(\Omega) := \inf \left\{ \liminf_{i\to\infty} \int_\Omega g_{u_i}\, d\mu : \, u_i \to u \text{ in } L^1_{\text{loc}}(\Omega) \right\},$$

where each g_{u_i} is an upper gradient of u_i in Ω. A function u is said to have bounded variation on Ω if $\|Du\|(\Omega) < \infty$.

On the real line $\mathbb{R}$, various function classes usually have simpler characterizations. For example, upon choosing a good representative, we can identify a Sobolev function $u \in W^{1,p}([a, b])$ with an absolutely continuous function with p-integrable derivative [7, Theorem 1, Page 163]. Functions of bounded variation on $\mathbb{R}$ can also be characterized by pointwise variation. Recall that the pointwise variation of a function $u\colon [a, b] \to \mathbb{R}$ is defined as

$$\mathrm{PV}(u, [a, b]) := \sup \left\{ \sum_{k=1}^{n-1} |u(t_k) - u(t_{k+1})|, \; a \le t_1 \le \ldots \le t_n \le b \right\}. \tag{1.1}$$

If $\Omega \subset \mathbb{R}$ is open, the pointwise variation $\mathrm{PV}(u, \Omega)$ is defined as $\sup \sum_{k=1}^{n} \mathrm{PV}(u, I_k)$, where $\{I_k\}_{k=1}^{n}$ is a finite collection of pairwise disjoint closed intervals in Ω and supremum is taken over all such collections in Ω. The essential variation $\mathrm{eV}(u, \Omega)$ is defined as

$$\mathrm{eV}(u, \Omega) := \inf \{ \mathrm{PV}(v, \Omega) : \, u = v \text{ a.e. in } \Omega \}.$$

For $u \in L^1_{\text{loc}}(\Omega)$, we have $\mathrm{eV}(u, \Omega) = \|Du\|(\Omega)$ [3, Theorem 3.27].

The above characterizations of function classes can be extended to general one-dimensional metric spaces. Let X be a complete and connected metric space with finite one-dimensional Hausdorff measure $\mathcal{H}^1(X) < \infty$. In [19], the notion of absolutely continuous functions is generalized and Newtonian Sobolev functions are characterized by these absolutely continuous functions. Functions of bounded variations on curves in metric measure spaces are studied by Martio [16, 17]. In this work, we investigate the pointwise variation characterizations of BV functions on the above one-dimensional space. We first give the definition:

Definition 1.2. Let X be a complete connected metric measure space with $\mathcal{H}^1(X) < \infty$. For a function v on X, we define the pointwise variation as

$$\mathrm{pV}(v, X) := \sup \left\{ \sum_j |v \circ \gamma_j(\ell_j) - v \circ \gamma_j(0)| \right\},$$

where the supremum is taken over all finite collections of pairwise disjoint injective arc-length parametrized curves $\gamma_j \colon [0, \ell_j] \to X$. Then we define

$$\mathrm{Var}(u, X) := \inf\{\mathrm{pV}(v, X),\ v = u \text{ a.e. on } X\}.$$

A function $u \colon X \to \mathbb{R}$ has bounded pointwise variation if $\mathrm{Var}(u, X) < \infty$.

It can be shown that when X is an interval, we have $\mathrm{Var}(u, X) = \mathrm{eV}(u, X)$.

Remark 1.1. In the above definition, one could replace $|v \circ \gamma_j(\ell_j) - v \circ \gamma_j(0)|$ with $\mathrm{PV}(v \circ \gamma_j, [0, \ell_j])$ for each simple curve. Lemma 3.1 shows that the two quantities are comparable.

We say that a function $\widetilde{u}$ is a good representative of u if $u = \widetilde{u}$ almost everywhere and $\mathrm{Var}(u, X) = \mathrm{pV}(\widetilde{u}, X)$. We show that every function u with $\mathrm{Var}(u, X) < \infty$ admits a good representative.

Lemma 1.1 (Existence of a good representative)**.** *Suppose that $(X, d, \mathcal{H}^1)$ is a complete and connected metric measure space with $\mathcal{H}^1(X) < \infty$. If $\mathrm{Var}(u, X) < \infty$, then there exists a function $\widetilde{u}$ on X with $\widetilde{u} = u$ a.e. and*

$$\mathrm{pV}(\widetilde{u}, X) = \mathrm{Var}(u, X) = \inf\{\mathrm{pV}(v, X) : v = u \textit{ a.e. on } X\}.$$

We show that the class of BV functions given by Definition 1.2 is equivalent to the BV functions given in Definition 1.1. The main theorem is stated below:

Theorem 1.1 (Main Theorem)**.** *Suppose that $(X, d, \mathcal{H}^1)$ is a complete and connected metric measure space with $\mathcal{H}^1(X) < \infty$. Let u be a function on X. Then the following hold:*

(1) *If $\|Du\|(X) < \infty$, then $\mathrm{Var}(u, X) \le \|Du\|(X)$.*
(2) *Suppose there exists a constant C_0 such that for all $x \in X$*

$$\liminf_{r \to 0} \frac{\mathcal{H}^1(B(x, r))}{r} < C_0 \tag{1.2}$$

holds. If $\mathrm{Var}(u, X) < \infty$, then $\|Du\|(X) < \infty$.

Remark 1.2. In particular, if X is complete, connected and Ahlfors 1-regular with $\mathcal{H}^1(X) < \infty$, a function u on X satisfies $\|Du\|(X) < \infty$ if and only if $\mathrm{Var}(u, X) < \infty$.

Remark 1.3. The density upper bound (1.2) turns out to be essential in this characterization. Complete and connected metric spaces (X, d) with $\mathcal{H}^1(X) < \infty$ can be constructed such that a function u satisfies $\|Du\|(X) = \infty$ while $\mathrm{Var}(u, X) < \infty$, see Example 4.1 and Example 4.2.

The proof for the first part of the main theorem is standard and is given in Proposition 3.1. The second part requires a more delicate argument. Suppose u is a function with $\mathrm{Var}(u,X)<\infty$. We first use the existence of good representatives to show that $\mathrm{Var}(v,X)$ is lower semicontinuous with respect to convergence in $L^1(X)$. Then we prove the coarea inequality stated below, first for curve-continuous functions, i.e. functions that are continuous along every curve in X. A sequence of curve-continuous functions u_i approximating u in $L^1(X)$ can be constructed such that the limit superior of $\mathrm{pV}(u_i,X)$ is bounded above by $C_1\,\mathrm{Var}(u,X)$, where C_1 is a constant. These facts imply the following result; χ_E denotes the characteristic function of $E\subset X$.

Lemma 1.2 (Co-area Inequality)**.** *Let $(X,d,\mathcal{H}^1)$ be a complete and connected metric measure space with $\mathcal{H}^1(X)<\infty$. Suppose there exists a constant C_0 such that for all $x\in X$*

$$\liminf_{r\to 0}\frac{\mathcal{H}^1(B(x,r))}{r}<C_0$$

holds. Suppose $\mathrm{Var}(u,X)<\infty$*. Then*

$$C_1\,\mathrm{Var}(u,X)\geq \int_{\mathbb{R}}^{*}\mathrm{Var}(\chi_{\{u>t\}},X)\,dt.$$

Using also the BV coarea formula [18, Proposition 4.2] (see detailed statement (2.4) in Section 2), it now suffices to consider $u=\chi_E$ for $\mathrm{Var}(\chi_E,X)<\infty$. Hence the proof is completed by showing that $\|D\chi_E\|(X)$ is bounded above by $C_0\,\mathrm{Var}(\chi_E,X)$.

An interesting and important aspect of the theory of BV functions lies in the analysis of sets of finite perimeter, that is, sets whose characteristic functions are BV functions. For a set $E\subset\mathbb{R}^n$, Federer's characterization of sets of finite perimeter [8] states that E has finite perimeter if and only if the codimension one Hausdorff measure of its measure-theoretic boundary satisfies $\mathcal{H}(\partial^*E)<\infty$, see Section 4 for detailed definitions. Let (X,d,μ) be a complete and doubling metric measure space that supports a 1-Poincaré inequality and let $E\subset X$ be a measurable set. Ambrosio [1, Theorem 5.3] shows that if E has finite perimeter then $\mathcal{H}(\partial^*E)<\infty$. The converse implication of Federer's characterization in the general metric space setting is proved by the first author in [15, Theorem 1.1].

It has not been known so far whether the doubling and Poincaré inequality conditions on the underlying space are necessary when showing that the condition $\mathcal{H}(\partial^*E)<\infty$ implies that E is of finite perimeter. By constructing simple explicit examples of one-dimensional spaces, we show that these two conditions are really essential.

This paper is organized in the following way: preliminaries are covered in Section 2 and the proof of the main theorem is presented in Section 3. In Section 4, we construct two examples to show the necessity of the doubling condition and the Poincaré inequality in Federer's characterization.

2. Definitions and notation

Assume throughout the paper that $(X, d, \mathcal{H}^1)$ is a complete and connected metric space with $\mathcal{H}^1(X) < \infty$. If a property holds outside a set of $\mathcal{H}^1$-measure zero, we say that it holds almost everywhere, abbreviated a.e. The symbol C will denote a constant that only depends on the space X. We say that a measure μ is doubling if there exists a constant C such that $\mu(B(x, 2r)) \leq C\mu(B(x, r))$ for all open balls $B(x, r)$. The space X is Ahlfors s-regular if there is a constant C such that

$$C^{-1}r^s \leq \mu(B(x, r)) \leq Cr^s,$$

whenever $x \in X$ and $0 < r < \operatorname{diam}(X)$. If X is Ahlfors s-regular with respect to μ, we can replace μ by the s-dimensional Hausdorff measure $\mathcal{H}^s$ without losing essential information [12, Exercise 8.11].

A continuous mapping $\gamma\colon [a, b] \to X$ is said to be a rectifiable curve if it has finite length. A rectifiable curve always admits an arc-length parametrization (see e.g. [10, Theorem 3.2]). If $\gamma\colon [a, b] \to X$ is a rectifiable curve and $g\colon \gamma([a, b]) \to [0, \infty]$ is a Borel function, we define

$$\int_\gamma g\, ds := \int_0^\ell g(\widetilde{\gamma}(s))\, ds,$$

where $\widetilde{\gamma}\colon [0, \ell] \to X$ is the arc-length parametrization of γ. From now on we will assume all curves to be rectifiable and arc-length parametrized unless otherwise specified.

Definition 2.1 (Upper gradient)**.** Let $u\colon X \to \overline{\mathbb{R}}$. We say that a Borel function $g\colon X \to [0, \infty]$ is an upper gradient of u if

$$|u(\gamma(\ell_\gamma)) - u(\gamma(0))| \leq \int_\gamma g\, ds \tag{2.1}$$

for every curve γ. We use the conventions $\infty - \infty = \infty$ and $(-\infty) - (-\infty) = -\infty$. If $g\colon X \to [0, \infty]$ is a μ-measurable function and (2.1) holds for 1-almost every curve, we say that g is a 1-weak upper gradient of u. A property is said to hold for 1-almost every curve if there exists $\rho \in L^1(X)$ such that $\int_\gamma \rho\, ds = \infty$ for every curve γ for which the property fails.

For $1 \leq p < \infty$, the Newtonian Sobolev class $N^{1,p}(X)$ consists of those L^p-integrable functions on X for which there exists a p-integrable upper gradient.

The notation u_B stands for an integral average, that is,

$$u_B := \fint_B u\, d\mu := \frac{1}{\mu(B)} \int_B u\ d\mu.$$

A metric measure space supporting a Poincaré inequality is defined in the following way.

Definition 2.2 (Space supporting Poincaré inequality)**.** Let $1 \leq p < \infty$. A metric measure space (X, d, μ) is said to support a p-Poincaré inequality if there exists constants $C > 0$ and

$\lambda \geq 1$ such that the following holds for every pair of functions $u\colon X \to \overline{\mathbb{R}}$ and $g\colon X \to [0,\infty]$, where u is measurable and g is an upper gradient of u:

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq Cr \left(\fint_{B(x,\lambda r)} g^p \, d\mu \right)^{\frac{1}{p}}$$

for every ball $B(x,r)$.

A metric space X is quasiconvex if every two points can be joined by a curve with length comparable to the distance between these two points. If X is complete, doubling and supports a p-Poincaré inequality for $1 \leq p < \infty$, then X is quasiconvex [11, Proposition 4.4].

We recall the following generalization of the Euclidean area formula to the case of Lipschitz maps f from the Euclidean space $\mathbb{R}^n$ into a metric space X. The proof can be found in [14, Corollary 8].

Theorem 2.1 (Area formula)**.** *Let $f\colon \mathbb{R}^n \to X$ be Lipschitz. Then*

$$\int_{\mathbb{R}^n} g(x) J_n(mdf_x)\ dx = \int_X \sum_{x \in f^{-1}(y)} g(x) \, d\mathcal{H}^n(y)$$

for any Borel function $g\colon \mathbb{R}^n \to [0,\infty]$, and

$$\int_A g(f(x)) J_n(mdf_x)\ dx = \int_X g(y) \mathcal{H}^0(A \cap f^{-1}(y)) \, d\mathcal{H}^n(y)$$

for $A \subset \mathbb{R}^n$ measurable and any Borel function $g\colon X \to [0,\infty]$.

We apply the above theorem to an arc-length parametrized curve. Let $f = \gamma$ and $\gamma\colon [0,\ell] \to X$. In this case, $J_1(mdf_x)$ equals the metric derivative defined as

$$|\dot{\gamma}|(t) := \lim_{h \to 0} \frac{d(\gamma(t+h), \gamma(t))}{|h|},$$

and $|\dot{\gamma}|(t) = 1$ for almost every $t \in [0,\ell]$. Let $\Gamma = \gamma([0,\ell])$ and let $g\colon X \to [0,\infty]$ be a Borel function. It follows from Theorem 2.1 that

$$\int_0^\ell g(\gamma(s)) \, ds = \int_\Gamma g(y) \mathcal{H}^0([0,\ell] \cap \gamma^{-1}(y)) \, d\mathcal{H}^1(y). \tag{2.2}$$

A compact and connected 1-dimensional metric space admits a nice parametrization. The proofs of the following two classical results can be found in [4, Theorem 4.4.7, Theorem 4.4.8].

Theorem 2.2 (First Rectifiability Theorem)**.** *If E is complete and $C \subset E$ is a closed connected set such that $\mathcal{H}^1(C) < \infty$, then C is compact and connected by simple curves.*

Theorem 2.3 (Second Rectifiability Theorem)**.** *If E is complete, $C \subset E$ is closed and connected, and $\mathcal{H}^1(C) < \infty$, then there exist countably many arc-length parametrized simple* *curves* $\gamma_i\colon [0,\ell_i]\to C$ *such that*

$$\mathcal{H}^1\Big(C\setminus\bigcup_{i=1}^{\infty}\gamma_i([0,\ell_i])\Big)=0.$$

Given $u\in \mathrm{Lip}_{\mathrm{loc}}(X)$, we define the local Lipschitz constant by

$$\mathrm{Lip}\, u(x):=\limsup_{y\to x}\frac{|u(y)-u(x)|}{d(y,x)}. \tag{2.3}$$

Given an open set $\Omega\subset X$ and a function $u\in L^1_{\mathrm{loc}}(\Omega)$, we define the total variation of u in Ω by

$$\|Du\|(\Omega):=\inf\left\{\liminf_{i\to\infty}\int_\Omega g_{u_i}\,d\mu:\ u_i\in N^{1,1}_{\mathrm{loc}}(\Omega),\ u_i\to u\text{ in }L^1_{\mathrm{loc}}(\Omega)\right\},$$

where each g_i is a (1-weak) upper gradient of u_i in Ω. We say that a function $u\in L^1(\Omega)$ is of bounded variation, and denote $u\in \mathrm{BV}(\Omega)$, if $\|Du\|(\Omega)<\infty$. A μ-measurable set $E\subset X$ is said to be of finite perimeter if $\|D\chi_E\|(X)<\infty$, where χ_E is the characteristic function of E.

The following coarea formula is given in [18, Proposition 4.2]: if $\Omega\subset X$ is an open set and $u\in L^1_{\mathrm{loc}}(\Omega)$, then

$$\|Du\|(\Omega)=\int_{\mathbb{R}}^{*}\|D\chi_{\{u>t\}}\|(\Omega)\,dt, \tag{2.4}$$

where we abbreviate $\{u>t\}:=\{x\in\Omega:\ u(x)>t\}$. We use an upper integral since measurability is not clear, but if either side is finite, then both sides are finite and we also have measurability.

3. Proofs of the main results

Standing assumptions: We will assume throughout this section that $(X,d,\mathcal{H}^1)$ is a complete and connected metric measure space with $0<\mathcal{H}^1(X)<\infty$. By the First Rectifiability Theorem 2.2, it follows that X is compact.

3.1. Finite total variation implies finite pointwise variation.

We prove part (1) of Theorem 1.1 first.

Proposition 3.1. *Let u be a function on X such that $\|Du\|(X)<\infty$. Then* $\mathrm{Var}(u,X)\le \|Du\|(X)$.

Proof. From the definition of the total variation we find a sequence (u_i) such that $u_i\to u$ in $L^1(X)$ and

$$\lim_{i\to\infty}\int_X g_i\,d\mathcal{H}^1=\|Du\|(X), \tag{3.1}$$

where each g_i is an upper gradient of u_i. Passing to a subsequence (not relabeled), we also have $u_i\to u$ a.e. By the First Rectifiability Theorem 2.2, for every pair of points $x,y\in X$

we find a simple curve $\gamma\colon [0,\ell]\to X$ with $\gamma(0)=x$ and $\gamma(\ell)=y$, and then by (2.2),

$$|u_i(y)-u_i(x)|\le \int_\gamma g_i\,ds\le \int_X g_i\,d\mathcal{H}^1\to \|Du\|(X)\quad \text{as } i\to\infty.$$

Thus the functions u_i are uniformly bounded. Note that the sequence of Radon measures $g_i\,d\mathcal{H}^1$ has uniformly bounded mass, and so we know that passing to a subsequence (not relabeled) we have $g_i\,d\mathcal{H}^1 \overset{*}{\rightharpoonup} d\nu$ for some Radon measure ν on X [3, Theorem 1.59]. This reference also gives the lower semicontinuity

$$\nu(X)\le \liminf_{i\to\infty}\int_X g_i\,d\mathcal{H}^1=\|Du\|(X). \tag{3.2}$$

Moreover, for any compact set $K\subset X$ we have

$$\nu(K)\ge \limsup_{i\to\infty}\int_K g_i\,d\mathcal{H}^1; \tag{3.3}$$

see [3, Proposition 1.62] (and then in fact equality holds in (3.2)). Define $v(x):=\limsup_{i\to\infty}u_i(x)$ for every $x\in X$, so that $v=u$ $\mathcal{H}^1$-a.e., and v is bounded since the functions u_i are uniformly bounded. Now for every simple curve $\gamma\colon[0,\ell]\to X$ we have

$$\begin{aligned}
|v\circ\gamma(\ell)-v\circ\gamma(0)| &\le \limsup_{i\to\infty}|u_i\circ\gamma(\ell)-u_i\circ\gamma(0)|\\
&\le \limsup_{i\to\infty}\int_\gamma g_i\,ds\\
&= \limsup_{i\to\infty}\int_{\gamma([0,\ell])} g_i\,d\mathcal{H}^1\quad \text{by (2.2)}\\
&\le \nu(\gamma([0,\ell]))\quad \text{by (3.3)}.
\end{aligned}$$

It follows that for any finite collection of pairwise disjoint simple curves $\gamma_j\colon[0,\ell_j]\to X$,

$$\sum_j |v\circ\gamma_j(\ell_j)-v\circ\gamma_j(0)|\le \sum_j \nu(\gamma_j([0,\ell_j]))\le \nu(X)\le \|Du\|(X)\quad\text{by (3.2)}.$$

It follows that $\mathrm{pV}(v,X)\le\|Du\|(X)$ and so $\mathrm{Var}(u,X)\le\|Du\|(X)$. □

3.2. **Finite pointwise variation implies finite total variation.** The proof of part (2) of Theorem 1.1 is more involved. We divide the argument into several parts.

3.2.1. *Existence of a good representative.* We first show that every u with $\mathrm{Var}(u,X)<\infty$ admits a good representative $\widetilde{u}$. As a result, $\mathrm{Var}(u,X)$ turns out to be lower semicontinuous with respect to convergence in $L^1(X)$.

Note that we can define an alternative version of the pointwise variation of a function v on X by

$$\mathrm{PV}(v,X):=\sup\left\{\sum_j \mathrm{PV}(v\circ\gamma_j)\right\},$$

where the supremum is taken over finite collections of pairwise disjoint simple curves $\gamma_j\colon [0,\ell_j]\to X$, and we denote $\mathrm{PV}(v\circ\gamma_j):=\mathrm{PV}(v\circ\gamma_j,[0,\ell_j])$; recall (1.1). Then obviously $\mathrm{pV}(v,X)\le \mathrm{PV}(v,X)$. Conversely, we have the following.

Lemma 3.1. *For any function v on X, we have $\mathrm{PV}(v,X)\le 2\,\mathrm{pV}(v,X)$.*

Proof. Consider a simple curve γ. Take a partition $0=t_0\le t_1\le\ldots\le t_n=\ell_\gamma$. Suppose n is odd (the case of even n is similar). Then the subcurves $\gamma|_{[t_k,t_{k+1}]}$, for $k=0,2,\ldots,n-1$, are disjoint, and so are the subcurves $\gamma|_{[t_k,t_{k+1}]}$ for $k=1,3,\ldots,n-2$. Let γ^k be $\gamma|_{[t_k,t_{k+1}]}$ reparametrized by arc-length. Then

$$\begin{aligned}&\sum_{k=0}^{n-1}|v(\gamma(t_k))-v(\gamma(t_{k+1}))|\\&\qquad=\sum_{k=0,2,\ldots,n-1}|v(\gamma^k(0))-v(\gamma^k(\ell_{\gamma^k}))|+\sum_{k=1,3,\ldots,n-2}|v(\gamma^k(0))-v(\gamma^k(\ell_{\gamma^k}))|.\end{aligned}$$

Taking supremum over all partitions, we get $\mathrm{PV}(v\circ\gamma,[0,\ell_\gamma])\le 2\,\mathrm{pV}(v,X)$. If we consider collections of pairwise disjoint simple curves γ_j, and if we do the above for each γ_j, we obtain that $\mathrm{PV}(v,X)\le 2\,\mathrm{pV}(v,X)$. $\square$

Next we show that we can find a *good representative* $\widetilde{u}$ of any function u, with $\mathrm{pV}(\widetilde{u},X)=\mathrm{Var}(u,X)$. In proving this we will take inspiration from Martio [16]. Given a function v on X and a set $D\subset X$, we define

$$\mathrm{pV}_D(v,X):=\sup\left\{\sum_j|v\circ\gamma_j(\ell_j)-v\circ\gamma_j(0)|\right\},$$

where the supremum is taken over finite collections of pairwise disjoint simple curves $\gamma_j\colon[0,\ell_j]\to X$ with endpoints $\gamma_j(0),\gamma_j(\ell_j)\in D$.

Proposition 3.2. *Let $D\subset X$ be an arbitrary set with $\mathcal{H}^1(X\setminus D)=0$. Suppose $\mathrm{pV}_D(v,X)<\infty$. Then there exists a function v_e on X such that $v_e=v$ on D and $\mathrm{pV}(v_e,X)=\mathrm{pV}_D(v,X)$.*

Proof. If $x\in D$, define $v_e(x)=v(x)$. Fix a point $z_0\in D$. For any point $x\in X\setminus D$, by the First Rectifiability Theorem (Theorem 2.2), there exists a simple curve $\gamma_x\colon[0,\ell_x]\to X$ with $\gamma_x(0)=x$ and $\gamma_x(\ell_x)=z_0$. We define

$$v_e(x):=\lim_{t\to 0^+,\,\gamma_x(t)\in D}v\circ\gamma_x(t).$$

The limit exists since the quantity

$$\sup\left\{\sum_{k=1}^{n-1}|v\circ\gamma_x(t_k)-v\circ\gamma_x(t_{k+1})|,\ 0\le t_1\le\ldots\le t_n\le\ell_x,\ \gamma_x(t_k)\in D\right\}$$

is finite, which follows from the condition $\mathrm{pV}_D(v, X) < \infty$ just as in Lemma 3.1. Then we show that $v_e\colon X \to \mathbb{R}$, with $v_e = v$ on D, satisfies $\mathrm{pV}(v_e, X) = \mathrm{pV}_D(v, X)$. It is clear that $\mathrm{pV}(v_e, X) \geq \mathrm{pV}_D(v, X)$. Conversely, let $\{\gamma_j\}_{j=1}^n$ be an arbitrary collection of pairwise disjoint curves. If all the endpoints $\gamma_j(0), \gamma_j(\ell_j) \in D$, then

$$\sum_{j=1}^{n} |v \circ \gamma_j(\ell_j) - v \circ \gamma_j(0)| = \sum_{j=1}^{n} |v_e \circ \gamma_j(\ell_j) - v_e \circ \gamma_j(0)|.$$

If there exists a point $p_j = \gamma_j(\ell_j) \in X \setminus D$ (or $\gamma_j(0) \in X \setminus D$, or both), then we denote the curve connecting z_0 and p_j in the definition of the function value of v_e at p_j by $\gamma_{p_j}\colon [0, \ell_{p_j}] \to X$. Let $\epsilon > 0$ be arbitrary. We discuss two cases:

(1) If there exists $\delta > 0$ such that γ_j intersects with γ_{p_j} only at p_j inside $B(p_j, \delta)$, then we define a simple curve $\widetilde{\gamma}_j\colon [0, \widetilde{\ell}_j] \to X$ by

$$\widetilde{\gamma}_j(t) := \begin{cases} \gamma_j(t) & \text{if } 0 \leq t \leq \ell_j \\ \gamma_{p_j}(t - \ell_j) & \text{if } \ell_j \leq t \leq \widetilde{\ell}_j \end{cases}$$

where $\widetilde{\ell}_j \leq \ell_j + \delta$. By choosing $\widetilde{\ell}_j$ sufficiently close to ℓ_j, we have that

$$|v \circ \widetilde{\gamma}_j(\widetilde{\ell}_j) - v_e \circ \gamma_j(\ell_j)| < \frac{\epsilon}{2n}.$$

Likewise, if $p_j = \gamma_j(0) \in X \setminus D$, we can also extend γ_j slightly to $\widetilde{\gamma}_j$ by attaching a small piece of γ_{p_j} at the endpoint such that

$$|v \circ \widetilde{\gamma}_j(0) - v_e \circ \gamma_j(0)| < \frac{\epsilon}{2n}.$$

(2) If for every $\delta > 0$ there exists $q \in B(p_j, \delta)$ with $q \neq p_j$ such that $q = \gamma_j(\widetilde{t}) = \gamma_{p_j}(t)$ for some $\widetilde{t}, t$, then we define $\widetilde{\gamma}_j\colon [0, \widetilde{\ell}_j] \to X$ as the restriction of γ_j to $[0, \widetilde{t}]$, so that

$$\begin{aligned} |v \circ \widetilde{\gamma}_j(\widetilde{\ell}_j) - v_e \circ \gamma_j(\ell_j)| &= |v \circ \gamma_j(\widetilde{t}) - v_e \circ \gamma_j(\ell_j)| \\ &= |v \circ \gamma_{p_j}(t) - v_e(p_j)| \\ &\leq \frac{\epsilon}{2n}, \end{aligned}$$

if we choose t sufficiently close to 0. A similar modification works for the case when $p_j = \gamma_j(0)$.

Then we get a new collection of curves $\{\widetilde{\gamma}_j\}_{j=1}^n$ defined as above if at least one of the endpoints of γ_j belong to $X \setminus D$. Furthermore, since the curves γ_j are pairwise disjoint, we can choose δ sufficiently small such that the curves $\widetilde{\gamma}_j$ are pairwise disjoint. Hence, we get that

$$\sum_{j=1}^{n} |v_e \circ \gamma_j(\ell_j) - v_e \circ \gamma_j(0)| \leq \sum_{j=1}^{n} |v \circ \widetilde{\gamma}_j(\widetilde{\ell}_j) - v \circ \widetilde{\gamma}_j(0)| + \epsilon.$$

This implies that $\mathrm{pV}(v_e, X) \leq \mathrm{pV}_D(v, X)$, and $\mathrm{pV}(v_e, X) = \mathrm{pV}_D(v, X)$ follows. □

Proposition 3.3. *Suppose* $\operatorname{Var}(u,X)<\infty$. *Then there exists a function* $\widetilde{u}$ *on* X *with* $\widetilde{u}=u$ *a.e. and*

$$\mathrm{pV}(\widetilde{u},X)=\operatorname{Var}(u,X)=\inf\{\mathrm{pV}(v,X):\, v=u \text{ a.e. on } X\}.$$

Proof. Take a function $v=u$ a.e. with $\mathrm{pV}(v,X)<\infty$. Let $u_i\colon X\to\mathbb{R}$ be a sequence such that $u_i=v$ on D_i with $\mathcal{H}^1(X\setminus D_i)=0$ and $\mathrm{pV}(v_i,X)\to\operatorname{Var}(u,X)$. Let $D_0:=\bigcap_i D_i$. Then $u_i=v$ on D_0 and $\mathcal{H}^1(X\setminus D_0)=0$. By Proposition 3.2 there exists $\widetilde{u}\colon X\to\mathbb{R}$ such that $\widetilde{u}=v$ on D_0 and

$$\mathrm{pV}(\widetilde{u},X)=\mathrm{pV}_{D_0}(v,X)=\mathrm{pV}_{D_0}(u_i,X)\le \mathrm{pV}(u_i,X)\to\operatorname{Var}(u,X)\quad\text{as } i\to\infty.$$

□

We have the following lower semicontinuity results.

Proposition 3.4. *Suppose* $D\subset X$ *and* $v_i(x)\to v(x)$ *for all* $x\in D$. *Then*

$$\mathrm{pV}_D(v,X)\le\liminf_{i\to\infty}\mathrm{pV}_D(v_i,X).$$

Next suppose $u_i\to u$ *in* $L^1(X)$. *Then*

$$\operatorname{Var}(u,X)\le\liminf_{i\to\infty}\operatorname{Var}(u_i,X).$$

Proof. The first claim is easy to check. To prove the second, we can assume that the right-hand side is finite and in fact that $\operatorname{Var}(u_i,X)<\infty$ for each $i\in\mathbb{N}$, and then we can choose good representatives $\widetilde{u_i}$. Passing to a subsequence (not relabeled) we have $\widetilde{u_i}(x)\to u(x)$ for every $x\in D$ with $\mathcal{H}^1(X\setminus D)=0$. By the first claim,

$$\begin{aligned}\mathrm{pV}_D(u,X)&\le\liminf_{i\to\infty}\mathrm{pV}_D(\widetilde{u_i},X)\\&\le\liminf_{i\to\infty}\mathrm{pV}(\widetilde{u_i},X)\\&=\liminf_{i\to\infty}\operatorname{Var}(u_i,X)<\infty.\end{aligned}\tag{3.4}$$

By Proposition 3.2, there exists an extension u_e for u restricted to D satisfying $u_e=u$ on D and $\mathrm{pV}(u_e,X)=\mathrm{pV}_D(u,X)$. In particular, $u_e=u$ a.e. on X. We get

$$\begin{aligned}\operatorname{Var}(u,X)&=\inf\{\mathrm{pV}(v,X):\, v=u \text{ a.e. on } X\}\\&\le\mathrm{pV}(u_e,X)\\&=\mathrm{pV}_D(u,X)\\&=\liminf_{i\to\infty}\operatorname{Var}(u_i,X)\end{aligned}$$

by (3.4). □

3.2.2. *Approximation by curve-continuous functions.* We say that a function v on X is curve-continuous if $v\circ\gamma$ is continuous for every curve γ in X. In this part, we exploit the nice properties of curve-continuous functions to show that every function with $\operatorname{Var}(u,X)<\infty$

is $\mathcal{H}^1$-measurable and it can be approximated in $L^1(X)$ by a sequence of curve-continuous functions u_i such that

$$\limsup_{i\to\infty} \mathrm{pV}(u_i, X) \le C_1 \operatorname{Var}(u, X)$$

for some constant C_1 depending only on C_0 in the density upper bound condition (1.2). We first show that every curve-continuous function is $\mathcal{H}^1$ measurable.

Lemma 3.2. *Let v be a curve-continuous function on X. Then v is $\mathcal{H}^1$-measurable.*

Proof. Let $t \in \mathbb{R}$. It suffices to show that $\{v \ge t\}$ is $\mathcal{H}^1$-measurable. By curve-continuity, for each curve $\gamma\colon [0,\ell] \to X$ the set $\gamma([0,\ell]) \cap \{v \ge t\}$ is compact. By the Second Rectifiability Theorem 2.3, there exist curves $\gamma_j\colon [0,\ell_j] \to X$, $j \in \mathbb{N}$, such that

$$\mathcal{H}^1\left(X \setminus \bigcup_{j=1}^{\infty} \gamma_j([0,\ell_j])\right) = 0.$$

The set $\bigcup_{j=1}^{\infty}(\gamma_j([0,\ell_j]) \cap \{v \ge t\})$ is a Borel set and differs from $\{v \ge t\}$ only by a set of $\mathcal{H}^1$-measure zero. □

For a function v on X and $t \in \mathbb{R}$, $r > 0$, we define the truncations $v_t := \min\{t, v\}$ and $v_{t,t+r} := \max\{t, \min\{t + r, v\}\}$.

Lemma 3.3. *Let v be a curve-continuous function on X with $\mathrm{pV}(v, X) < \infty$ and let $t \in \mathbb{R}$, $r > 0$. Then*

$$\mathrm{pV}(v_t, X) + \mathrm{pV}(v_{t,t+r}, X) \le \mathrm{pV}(v_{t+r}, X).$$

Proof. Consider a curve γ used in estimating $\mathrm{pV}(v_t, X) < \infty$. Note that $v_t \equiv t$ in $\{v \ge t\}$. Thus, by also reversing direction if necessary, we can assume that $\gamma(0) \in \{v < t\}$. Suppose also $\gamma(\ell_\gamma) \in \{v < t\}$, but γ intersects $\{v \ge t\}$. Let s_1, s_2 be the smallest and largest number, respectively, for which $\gamma(s_1), \gamma(s_2) \in \{v \ge t\}$; these exist by the curve-continuity. If $\varepsilon > 0$, by curve-continuity we find $\widetilde{s}_1 < s_1, \widetilde{s}_2 > s_2$ such that $v_t(\gamma(\widetilde{s}_1)) > t - \varepsilon$ and $v_t(\gamma(\widetilde{s}_2)) > t - \varepsilon$. Then for the subcurves $\gamma_1 := \gamma|_{[0,\widetilde{s}_1]}$ and $\gamma_2 := \gamma|_{[\widetilde{s}_2,\ell_\gamma]}$ (reparametrized by arc-length) we have

$$|v_t(\gamma_1(0)) - v_t(\gamma_1(\ell_{\gamma_1}))| \ge |v_t(\gamma(0)) - t| - \varepsilon$$

and

$$|v_t(\gamma_2(0)) - v_t(\gamma_2(\ell_{\gamma_2}))| \ge |v_t(\gamma(\ell_\gamma)) - t| - \varepsilon.$$

Thus

$$|v_t(\gamma_1(0)) - v_t(\gamma_1(\ell_{\gamma_1}))| + |v_t(\gamma_2(0)) - v_t(\gamma_2(\ell_{\gamma_2}))| \ge |v_t(\gamma(0)) - v_t(\gamma(\ell_\gamma))| - 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that in the definition of $\mathrm{pV}(v, X)$, we can replace the curve γ by two curves that are contained in $\{v < t\}$. Similarly, if $\gamma(0) \in \{v < t\}$ and $\gamma(\ell_\gamma) \in \{v \ge t\}$, we can replace such γ by one subcurve that is in $\{v < t\}$.

Now fix $\varepsilon > 0$ and take a collection of pairwise disjoint simple curves γ_j contained inside $\{v < t\}$ such that

$$\sum_{j=1}^{N_1} |v_t \circ \gamma_j(\ell_j) - v_t \circ \gamma_j(0)| + \varepsilon > \mathrm{pV}(v_t, X).$$

Analogously, we find a collection of pairwise disjoint simple curves γ_j contained inside $\{v > t\}$ such that

$$\sum_{j=N_1+1}^{N_2} |v_{t,t+r} \circ \gamma_j(\ell_j) - v_{t,t+r} \circ \gamma_j(0)| + \varepsilon > \mathrm{pV}(v_{t,t+r}, X).$$

Now the curves γ_j, $j = 1, \ldots, N_2$, are pairwise disjoint, and thus

$$\begin{aligned}
&\mathrm{pV}(v_t, X) + \mathrm{pV}(v_{t,t+r}, X) \\
&\quad\le \sum_{j=1}^{N_1} |v_t \circ \gamma_j(\ell_j) - v_t \circ \gamma_j(0)| + \sum_{j=N_1+1}^{N_2} |v_{t,t+r} \circ \gamma_j(\ell_j) - v_{t,t+r} \circ \gamma_j(0)| + 2\varepsilon \\
&\quad= \sum_{j=1}^{N_2} |v_{t+r} \circ \gamma_j(\ell_j) - v_{t+r} \circ \gamma_j(0)| + 2\varepsilon \\
&\quad\le \mathrm{pV}(v_{t+r}, X) + 2\varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$, we get $\mathrm{pV}(v_t, X) + \mathrm{pV}(v_{t,t+r}, X) \le \mathrm{pV}(v_{t+r}, X)$. □

Lemma 3.4. *Let v be a curve-continuous function on X and $t \in \mathbb{R}$, $r > 0$. Then*

$$\mathrm{pV}(\chi_{\{v>t\}}, X) \le \liminf_{r\to 0} \frac{1}{r} \mathrm{pV}(v_{t,t+r}, X).$$

Proof. Let $\gamma\colon [0, \ell] \to X$ be a simple curve. We have for every $s \in [0, \ell]$

$$\chi_{\{v>t\}}(\gamma(s)) = \lim_{r\to 0} \frac{v_{t,t+r}(\gamma(s)) - t}{r}.$$

In fact, if $v(\gamma(s)) \le t$, then $\chi_{\{v>t\}}(\gamma(s)) = 0$ and $v_{t,t+r}(\gamma(s)) = t$. If $v(\gamma(s)) > t$, then $\chi_{\{v>t\}}(\gamma(s)) = 1$. Choose r_0 sufficiently small such that $v(\gamma(s)) \ge t + r$ for all $r \le r_0$ and then $v_{t,t+r}(\gamma(s)) = t + r$.

Now

$$|\chi_{\{v>t\}} \circ \gamma(\ell) - \chi_{\{v>t\}} \circ \gamma(0)| = \lim_{r\to 0} r^{-1} |v_{t,t+r} \circ \gamma(\ell) - v_{t,t+r} \circ \gamma(0)|.$$

Let $\varepsilon > 0$. Then take a collection of pairwise disjoint injective curves γ_j such that

$$\begin{aligned}
\min\{\mathrm{pV}(\chi_{\{v>t\}}, X), \varepsilon^{-1}\} &\le \sum_{j=1}^{N} |\chi_{\{v>t\}} \circ \gamma_j(\ell_j) - \chi_{\{v>t\}} \circ \gamma_j(0)| + \varepsilon \\
&= \sum_{j=1}^{N} \lim_{r\to 0} r^{-1} |v_{t,t+r} \circ \gamma_j(\ell_j) - v_{t,t+r} \circ \gamma_j(0)| + \varepsilon \\
&= \lim_{r\to 0} r^{-1} \sum_{j=1}^{N} |v_{t,t+r} \circ \gamma_j(\ell_j) - v_{t,t+r} \circ \gamma_j(0)| + \varepsilon \\
&\le \liminf_{r\to 0} r^{-1}\, \mathrm{pV}(v_{t,t+r}, X) + \varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$, we get the result. □

For any functions v, w on X, we clearly have the subadditivity

$$\mathrm{pV}(v + w, X) \le \mathrm{pV}(v, X) + \mathrm{pV}(w, X). \tag{3.5}$$

Define the *inner metric* d_{in} by

$$d_{in}(x, y) := \inf\{\ell_\gamma :\ \gamma \text{ is a curve such that } \gamma(0) = x,\ \gamma(\ell_\gamma) = y\}, \quad x, y \in X.$$

Denote a ball with respect to the inner metric by $B_{in}(x, r)$.

Proposition 3.5. *Suppose there exists a constant C_0 such that for all $x \in X$*

$$\liminf_{r\to 0} \frac{\mathcal{H}^1(B(x, r))}{r} < C_0$$

holds. Suppose $\mathrm{Var}(u, X) < \infty$*. Then u is $\mathcal{H}^1$-measurable and there exists a sequence of curve-continuous functions $u_i \to u$ in $L^1(X)$ such that*

$$\limsup_{i\to\infty} \mathrm{pV}(u_i, X) \le C_1 \mathrm{Var}(u, X).$$

for a constant C_1 that depends only on C_0.

Proof. By Proposition 3.3 we find a good representative v of u. Note that v is necessarily bounded; if it were not, we could fix a point x_0 and find points x_j with $|v(x_j)| \to \infty$ as $j \to \infty$, and join x_0 to each x_j with a curve γ_j (by the First Rectifiability Theorem), to get

$$\mathrm{pV}(v, X) \ge |v(\gamma_j(\ell_{\gamma_j})) - v(\gamma_j(0))| = |v(x_j) - v(x_0)| \to \infty \quad \text{as } j \to \infty.$$

Fix $\varepsilon > 0$. Consider all the points where v is not curve-continuous; such points are contained in the "jump sets", defined for $\kappa > 0$ by

$$\begin{aligned}
J_{v,\kappa} := \{x \in X :\ &\text{for all } \delta > 0 \text{ there exist pairwise disjoint curves } \gamma_j \subset B_{in}(x, \delta) \\
&\text{such that } \sum_j |v(\gamma_j(\ell_j)) - v(\gamma_j(0))| \ge \kappa\}.
\end{aligned} \tag{3.6}$$

We can see that each $J_{v,\kappa}$ is finite (else we would get $\mathrm{pV}(v,X)=\infty$). Let also $J_v := \bigcup_{\kappa>0} J_{v,\kappa}$. For every $x\in J_v$, we define the "size of the jump"

$$J_v(x) := \sup\{\kappa>0:\, x\in J_{v,\kappa}\}.$$

Let $\varepsilon>0$. The set J_v is at most countable, and so we find an open set $W_\varepsilon \supset J_v$ with $\mathcal{H}^1(W_\varepsilon)<\varepsilon$.

Let x_k be an enumeration of all the points in J_v, with the jumps $J_v(x_k)$ in decreasing order. Note first that by choosing suitable short curves near the jump points, we find that

$$\mathrm{pV}(v,X) \ge \sum_{k=1}^{\infty} J_v(x_k). \tag{3.7}$$

We modify v as follows. We find $r_1>0$ such that $B_1 = B_{in}(x_1,r_1)\subset W_\varepsilon$ and, using also (1.2) (below $\mathrm{pV}(v,2B_1)$ means that all the curves considered are inside $2B_1 = B_{in}(x_1,2r_1)$)

$$\mathrm{pV}(v,2B_1) \le 2J_v(x_1) \quad \text{and} \quad \frac{\mathcal{H}^1(2B_1)}{r_1} < 2C_0. \tag{3.8}$$

Choose a function η_1 that is r_1^{-1}-Lipschitz with respect to d_{in}, with $\eta_1=1$ in B_1 and $\eta_1=0$ outside $2B_1$. Define (v_{B_1} denotes integral average)

$$w_1 := v(1-\eta_1) + \eta_1\cdot v_{B_1}.$$

Note that $J_{w_1}\subset J_v\setminus\{x_1\}$ and that

$$J_{w_1}(x_k) \le J_v(x_k) \quad \text{for all } k\ge 2. \tag{3.9}$$

Note also that $w_1 = v+\eta_1(v_{B_1}-v)$ and consider $\mathrm{pV}(\eta_1(v_{B_1}-v),X)$. Let γ_j be pairwise disjoint simple curves. Note that $\eta_1(v_{B_1}-v)\neq 0$ only inside the ball $2B_1$. By splitting the curves γ_j into subcurves if necessary, we can assume that each of them is contained inside the ball $2B_1$. Then we have

$$\begin{aligned}
&|(\eta_1(v_{B_1}-v))(\gamma_j(\ell_j)) - (\eta_1(v_{B_1}-v))(\gamma_j(0))| \\
&\quad\le |\eta_1(\gamma_j(\ell_j))(v_{B_1}-v)(\gamma_j(\ell_j)) - \eta_1(\gamma_j(\ell_j))(v_{B_1}-v)(\gamma_j(0))| \\
&\qquad + |\eta_1(\gamma_j(\ell_j))(v_{B_1}-v)(\gamma_j(0)) - \eta_1(\gamma_j(0))(v_{B_1}-v)(\gamma_j(0))| \\
&\quad\le |v(\gamma_j(\ell_j)) - v(\gamma_j(0))| + |\eta_1(\gamma_j(\ell_j)) - \eta_1(\gamma_j(0))| \sup_{2B_1}|v_{B_1}-v| \\
&\quad\le |v(\gamma_j(\ell_j)) - v(\gamma_j(0))| + |\eta_1(\gamma_j(\ell_j)) - \eta_1(\gamma_j(0))|\cdot 2J_v(x_1) \quad \text{by (3.8)} \\
&\quad\le |v(\gamma_j(\ell_j)) - v(\gamma_j(0))| + r_1^{-1}\ell_{\gamma_j}\cdot 2J_v(x_1).
\end{aligned}$$

Thus

$$\begin{aligned}\sum_j &|(\eta_1(v_{B_1}-v))(\gamma_j(\ell_j))-(\eta_1(v_{B_1}-v))(\gamma_j(0))|\\ &\le \sum_j |v(\gamma_j(\ell_j))-v(\gamma_j(0))|+r_1^{-1}\mathcal{H}^1(2B_1)2J_v(x_1)\le (2+4C_0)J_v(x_1)\quad \text{by (3.8)}\end{aligned}$$

and so

$$\mathrm{pV}(\eta_1(v_{B_1}-v),X)\le (2+4C_0)J_v(x_1).$$

Finally, by (3.5),

$$\mathrm{pV}(w_1,X)\le \mathrm{pV}(v,X)+\mathrm{pV}(\eta_1(v_{B_1}-v),X)\le \mathrm{pV}(v,X)+(2+4C_0)J_v(x_1). \tag{3.10}$$

Now we can do this inductively. For each $k\in\mathbb{N}$, provided that $x_{k+1}\in J_{w_k}$ (if not, we just let $w_{k+1}=w_k$) we choose $r_{k+1}>0$ such that $2B_{k+1}=B_{in}(x_{k+1},2r_{k+1})\subset W_\varepsilon$ and

$$\mathrm{pV}(w_k,2B_{k+1})\le 2J_{w_k}(x_{k+1})\quad\text{and}\quad \frac{\mathcal{H}^1(2B_{k+1})}{r_{k+1}}<2C_0.$$

As above, we choose a cutoff function η_{k+1} and then define

$$w_{k+1}:=w_k(1-\eta_{k+1})+\eta_{k+1}\cdot (w_k)_{B_{k+1}}.$$

We claim that for all $k\in\mathbb{N}$, we have

$$\mathrm{pV}(w_k,X)\le \mathrm{pV}(v,X)+(2+4C_0)\sum_{m=1}^{k}J_v(x_m)$$

and that

$$J_{w_k}(x_m)\le J_v(x_m)\quad\text{for all } m\ge k+1. \tag{3.11}$$

We have shown these to be true for $k=1$ (recall also (3.9)), and (3.11) is easily seen to hold with k replaced by $k+1$. Moreover,

$$\begin{aligned}\mathrm{pV}(w_{k+1},X)&\le \mathrm{pV}(w_k,X)+(2+4C_0)J_{w_k}(x_{k+1})\quad \text{(just as in (3.10))}\\ &\le \mathrm{pV}(v,X)+(2+4C_0)\sum_{m=1}^{k}J_v(x_m)+(2+4C_0)J_{w_k}(x_{k+1})\quad \text{by ind. hyp.}\\ &\le \mathrm{pV}(v,X)+(2+4C_0)\sum_{m=1}^{k+1}J_v(x_m)\quad \text{by (3.11).}\end{aligned}$$

Then let $w:=\lim_{k\to\infty}w_k$. Note that the convergence is uniform, in particular pointwise, since

$$|w_{k+1}-w_k|\le 2J_v(x_{k+1})$$

and recalling (3.7). Now by Proposition 3.4 and (3.7),

$$\begin{aligned}\mathrm{pV}(w,X) &\le \liminf_{k\to\infty} \mathrm{pV}(w_k,X)\\ &\le \mathrm{pV}(v,X) + (2+4C_0)\sum_{k=1}^{\infty} J_v(x_k) \le (3+4C_0)\,\mathrm{pV}(v,X).\end{aligned}$$

Since w_k has jump discontinuities on curves with jump size at most $J_v(x_{k+1}) \to 0$ as $k\to\infty$, and since $w_k \to w$ uniformly, we see that w is curve-continuous.

Recall that w also depends on $\varepsilon > 0$, with $w = v$ outside the open set W_ε with $\mathcal{H}^1(W_\varepsilon) < \varepsilon$. Recall also that v is bounded, and furthermore it is easy to check from the construction that $\inf_X v \le w \le \sup_X v$. Choosing $\varepsilon = 1/i$ and letting u_i be the corresponding curve-continuous function w, we now get $u_i \to u$ a.e., and so u is $\mathcal{H}^1$-measurable by Lemma 3.2, and then $u_i \to u$ in $L^1(X)$ and

$$\limsup_{i\to\infty} \mathrm{pV}(u_i,X) \le (3+4C_0)\,\mathrm{pV}(v,X) = (3+4C_0)\,\mathrm{Var}(u,X).$$

□

3.2.3. *Coarea inequality and the conclusion.* In the last part, we will show a coarea inequality and prove the implication from sets with finite pointwise variation to finite total variation. First we show the following coarea inequality.

Proposition 3.6. *Suppose there exists a constant C_0 such that for all $x \in X$*

$$\liminf_{r\to 0} \frac{\mathcal{H}^1(B(x,r))}{r} < C_0$$

holds. Suppose $\mathrm{Var}(u,X) < \infty$. *Then*

$$C_1\,\mathrm{Var}(u,X) \ge \int_{\mathbb{R}}^{*} \mathrm{Var}(\chi_{\{u>t\}},X)\,dt.$$

Note that we use an upper integral since measurability is not clear.

Proof. First assume that u is curve-continuous and that $\mathrm{pV}(u,X) < \infty$. Define (recall that $u_t = \min\{t,u\}$)

$$m(t) := \mathrm{pV}(u_t,X), \quad t \in \mathbb{R}.$$

Then m is an increasing function and so

$$\mathrm{pV}(u,X) \ge \int_{-\infty}^{\infty} m'(t)\,dt.$$

Let $\varepsilon > 0$. Now by Lemma 3.3,

$$m(t+r) - m(t) \ge \mathrm{pV}(u_{t,t+r},X).$$

Furthermore, Lemma 3.4 implies that

$$\liminf_{r\to 0} \frac{m(t+r)-m(t)}{r} \ge \liminf_{r\to 0} \frac{\mathrm{pV}(u_{t,t+r},X)}{r} \ge \mathrm{pV}(\chi_{\{u>t\}},X).$$

Thus we have

$$\mathrm{pV}(u,X) \ge \int_{\mathbb{R}}^{*} \mathrm{pV}(\chi_{\{u>t\}},X)\,dt \ge \int_{\mathbb{R}}^{*} \mathrm{Var}(\chi_{\{u>t\}},X)\,dt. \tag{3.12}$$

Now for a general function u on X with $\mathrm{Var}(u,X)<\infty$, by Proposition 3.5 we find a sequence of curve-continuous functions u_i with $u_i \to u$ in $L^1(X)$ and

$$\limsup_{i\to\infty} \mathrm{pV}(u_i,X) \le C_1\,\mathrm{Var}(u,X).$$

For every $x\in X$,

$$\int_{-\infty}^{\infty} |\chi_{\{u_i>t\}}(x)-\chi_{\{u>t\}}(x)|\,dt = \int_{\min\{u_i(x),u(x)\}}^{\max\{u_i(x),u(x)\}} dt = |u_i(x)-u(x)|.$$

Hence by Fubini's theorem (recall the measurability statement of Proposition 3.5)

$$\begin{aligned}\int_X |u_i-u|\,d\mathcal{H}^1 &= \int_X \int_{-\infty}^{\infty} |\chi_{\{u_i>t\}}(x)-\chi_{\{u>t\}}(x)|\,dt\,d\mathcal{H}^1(x)\\ &= \int_{-\infty}^{\infty} \int_X |\chi_{\{u_i>t\}}(x)-\chi_{\{u>t\}}(x)|\,d\mathcal{H}^1(x)\,dt.\end{aligned}$$

Thus $\|\chi_{\{u_i>t\}}-\chi_{\{u>t\}}\|_{L^1(X)} \to 0$ in $L^1(\mathbb{R})$ and so we can find a subsequence of u_i (not relabeled) such that

$$\|\chi_{\{u_i>t\}}-\chi_{\{u>t\}}\|_{L^1(X)} \to 0 \quad \text{for a.e. } t\in\mathbb{R}.$$

Then for such t, by the lower semicontinuity of Proposition 3.4,

$$\mathrm{Var}(\chi_{\{u>t\}},X) \le \liminf_{i\to\infty} \mathrm{Var}(\chi_{\{u_i>t\}},X).$$

We find measurable functions $h_i \ge \chi_{\{u_i>t\}}$ on $\mathbb{R}$ such that

$$\liminf_{i\to\infty} \int_{-\infty}^{\infty} h_i(t)\,dt = \liminf_{i\to\infty} \int_{\mathbb{R}}^{*} \mathrm{Var}(\chi_{\{u_i>t\}},X)\,dt.$$

Then by Fatou's lemma

$$\begin{aligned}\int_{\mathbb{R}}^{*} \mathrm{Var}(\chi_{\{u>t\}},X)\,dt &\le \int_{\mathbb{R}}^{*} \liminf_{i\to\infty} \mathrm{Var}(\chi_{\{u_i>t\}},X)\,dt\\ &\le \int_{\mathbb{R}} \liminf_{i\to\infty} h_i(t)\,dt\\ &\le \liminf_{i\to\infty} \int_{\mathbb{R}} h_i(t)\,dt\\ &= \liminf_{i\to\infty} \int_{\mathbb{R}}^{*} \mathrm{Var}(\chi_{\{u_i>t\}},X)\,dt\\ &\le \liminf_{i\to\infty} \mathrm{pV}(u_i,X) \quad \text{by (3.12)}\\ &\le C_1\,\mathrm{Var}(u,X).\end{aligned}$$

□

Due to the above coarea inequality, it will suffice to consider characteristic functions $u = \chi_E$ for $E \subset X$.

Proposition 3.7. *Suppose there exists a constant C_0 such that for all $x \in X$*

$$\liminf_{r\to 0} \frac{\mathcal{H}^1(B(x,r))}{r} < C_0$$

holds. Let $E \subset X$. Then $\|D\chi_E\|(X) \le C_0 \operatorname{Var}(\chi_E, X)$.

Proof. We can assume that $\operatorname{Var}(\chi_E, X) < \infty$. By Proposition 3.3 we find a good representative v of χ_E, so that $\operatorname{pV}(v, X) = \operatorname{Var}(\chi_E, X)$. Let $D := \{x \in X : v(x) \in \{0, 1\}\}$, so that $\mathcal{H}^1(X \setminus D) = 0$. By Proposition 3.2 and its proof, we know that there is a function v_e on X with $v_e = v$ on D, taking only the values $0, 1$, with $\operatorname{pV}(v_e, X) = \operatorname{pV}_D(v, X) \le \operatorname{pV}(v, X)$ and so in fact $\operatorname{pV}(v_e, X) = \operatorname{Var}(\chi_E, X)$. In conclusion, we can take the good representative to be χ_F for $F \subset X$, and then $\operatorname{pV}(\chi_F, X) = \operatorname{Var}(\chi_E, X)$.

Recall the definition of the jump set from (3.6); it is not difficult to see that now

$$J_{\chi_F} = \{x \in X : \text{for all } \delta > 0 \,\text{there exists a curve } \gamma \subset B_{in}(x, \delta) \\ \text{that intersects both } F \text{ and } X \setminus F\}.$$

We call this the "curve boundary" $\partial^c F := J_{\chi_F}$. Clearly any curve intersecting both F and $X \setminus F$ needs to intersect also $\partial^c F$. Now if $\mathcal{H}^0(\partial^c F) = \infty$, then we can pick arbitrarily many disjoint curves $\gamma \colon [0, \ell] \to X$ with $|\chi_F(\gamma(\ell)) - \chi_F(\gamma(0))| = 1$ and thus $\operatorname{pV}(\chi_F, X) = \infty$. But since $\operatorname{pV}(\chi_F, X) < \infty$, actually $\mathcal{H}^0(\partial^c F) < \infty$. In other words, $\partial^c F = \{x_1, \ldots, x_N\}$ with $N \le \operatorname{pV}(\chi_F, X) = \operatorname{Var}(\chi_E, X)$.

Take a sequence $\delta_i \searrow 0$ such that the balls $B(x_j, \delta_1)$, $j = 1, \ldots, N$, are pairwise disjoint. Fix $i \in \mathbb{N}$. By (1.2), for each $j = 1, \ldots, N$ we find $\delta_{j,i} \in (0, \delta_i)$ such that

$$\frac{\mathcal{H}^1(B(x_j, \delta_{j,i}))}{\delta_{j,i}} < C_0. \tag{3.13}$$

For each $j = 1, \ldots, N$, let $\eta_{j,i}$ be a $1/\delta_{j,i}$-Lipschitz function with $\eta_{j,i}(x_j) = 1$ and $\eta_{j,i} = 0$ outside $B(x_j, \delta_{j,i})$. Define

$$v_i := \max\{\eta_{1,i}, \ldots, \eta_{N,i}\} \quad \text{and} \quad u_i := \max\{\chi_F, \eta_{1,i}, \ldots, \eta_{N,i}\}.$$

Let

$$g_i := \sum_{j=1}^{N} \frac{\chi_{B(x_j,\delta_{j,i})}}{\delta_{j,i}}.$$

Note that since the pointwise Lipschitz constant (2.3) is an upper gradient [5, Proposition 1.14], and by [5, Corollary 2.21], we know that $\chi_{B(x_j,\delta_{j,i})}/\delta_{j,i}$ is a 1-weak upper gradient of $\eta_{j,i}$ (recall Definition 2.1). Then g_i is a 1-weak upper gradient of v_i.

Then we can verify that g_i is a 1-weak upper gradient of u_i. For this we need to check three cases for a curve $\gamma \colon [0, \ell] \to X$ with end points $\gamma(0) = x$ and $\gamma(\ell) = y$. We can assume that the pair (v_i, g_i) satisfies the upper gradient inequality on the curve γ as well as all of its subcurves [5, Lemma 1.40]. The first case is $x, y \in F$, where

$$|u_i(x) - u_i(y)| = 0 \le \int_\gamma g_i\, ds.$$

The second case is $x, y \in X \setminus F$. Here

$$|u_i(x) - u_i(y)| = |v_i(x) - v_i(y)| \le \int_\gamma g_i\, ds.$$

The third case is $x \in F$ and $y \in X \setminus F$. As mentioned before, γ now necessarily intersects $\partial^c F$. Thus there is some $t \in [0, \ell]$ such that $\gamma(t) \in \partial^c F$, and thus $\gamma(t) = x_j$ for some j. Note that $u_i(\gamma(0)) = 1$, $u_i(\gamma(t)) = v_i(\gamma(t)) = 1$, and $u_i(\gamma(\ell)) = v_i(\gamma(\ell))$. It follows that

$$\begin{aligned} |u_i(\gamma(\ell)) - u_i(\gamma(0))| &\le |u_i(\gamma(\ell)) - u_i(\gamma(t))| + |u_i(\gamma(t)) - u_i(\gamma(0))| \\ &= |v_i(\gamma(\ell)) - v_i(\gamma(t))| \le \int_\gamma g_i\, ds. \end{aligned}$$

In conclusion, g_i is a 1-weak upper gradient of u_i. It is easy to see that also $u_i \to \chi_E$ in $L^1(X)$. Now we have, using (3.13),

$$\|D\chi_E\|(X) \le \liminf_{i\to\infty} \int_X g_i\, d\mathcal{H}^1 \le \liminf_{i\to\infty} \sum_{j=1}^N \frac{\mathcal{H}^1(B(x_j, \delta_{j,i}))}{\delta_{j,i}} \le C_0 N \le C_0 \operatorname{Var}(\chi_E, X).$$

□

Proposition 3.8. *Suppose there exists a constant C_0 such that for all $x \in X$*

$$\liminf_{r\to 0} \frac{\mathcal{H}^1(B(x,r))}{r} < C_0$$

holds. Suppose $\operatorname{Var}(u, X) < \infty$. *Then* $\|Du\|(X) \le C \operatorname{Var}(u, X)$.

Proof. From $\operatorname{Var}(u, X) < \infty$ it follows that u is essentially bounded, and u is $\mathcal{H}^1$-measurable by Proposition 3.5. Combined with the fact that $\mathcal{H}^1(X) < \infty$, we get $u \in L^1(X)$. By the BV coarea formula (2.4), Proposition 3.7, and the coarea inequality of Proposition 3.6, it follows that

$$\|Du\|(X) = \int_{\mathbb{R}}^{*} \|D\chi_{\{u>t\}}\|(X)\, dt \le C_0 \int_{\mathbb{R}}^{*} \operatorname{Var}(\chi_{\{u>t\}}, X)\, dt \le C_0 C_1 \operatorname{Var}(u, X).$$

□

Theorem 1.1 follows by combining Proposition 3.1 and Proposition 3.8.

4. Federer's characterization of sets of finite perimeter

Let us briefly consider a more general metric space (X, d, μ), where μ is a Radon measure. The *codimension one Hausdorff measure* is defined for any set $A \subset X$ by

$$\mathcal{H}(A) := \lim_{R\to 0} \mathcal{H}_R(A)$$

with

$$\mathcal{H}_R(A) := \inf\left\{\sum_{i\in I}\frac{\mu(B(x_i,r_i))}{r_i}\,:\, A\subset\bigcup_{i\in I}B(x_i,r_i),\, r_i\le R\right\},$$

where $I\subset\mathbb{N}$ is a finite or countable index set. Note that in an Ahlfors one-regular space, $\mathcal{H}$ is comparable to $\mathcal{H}^0$.

Given any set $E\subset X$, the measure-theoretic boundary ∂^*E is the set of points $x\in X$ for which

$$\limsup_{r\to 0}\frac{\mu(B(x,r)\cap E)}{\mu(B(x,r))}>0\quad\text{and}\quad\limsup_{r\to 0}\frac{\mu(B(x,r)\setminus E)}{\mu(B(x,r))}>0.$$

Recall from the Introduction that if (X,d,μ) is a complete metric space such that μ is doubling and the space supports a 1-Poincaré inequality, then the condition $\mathcal{H}(\partial^*E)<\infty$ for a measurable set $E\subset X$ implies that $\|D\chi_E\|(X)<\infty$. This is the "if" direction of Federer's characterization of sets of finite perimeter.

Define a space as a subset of $\mathbb{R}^2$ as follows. First define for each $j\in\mathbb{N}$

$$A_j := \bigcup_{k=0}^{2^j-1} I_k^j,$$

where

$$I_k^j := \left\{\left(t\cos\left(\frac{k\pi}{2^j}\right), t\sin\left(\frac{k\pi}{2^j}\right)\right)\in\mathbb{R}^2\,:\, t\in[-1,1]\right\}$$

is a line segment passing through the origin with length $\mathcal{H}^1(I_k^j)=2$. The angle between I_k^j and the positive x-axis is $\frac{k\pi}{2^j}$ and the angle between I_k^j and I_{k-1}^j is $\frac{\pi}{2^j}$. For any set $A\subset\mathbb{R}^2$ and $a>0$, we let

$$aA := \{(ax,ay)\,:\,(x,y)\in A\}.$$

Then consider $\widetilde{A}_j := 2^{-2j-1}A_j$ for each $j\in\mathbb{N}$. Note that $\widetilde{A}_j$ is a collection of 2^j line segments $\widetilde{I}_k^j$ with length $\mathcal{H}^1(\widetilde{I}_k^j)=2^{-2j}$.

Define

$$X := \bigcup_{j=1}^{\infty}\widetilde{A}_j. \tag{4.1}$$

We first show that the doubling condition is essential in the "if" direction of Federer's characterization.

Example 4.1. Equip the set X in (4.1) with the geodesic metric and the measure $\mathcal{H}^1$. We have

$$\mathcal{H}^1(X)\le\sum_{j=1}^{\infty}2^j\mathcal{H}^1(\widetilde{I}_k^j)=\sum_{j=1}^{\infty}2^{-j}=1.$$

Clearly, the density upper bound condition (1.2) no longer holds at 0. Moreover, $\mathcal{H}^1$ is not doubling: the doubling condition fails when we choose points x close to 0 with $0\in B(x,2r)$ and $0\notin B(x,r)$.

Now we show that this space does support a 1-Poincaré inequality. First consider a ball $B(0,r)$. Suppose u is a function on X with $u(0)=0$ and let g be an upper gradient of u. Every $x\in B(0,r)$ is connected to 0 by a line segment I. We have

$$\int_I g\,d\mathcal{H}^1 \ge |u(x)-u(0)| = |u(x)|.$$

Note that $B(0,r)$ consists of countably many line segments $\{I_j\}_{j=1}^{\infty}$ that have the origin as one end point (some may be half-open). By the above, we have

$$|u(x)| \le \int_{I_j} g\,d\mathcal{H}^1 \quad \text{for every } x\in I_j.$$

Thus

$$\begin{aligned}\int_{B(0,r)} |u|\,d\mathcal{H}^1 = \sum_{j=1}^{\infty}\int_{I_j}|u|\,d\mathcal{H}^1 &\le \sum_{j=1}^{\infty}\Big(\mathcal{H}^1(I_j)\int_{I_j} g\,d\mathcal{H}^1\Big)\\ &\le r\int_{B(0,r)} g\,d\mathcal{H}^1 \quad \text{since } \mathcal{H}^1(I_j)\le r \text{ for all } j\in\mathbb{N}.\end{aligned}$$

Now consider a general ball $B(x,r)$ and a function $u\in L^1(X)$ with upper gradient g. If $B(x,r)$ is contained in only one line segment, the Poincaré inequality obviously holds since it holds in $\mathbb{R}$. So we can assume that $0\in B(x,r)$. We can also assume that $\int_{B(0,2r)} g\,d\mathcal{H}^1<\infty$ and then u is a bounded function in $B(0,2r)$. Thus we can assume that $u(0)=0$. Now

$$\begin{aligned}\int_{B(x,r)} |u-u_{B(x,r)}|\,d\mathcal{H}^1 &\le 2\int_{B(x,r)} |u|\,d\mathcal{H}^1 \quad \text{(see e.g. [5, Lemma 4.17])}\\ &\le 2\int_{B(0,2r)} |u|\,d\mathcal{H}^1\\ &\le 4r\int_{B(0,2r)} g\,d\mathcal{H}^1\\ &\le 4r\int_{B(x,3r)} g\,d\mathcal{H}^1.\end{aligned}$$

Thus a 1-Poincaré inequality holds with $C_P=4$ and $\lambda=3$.

Next, for each $j\in\mathbb{N}$ choose

$$I_1^j=\{(t\cos(2^{-j}\pi), t\sin(2^{-j}\pi)),\, t\in[-1,1]\}$$

and then let

$$E:=\bigcup_{j=1}^{\infty}\widetilde{I}_1^j=\bigcup_{j=1}^{\infty}2^{-2j-1}I_1^j. \tag{4.2}$$

Consider any sequence $(u_i)\subset N^{1,1}(X)$ with $u_i\to\chi_E$ in $L^1(X)$, with upper gradients g_i. We can also assume that $u_i\to\chi_E$ a.e. Thus for each $j\in\mathbb{N}$ we can choose a point $x_j\in\widetilde{I}_1^j$, $x_j\neq 0$ and a point x_j' in $\widetilde{A}_j\setminus\widetilde{I}_1^j$ such that

(1) $u_i(x_j) \to 1$ as $i \to \infty$;
(2) $u_i(x_j') \to 0$ as $i \to \infty$;
(3) the curves γ_j joining x_j' and x_j only intersect at the origin.

Now

$$\int_X g_i \, d\mathcal{H}^1 \geq \sum_{j=1}^{\infty} \int_{\gamma_j} g_i \, d\mathcal{H}^1 \geq \sum_{j=1}^{\infty} |u_i(x_j') - u_i(x_j)| \to \infty \quad \text{as } i \to \infty.$$

Hence $\|D\chi_E\|(X) = \infty$.

It is easy to check that $0 \notin \partial^* E$ and then in fact $\partial^* E = \emptyset$. This shows that the "if" direction of Federer's characterization does not hold without the doubling condition.

On the other hand, $\mathrm{pV}(\chi_E, X) = 1$ since only a curve intersecting 0 can give nonzero variation. Thus we do need condition (1.2) in Proposition 3.7 and Proposition 3.8.

The following example shows that the Poincaré inequality cannot be dropped in the implication from $\mathcal{H}(\partial^* E) < \infty$ to $\|D\chi_E\|(X) < \infty$ either.

Example 4.2. Equip the set X in (4.1) with the metric inherited from $\mathbb{R}^2$ and the measure $\mathcal{H}^1$. In this case, we will show that $\mathcal{H}^1$ is doubling on X, but X does not support any Poincaré inequality since it is clearly not quasiconvex (recall Definition 2.2 and the paragraph after it). Let $x \in X$. If $x \neq 0$, we have $2^{-2k-3} \leq d(x, 0) \leq 2^{-2k-1}$ for some $k \in \mathbb{N}$. Suppose first that $r \leq 2^{-2k-4}$. Recalling the notation from the previous example, note that $\widetilde{A}_k$ consists of 2^k line segments, which are at angles $2\pi \times 2^{-k-1}$ from each other. By simple geometric reasoning we see that the ball $B(x, r/2)$ is intersected by at least

$$\frac{r}{2} \times 2^{2k-1} \times (2\pi \times 2^{-k-1})^{-1} \geq 2^{3k-4} r$$

line segments belonging to $\widetilde{A}_k$, each for a length at least $r/2$ inside $B(x, r)$. Thus

$$\mathcal{H}^1(B(x, r)) \geq 2^{3k-5} r^2.$$

To prove a converse estimate, suppose still that $2^{-2k-3} \leq d(x, 0) \leq 2^{-2k-1}$, and suppose that $2^{-3k-6} \leq r \leq 2^{-2k-4}$. We have $B(x, r) \cap \widetilde{A}_j = \emptyset$ for all $j \geq k + 2$. Note that $\widetilde{A}_{k+1}$ consists of 2^{k+1} line segments, which are at angles $2\pi \times 2^{-k-2}$ from each other. Thus we can see that there are at most

$$4r \times 2^{2k+4} \times (2\pi \times 2^{-k-2})^{-1} \leq 2^{3k+6} r$$

line segments intersecting $B(x, r)$, each for a length at most $2r$. Thus

$$\mathcal{H}^1(B(x, r)) \leq 2^{3k+7} r^2.$$

Thus in total

$$2^{3k-5} r^2 \leq \mathcal{H}^1(B(x, r)) \leq 2^{3k+7} r^2, \tag{4.3}$$

where the first inequality holds for all $r \leq 2^{-2k-4}$ and the second for all $2^{-3k-6} \leq r \leq 2^{-2k-4}$.

Moreover, for every $k \in \mathbb{N}$,

$$\mathcal{H}^1(B(0, 2^{-2k-1})) \geq 2^{-2k-1}\mathcal{H}^1(A_k) = 2^{-2k-1}2^{k+1} = 2^{-k}$$

and so

$$2^{-k} \leq \mathcal{H}^1(B(0, 2^{-2k-1})) \leq \sum_{j=k}^{\infty} 2^{-2j-1}\mathcal{H}^1(A_j) = \sum_{j=k}^{\infty} 2^{-2j-1}2^{j+1} = 2^{-k+1}. \tag{4.4}$$

From these, the doubling condition for balls centered at 0 easily follows. Now assume again that $x \neq 0$, so that $2^{-2k-3} \leq d(x, 0) \leq 2^{-2k-1}$ for a given $k \in \mathbb{N}$. We consider four cases:

(1) If $R < 2^{-3k-4}$, then $B(x, 2R)$ consists of just one line segment and so

$$\mathcal{H}^1(B(x, 2R)) = 2\mathcal{H}^1(B(x, R)).$$

(2) If $2^{-3k-4} \leq R \leq 2^{-2k-5}$, then by (4.3),

$$2^{3k-5}R^2 \leq \mathcal{H}^1(B(x, R)) \quad \text{and} \quad \mathcal{H}^1(B(x, 2R)) \leq 2^{3k+7}(2R)^2,$$

and so we have

$$\mathcal{H}^1(B(x, 2R)) \leq 2^{14}\mathcal{H}^1(B(x, R)).$$

(3) If $2^{-2k-5} < R \leq 2^{-2k+1}$, then applying (4.3) with $r = 2^{-2k-5}$,

$$\mathcal{H}^1(B(x, R)) \geq \mathcal{H}^1(B(x, 2^{-2k-5})) \geq 2^{3k-5}(2^{-2k-5})^2 = 2^{-k-15}$$

and by (4.4),

$$\mathcal{H}^1(B(x, 2R)) \leq \mathcal{H}^1(B(0, 2^{-2k+2})) \leq 2^{-k+3},$$

and so we have

$$\mathcal{H}^1(B(x, 2R)) \leq 2^{18}\mathcal{H}^1(B(x, R)).$$

(4) If $2^{-2k+1} < R \leq 2^{-2}$ with $k \geq 2$ (note that $\operatorname{diam} X = 2^{-2}$), we choose $j \leq k$ such that $2^{-2j+1} < R \leq 2^{-2j+3}$. Note that $B(0, R/2) \subset B(x, R) \subset B(x, 2R) \subset B(0, 4R)$. Now by (4.4),

$$\mathcal{H}^1(B(x, R)) \geq \mathcal{H}^1(B(0, R/2)) \geq \mathcal{H}^1(B(0, 2^{-2j})) \geq 2^{-j}$$

and

$$\mathcal{H}^1(B(x, 2R)) \leq \mathcal{H}^1(B(0, 4R)) \leq \mathcal{H}^1(B(0, 2^{-2j+5})) \leq 2^{-j+4}.$$

Thus

$$\mathcal{H}^1(B(x, 2R)) \leq 2^4\mathcal{H}^1(B(x, R)).$$

In total, the doubling condition always holds with doubling constant 2^{18}, when $x \neq 0$.

Finally, define the set E as in (4.2). As before, we obtain that $\|D\chi_E\|(X) = \infty$, $\mathrm{pV}(\chi_E, X) = 1$, and $\partial^* E = \emptyset$. Thus again we see that the "if" direction of Federer's characterization does not hold, and that condition (1.2) is needed in Proposition 3.7 and Proposition 3.8.

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