# HORIZONTAL CONVEX ENVELOPE IN THE HEISENBERG GROUP AND APPLICATIONS TO SUB-ELLIPTIC EQUATIONS

QING LIU AND XIAODAN ZHOU

Abstract. This paper introduces in a natural way a notion of horizontal convex envelopes of continuous functions in the Heisenberg group. We provide a convexification process to find the envelope in a constructive manner. We also apply the convexification process to show h-convexity of viscosity solutions to a class of fully nonlinear elliptic equations in the Heisenberg group satisfying a certain symmetry condition. Our examples show that in general one cannot expect h-convexity of solutions without the symmetry condition.

## 1. Introduction

The convex envelope of a given continuous function in $\mathbb{R}^N$ is a powerful tool in analysis and partial differential equations. In this paper, we exploit its sub-Riemannian counterpart, introducing the notion of convex envelope in the first Heisenberg group $\mathbb{H}$ and discussing its applications in the study of fully nonlinear sub-elliptic partial differential equations.

1.1. **Background and motivation.** In order to explain the motivation of our work, let us first briefly recall the definition, properties and several applications of the convex envelope in $\mathbb{R}^N$. For any given function $u \in C(\mathbb{R}^N)$ that is bounded below. There are at least two ways to define the Euclidean convex envelope, which we denote by $\Gamma_E u$. The first is to consider the largest convex function majorized by $u$, that is,

$$(\Gamma_E u)(p) := \sup \left\{ v(p) : v \text{ is convex and } v \le u \text{ in } \mathbb{R}^N \right\} \tag{1.1}$$

for all $p \in \mathbb{R}^N$.

An equivalent way of defining the convex envelope is to convexify pointwise the given function $u$; namely, we have

$$(\Gamma_E u)(p) = \inf \left\{ \sum_{i=1}^{N+1} c_i u(p_i) : \ c_i \in [0,1],\ p_i \in \mathbb{R}^N \ (i = 1, 2, \ldots, N+1), \right. \\ \left. \sum_i c_i = 1,\ \sum_i c_i p_i = p \right\} \tag{1.2}$$

for all $p \in \mathbb{R}^N$. Compared to (1.1), the definition in (1.2) is more constructive and more likely to be used in practical computations.

Besides the equivalent definitions above, there is a characterization of the convex envelope in terms of a nonlinear obstacle problem, recently proposed by [37, 38]. More

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precisely, in view of [37, Theorem 2], the convex envelope $\Gamma_E$ can be characterized as a maximal viscosity solution of

$$\max\{-\lambda_E^*[v](x), v-u\} = 0 \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

that is,

$$(\Gamma_E u)(p) = \sup\{v(p) : v \text{ is a subsolution of (1.3)}\}.$$

Here $\lambda_E^*[v]$ denotes the least eigenvalue of $\nabla^2 v$ for any $v \in C^2(\mathbb{R}^N)$. This can be viewed as an alternative expression of (1.1).

As an important tool, the convex envelope in $\mathbb{R}^N$ is extensively studied and widely applied in different contexts. One of important applications appears in the Alexandrov-Bakelman-Pucci (ABP) estimate for elliptic partial differential equations (see for instance [7, Definition 3.1, Theorem 3.2]). The convex envelope is used to describe the coincidence set $\{p : \Gamma_E u(p) = u(p)\}$ contained in the domain, which leads to an accurate form of the estimate.

Moreover, one can also use the convex envelope to show convexity of solutions to general fully nonlinear elliptic equations in the form

$$F(p, u(p), \nabla u(p), \nabla^2 u(p)) = 0 \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

where $F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathbf{S}^N \to \mathbb{R}$ is a continuous elliptic operator. Here $\mathbf{S}^N$ denotes the set of all $n \times n$ symmetric matrices.

Two methods are well known to prove spatial convexity of the unique solution to an elliptic or parabolic equation. One method is based on the so-called convexity (concavity) maximum principle. For more details, we refer to [28, 25, 26] on this method for classical solutions and to [18] for a generalized result using viscosity solutions.

The other method, proposed in [1], is to employ the minimization in (1.2) to find the relation between the first and second derivatives of $\Gamma_E u$ at $p$ and those of $u$ at $p_i$. Here we assume that the infimum in (1.2) can be attained at $p_i$ $(i = 1, 2, \ldots, N+1)$. Roughly speaking, assuming that these functions are smooth, we can easily see that

$$\begin{gathered} \nabla \Gamma_E u(p) = \nabla u(p_i), \quad \text{for } i = 1, 2, \ldots, N+1, \text{ and} \\ \nabla^2 \Gamma_E u(p) \geq \sum_i c_i \nabla^2 u(p_i). \end{gathered} \tag{1.5}$$

Then under necessary regularities and assumptions on the operator $F$, one can use this relation to show that the convex envelope of any viscosity supersolution remains to be a viscosity supersolution [1, Proposition 3]. Such a supersolution preserving property enables us to obtain the convexity of the solution immediately if the comparison principle for the equation is known to hold. We refer the reader also to [23] and recent work [21, 11] for more applications of this method in the Euclidean space.

In the sub-Riemannian setting, an intrinsic notion of convex functions is available. The so-called horizontal convex (h-convex) functions on the Heisenberg group is introduced in [32] and on general Carnot groups in [12]. For a smooth function $u$ in $\mathbb{H}$, the h-convexity of $u$ simply requires $u$ to satisfy

$$(\nabla_H^2 u)^\star \geq 0 \quad \text{in } \mathbb{H},$$

where $(\nabla_H^2 u)^\star$ stands for the symmetrized horizontal Hessian of $u$. When $u$ is only a continuous function, then one should interpret the inequality above in the viscosity sense. Regularity properties of h-convex functions can also be found in [3] for the Heisenberg group and in [39, 33, 24, 34, 4] for more general sub-Riemannian manifolds.

Such a convexity notion enables us to naturally consider the corresponding envelope for a given continuous function $u : \mathbb{H} \to \mathbb{R}$, following the definition (1.1) in the Euclidean case. It is thus expected that the horizontal convex envelope can shed light on the above problems in the Heisenberg group. We remark that there are results related to ABP estimate in the sub-Riemannian circumstances such as [13, 20, 17, 2] but mainly for h-convex functions.

We are thus more interested in the question: under what assumptions on the elliptic or parabolic equations are their solutions h-convex in space? As one of possible important applications, we aim to understand whether the h-convexity preserving property holds for the horizontal mean curvature flow in the Heisenberg group. Well-posedness for the level-set horizontal mean curvature flow equation is addressed in [9, 16, 5]. Under additional symmetry assumptions on solutions, the h-convexity preserving property for a class of semilinear parabolic equations is shown in [29] by extending a convexity maximum principle to the Heisenberg group.

In this work, we focus our attention on fully nonlinear elliptic equations in the Heisenberg group in the form

$$F(p, u, \nabla_H u, (\nabla_H^2 u)^\star) = 0 \quad \text{in } \mathbb{H}, \tag{1.6}$$

where $\nabla_H u$ denotes the horizontal gradient of $u$. It is of our curiosity whether an analogue of the result in [1] can bring us better convexity results for such general sub-elliptic equations. Our main purpose is therefore to study fundamental properties of the h-convex envelope and adopt them to understand geometric properties of (1.6). Let us mention that starshapedness of level sets of solutions to general elliptic equations in Carnot groups is recently studied in [14].

It is worth remarking that another interesting question concerns the continuous differentiability of the h-convex envelope. The regularity issue in the Euclidean space is addressed in [19, 27]. We discuss the same question in the Heisenberg group in our forthcoming work [31].

1.2. **Main results.** As mentioned above, the most reasonable way to define the horizontal convex envelope (h-convex envelope), denoted by $\Gamma u$, is clearly to take the greatest h-convex function majorized by $u$, i.e.,

$$(\Gamma u)(p) := \sup \{v(p) : v \text{ is h-convex and } v \leq u \text{ in } \mathbb{H}\} \tag{1.7}$$

for all $p \in \mathbb{H}$; see also Definition 3.1. The corresponding construction of $\Gamma u$ as in (1.2) is not as straightforward as its definition. One may still attempt to convexify $u$ at each $p \in \mathbb{H}$ by setting

$$S[u](p) := \inf \left\{ \sum_i c_i u(p_i) : \ c_i \in [0,1],\ p_i \in \mathbb{H}_p\ (i = 1,2,3),\ \sum_i c_i = 1,\ \sum_i c_i p_i = p \right\}, \tag{1.8}$$

where $\mathbb{H}_p$ denotes the horizontal plane passing through $p$. However, $S[u]$ is in fact not necessarily h-convex in $\mathbb{H}$ in general, as shown in Examples 4.5, 4.14 and 4.15. Moreover, our examples also show that, without coercivity condition on $u$, $S[u]$ may not be continuous in spite of the continuity of $u$. The operator $S$ only partially convexifies the function $u$ and the situation is thus totally different from the Euclidean case.

It turns out that, in order to construct $\Gamma u$, one needs to iterate the operator $S$; namely, we show that

$$S^n[u] \to \Gamma u \quad \text{ pointwise in } \mathbb{H} \text{ as } n \to \infty \tag{1.9}$$

provided that $u$ is continuous and bounded below in $\mathbb{H}$. The convergence is locally uniformly if $u$ is further assumed to be coercive in $\mathbb{H}$, i.e.,

$$\inf_{|p|\geq R} \frac{u(p)}{|p|} \to \infty \quad \text{ as } R \to \infty. \tag{1.10}$$

See Theorem 4.8 and Remark 4.9 for precise statements.

It causes much difficulty that the operator $S$ needs to be implemented multiple times to get the envelope $\Gamma u$. We do not know whether the iteration process can be completed in finite times. Example 4.13 indicates that $\Gamma u$ can be obtained in one step for certain functions $u$ while Example 4.14 shows that two steps are needed sometimes. It is not clear to us how the total number of the necessary iteration is related to the structure of the function $u$.

We next discuss the application of the h-convex envelope in relation to the h-convexity of solutions to elliptic equations in the Heisenberg group. It was pointed out in our earlier work [29] that the horizontal convexity preserving property fails to hold even for linear transport equations. We now can give an example, showing that the h-convexity also fails for solutions of linear sub-elliptic equations in the form of

$$u - \Delta_H u + \langle \zeta, \nabla_H u\rangle = f(p) \quad \text{in } \mathbb{H}, \tag{1.11}$$

where $\zeta \in \mathbb{R}^2$ and $f \in C(\mathbb{H})$ is given. Here $\Delta_H u$ denotes the horizontal Laplacian of $u$. Note that for the same type of equations in $\mathbb{R}^N$, i.e,

$$u - \Delta u + \langle \zeta, \nabla u\rangle = f(p) \quad \text{ in } \mathbb{R}^N,$$

where in this case $\zeta \in \mathbb{R}^N$ is given, any smooth solution is convex provided that $f$ is convex in $\mathbb{R}^N$; the proof is merely an application of the maximum principle to the function $\left\langle \nabla^2 u\ \eta, \eta\right\rangle$ for any fixed $\eta \in \mathbb{R}^N$.

However, similar convexity results cannot be expected for (1.11), since horizontal differentiation is not commutative in general.

*Example* 1.1 (Failure of h-convexity). Consider the linear equation (1.11) with $\zeta = (0, 2)$ and

$$f(x,y,z) = 2xz + x^2 y + \frac{1}{4}x^4 + \frac{3}{2}y^2 + 6y - 1$$

for $(x,y,z) \in \mathbb{H}$. By direct calculations, one can verify that

$$u(x,y,z) = 2xz + x^2 y + \frac{1}{4}x^4 - x^2 + \frac{3}{2}y^2 \tag{1.12}$$

is the unique solution. (The uniqueness of solutions to this equation with polynomial growth at infinity is due to [22, Theorem 7.4].) Note that

$$(\nabla_H^2 f)^\star(x,y,z) = \begin{pmatrix} 3x^2 & 3x \\ 3x & 3 \end{pmatrix}$$

but

$$(\nabla_H^2 u)^\star(x,y,z) = \begin{pmatrix} 3x^2 - 2 & 3x \\ 3x & 3 \end{pmatrix},$$

which reveals that $u$ is not h-convex at the origin although $f$ is h-convex in $\mathbb{H}$.

This example suggests that more restrictive assumptions on $F$ are needed if one wants to prove h-convexity of the solutions of (1.6). A typical result we can show is as follows.

**Theorem 1.2** (H-convexity for semilinear ellitpic equations)**.** *Let $\alpha, \beta \geq 0$. Assume that $f \in C(\mathbb{H})$ is h-convex in $\mathbb{H}$ and symmetric with respect to z-axis, i.e.,*

$$u(x, y, z) = u(-x, -y, z), \quad \text{for all } (x, y, z) \in \mathbb{H}. \tag{1.13}$$

*Let $\mathcal{A} \subset \mathbb{R}^2$ be a compact set symmetric with respect to the origin, that is, $\zeta \in \mathcal{A}$ implies $-\zeta \in \mathcal{A}$. If $u$ is a coercive solution of the semilinear equation*

$$u = \alpha \Delta_H u + \beta \sup_{\zeta \in \mathcal{A}} \langle \zeta, \nabla_H u \rangle + f(p) \quad \text{in } \mathbb{H}, \tag{1.14}$$

*then $\Gamma u$ is a supersolution. In particular, the unique solution of* (1.14) *with polynomial growth near space infinity is h-convex in $\mathbb{H}$.*

The polynomial growth condition here is again used to guarantee the uniqueness of solutions due to the comparison result in [22, Theorem 7.4]; see Section 2.2 for clarification.

Note that Theorem 1.2 is only a special case of our main result, where we prove the h-convexity of the solution to (1.6) with a general nonlinear concave symmetric operator $F$, assuming that the comparison principle holds; see Theorem 5.7. We emphasize that $F$ here is assumed to be concave with respect to all arguments, which is stronger than the condition in the Euclidean case. The reasons why the symmetry and the strong concavity assumptions are needed will be clarified in a moment.

Compared to the left invariant h-convexity, it is in fact easier to obtain the desired results by considering the right invariant case instead. To see this, we introduce the right invariant counterpart $\tilde{S}$ of the partial convexification operator $S$, given by

$$\tilde{S}[u](p) := \inf \left\{ \sum_i c_i u(p_i) : \ c_i \in [0, 1], \ p_i \in \tilde{\mathbb{H}}_p \ (i = 1, 2, 3), \ \sum_i c_i = 1, \ \sum_i c_i p_i = p \right\}, \tag{1.15}$$

where $\tilde{\mathbb{H}}_p$ stands for the right invariant horizontal plane passing though $p \in \mathbb{H}$. We then show, in Theorem 5.1, that $\tilde{S}$ has the supersolution preserving property for a class of concave fully nonlinear elliptic operators, i.e., $\tilde{S}[u]$ is a supersolution if $u$ is a supersolution.

Our proof of Theorem 5.1 is essentially an adaptation of the argument [1] to the sub-Riemannian setting. However, in contrast to the situation in [1], we here have an additional constraint condition due to the extra requirement $p_i \in \tilde{\mathbb{H}}_p$ in (1.15). Roughly speaking, for any fixed $p \in \mathbb{H}$ and minimizers $p_i \in \mathbb{H}$ in (1.15), in order to find the relation between $\big((\nabla_H \Gamma u)(p), (\nabla_H^2 \Gamma u)^\star(p)\big)$ and $\big(\nabla_H u(p_i), (\nabla_H^2 u)^\star(p_i)\big)$, we append to the standard minimization an extra term penalizing the distance between $p_i$ and the horizontal plane through $\sum_i p_i$. We refer the reader to Section 5.1 for technical details. In summary, our idea more closely resembles the method of Lagrange multipliers rather than direct unconstrained minimization.

It is the extra penalty term that requires us to impose the concavity of $\xi \mapsto F(p, r, \xi, A)$, which is not needed in the Euclidean case. Roughly speaking, we are not able to obtain the equality as in (1.5) for the horizontal gradients but instead we get

$$\nabla_H \Gamma u(p) = \sum_i c_i \nabla_H u(p_i),$$

which demands the concavity of $F$ in the horizontal gradient to conclude. In addition, we have an example, Example 5.11, showing that such a strong concavity condition is necessary. This condition unfortunately excludes possible applications of our approach to the mean curvature operator and p-Laplacians in the Heisenberg group.

Let us briefly discuss the symmetry assumption on $F$. Notice that due to the presence of Example 1.1, even in the simpler case (1.14), it seems necessary to assume the function $f$ and the term involving $\nabla_H u$ to be symmetric. The additional symmetry on the elliptic operator implies that the unique solution $u$ is symmetric about the $z$-axis. This further yields $S[u] = \tilde{S}[u]$ in $\mathbb{H}$; see Theorem 4.11 for details. We thus have $S[u] = u$, which concludes the proof of Theorem 1.2. By iterating this argument for $S^n[u]$ and passing to the limit as $n \to \infty$, we can also prove a symmetric supersolution preserving property, namely, if $u$ is a symmetric supersolution, then so is $\Gamma u$. This property is addressed for the general equation (1.6) in Theorem 5.4.

We finally mention that one can also obtain the Euclidean convexity of the unique solution to (1.6) under weaker structure assumptions on $F$. In particular, the symmetry condition is no longer needed in this case. Our result is consistent with that in [1] and our proof is based on slight modification of the arguments used for Theorem 5.1 but applied to the Euclidean convex envelope. See Section 5.4 for more explanations.

The rest of the paper is organized in the following way. In Section 2, we give a brief review of the Heisenberg group and the theory of viscosity solutions. We also recall basic notions and regularity results related to the horizontal convex functions in Section 2.3. The definition of h-convex envelop is given in Section 3. We introduce the iterated convexification process and give several concrete examples of the envelope in Section 4. Section 5 is devoted to our main results with detailed discussion on the application of the h-convex envelope to the study on sub-elliptic PDEs.

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## 2. Preparations

2.1. **Preliminaries on the Heisenberg group.** Recall that the Heisenberg group $\mathbb{H}$ is $\mathbb{R}^3$ endowed with the non-commutative group multiplication

$$(x_p, y_p, z_p) \cdot (x_q, y_q, z_q) = \left(x_p + x_q, y_p + y_q, z_p + z_q + \frac{1}{2}(x_p y_q - x_q y_p)\right),$$

for all $p = (x_p, y_p, z_p)$ and $q = (x_q, y_q, z_q)$ in $\mathbb{H}$. Note that the group inverse of $p = (x_q, y_q, z_q)$ is $p^{-1} = (-x_q, -y_q, -z_q)$. The Korányi gauge is given by

$$|p|_G = ((p_1^2 + p_2^2)^2 + 16p_3^2)^{1/4},$$

and the left-invariant Korányi or gauge metric is

$$d_L(p, q) = |p^{-1} \cdot q|_G.$$

We denote by $B_R(p)$ the gauge ball centered at $p$ with radius $R > 0$. We denote the gauge ball centered at the origin simply by $B_R$.

The Lie algebra of $\mathbb{H}$ is generated by the left-invariant vector fields

$$X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z}.$$

One may easily verify the commuting relation $Z = [X, Y] = XY - YX$.

The horizontal gradient of $u$ is given by

$$\nabla_H u = (Xu, Yu)$$

and the symmetrized second horizontal Hessian $(\nabla_H^2 u)^* \in \mathbf{S}^2$ is given by

$$(\nabla_H^2 u)^\star := \begin{pmatrix} X^2 u & (XYu + YXu)/2 \\ (XYu + YXu)/2 & Y^2 u \end{pmatrix}.$$

Here $\mathbf{S}^n$ denotes the set of all $n \times n$ symmetric matrices.

Define

$$\mathbb{H}_0 = \{h \in \mathbb{H} : h = (x, y, 0) \ \text{ for } x, y \in \mathbb{R}\}.$$

For any $p \in \mathbb{H}$, we call

$$\mathbb{H}_p = \{p \cdot h : \ h \in \mathbb{H}_0\}$$

the horizontal plane through $p$. The horizontal plane through $p = (x_p, y_p, z_p)$ can be expressed by the following equation:

$$y_p x - x_p y + 2z - 2z_p = 0. \tag{2.1}$$

*Remark* 2.1. Later we will also use the right invariant vector fields in $\mathbb{H}$ given by

$$\tilde{X} = \frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial z}, \quad \tilde{Y} = \frac{\partial}{\partial y} - \frac{x}{2}\frac{\partial}{\partial z}, \quad \tilde{Z} = \frac{\partial}{\partial z}.$$

Accordingly, the right invariant horizontal gradient

$$\tilde{\nabla}_H u = (\tilde{X}u, \tilde{Y}u)$$

and symmetrized Hessian

$$(\tilde{\nabla}_H^2 u)^\star := \begin{pmatrix} \tilde{X}^2 u & (\tilde{X}\tilde{Y}u + \tilde{Y}\tilde{X}u)/2 \\ (\tilde{X}\tilde{Y}u + \tilde{Y}\tilde{X}u)/2 & \tilde{Y}^2 u \end{pmatrix}.$$

We also write $\tilde{\mathbb{H}}_p$ to denote the right invariant horizontal plane, that is,

$$\tilde{\mathbb{H}}_p = \{h \cdot p : h \in \mathbb{H}_0\}. \tag{2.2}$$

We can write an analogue of the plane equation (2.1) for $\tilde{\mathbb{H}}_p$ as below:

$$y_p x - x_p y - 2z + 2z_p = 0. \tag{2.3}$$

2.2. **Viscosity solutions.** Viscosity solutions are known to have many applications in fully nonlinear equations in the Euclidean space; see [10] for an introduction. We refer to [6, 36] and many others for generalization on the sub-Riemannian manifolds.

Let us consider (1.6), where $F : \mathbb{H} \times \mathbb{R} \times \mathbb{R}^2 \times \mathbf{S}^2 \to \mathbb{R}$ is a continuous operator satisfying the following assumptions.

(A1) $F$ is (degenerate) elliptic; namely,

$$F(p, r, \xi, A_1) \leq F(p, r, \xi, A_2)$$

for all $p \in \mathbb{H}$, $r \in \mathbb{R}$, $\xi \in \mathbb{R}^2$ and $A_1, A_2 \in \mathbf{S}^2$ with $A_1 \geq A_2$.

(A2) $F$ is proper; namely,

$$F(p, r_1, \xi, A) \geq F(p, r_2, \xi, A)$$

for all $p \in \mathbb{H}, \xi \in \mathbb{R}^2, A \in \mathbf{S}^2$ and $r_1, r_2 \in \mathbb{R}$ with $r_1 \geq r_2$.

We begin with a definition for viscosity solutions of (1.6) below. Denote by $USC(\mathbb{H})$ (resp., $LSC(\mathbb{H})$) the class of upper semicontinuous (resp., lower semicontinuous) functions in $\mathbb{H}$.

**Definition 2.2** (Definition of viscosity solutions)**.** Let $\Omega$ be a domain in $\mathbb{H}$. A locally bounded function $u \in USC(\Omega)$ (resp., $u \in LSC(\Omega)$) is said to be a viscosity subsolution (resp., supersolution) of (1.6) in $\Omega$ if whenever there exist $\varphi \in C^2(\Omega)$ and $p_0 \in \Omega$ such that $u - \varphi$ attains a (strict) maximum (resp., minimum) in $\Omega$ at $p_0$, we have

$$F\left(p_0, u(p_0), \nabla_H \varphi(p_0), (\nabla_H^2 \varphi)^\star(p_0)\right) \le 0$$
$$\left(\text{resp., } F\left(p_0, u(p_0), \nabla_H \varphi(x_0), (\nabla_H^2 \varphi)^\star(p_0)\right) \ge 0\right).$$

A bounded continuous function $u$ is called a viscosity solution of (1.6) if it is both a subsolution and a supersolution.

As a standard remark, one can use the so-called subelliptic semijets to give an alternative definition of sub- and supersolutions; we refer the reader to [6, 36] for details. Hereafter let us denote by $J_H^{2,\pm} u(p)$ the semijets of $u$ at a given point $p \in \mathbb{H}$. Moreover, we use $\overline{J}_H^{2,\pm} u(p)$ to denote the "closure" of $J_H^{2,\pm} u(p)$. We can equivalently define a supersolution by requiring that, for any $p_0 \in \mathbb{H}$,

$$F(p_0, u(p_0), \xi, A) \ge 0$$

holds provided that $(\xi, A) \in \overline{J}_H^{2,-} u(p_0)$; see [6, Proposition 3.1]. One can give an equivalent definition for subsolutions analogously.

*Remark* 2.3. Throughout this work, we always assume that a comparison principle holds for (1.6), since it is not our main concern here. Let us recall the standard comparison principle states that any subsolution $u$ and any supersolution $v$ of (1.6) satisfies $u \le v$ in $\mathbb{H}$. However, it is worth remarking that establishing a comparison principle for fully nonlinear elliptic equations in the whole space is completely nontrivial even in the Euclidean space. One usually needs to impose additional assumptions on the growth rate of sub- and supersolutions near space infinity.

However, on the other hand, a comparison principle is available in [22] for a special class of operators

$$F(p, r, \xi, A) = r - \alpha \operatorname{tr} A - \beta \sup_{\zeta \in \mathcal{A}} \langle \zeta, \xi \rangle - f(p) \tag{2.4}$$

for $(p, r, \xi, A) \in \mathbb{H} \times \mathbb{R} \times \mathbb{R}^2 \times \mathbf{S}^2$, where $\alpha, \beta \ge 0$, $\mathcal{A}$ is a compact subset of $\mathbb{R}^2$ and $f \in C(\mathbb{H})$ is given. Indeed, in this case one can write (1.6) in the Euclidean coordinates and apply [22, Theorem 7.4] to get $u \le v$ in $\mathbb{H}$ if the subsolution $u$ and the supersolution $v$ have polynomial growth at space infinity, namely, there exists $k > 0$ such that

$$\sup_{p \in \mathbb{H}} \frac{u(p)}{|p|^k + 1} < \infty \quad \text{and} \quad \inf_{p \in \mathbb{H}} \frac{v(p)}{|p|^k + 1} > -\infty.$$

Viscosity solutions to the parabolic equation,

$$u_t + F(p, u, \nabla_H u, (\nabla_H^2 u)^\star) = 0 \quad \text{ in } \mathbb{H} \times (0, \infty), \tag{2.5}$$

can be similarly defined.

2.3. **Horizontal convexity.** In what follows, we review basic results on the notion of convex functions in the Heisenberg group; more details can be found in [32, 12].

**Definition 2.4** ([32, Definition 4.1])**.** Let $\Omega$ be an open set in $\mathbb{H}$ and $u : \Omega \to \mathbb{R}$ be an upper semicontinuous function. The function $u$ is said to be horizontally convex or h-convex in $\Omega$, if for every $p \in \mathbb{H}$ and $h \in \mathbb{H}_0$ such that $[p \cdot h^{-1}, p \cdot h] \subset \Omega$, we have

$$u(p \cdot h^{-1}) + u(p \cdot h) \ge 2u(p). \tag{2.6}$$

*Remark* 2.5. It turns out that an upper semicontinuous function $u : \Omega \to \mathbb{R}$ is h-convex if and only if

$$u(p) \leq \sum_{i=1,2,3} c_i u(p_i) \tag{2.7}$$

for all $p \in \Omega$, $c_i \in [0, 1]$ and $p_i \in \mathbb{H}_p \cap \Omega$ $(i = 1, 2, 3)$ satisfying

$$\sum_{i=1,2,3} c_i = 1, \quad \sum_{i=1,2,3} c_i p_i = p. \tag{2.8}$$

The proof of (2.6) using (2.7) is straightforward and it thus suffices to prove the reverse implication. To see this, we recall from [8, Theorem 4.1] that $u : \Omega \to \mathbb{R}$ is h-convex if and only if for any $p \in \Omega$ there exists $\xi \in \mathbb{H}_0$ such that

$$u(p \cdot h) \geq u(p) + \langle h, \xi \rangle \tag{2.9}$$

holds for all $h \in \mathbb{H}_0$ and $p \cdot h \in \Omega$. This sub-differential characterization is also observed in [34, Remark 3.7] for h-convex functions on more general stratified groups. For $c_i \in [0, 1]$ and $p_i \in \mathbb{H}_p \cap \Omega$ $(i = 1, 2, 3)$ satisfying (2.8), we can find $h_i \in \mathbb{H}_0$ such that $p_i = p \cdot h_i$ and $\sum_{i=1,2,3} c_i h_i = 0$. Then applying (2.9) with $h = h_i$ and summing up the inequalities with factors $c_i$, we obtain

$$\sum_{i=1,2,3} c_i u(p_i) \geq \sum_{i=1,2,3} c_i \left(u(p) + \langle \xi, h_i \rangle\right) \geq u(p),$$

which leads to (2.7).

One may also define convexity of a function in the following way.

**Definition 2.6** ([32, Definition 3.3])**.** Let $\Omega$ be an open set in $\mathbb{H}$ and $u : \Omega \to \mathbb{R}$ be an upper semicontinuous function. The function $u$ is said to be v-convex in $\Omega$ if

$$(\nabla_H^2 u)^\star(p) \geq 0 \quad \text{for all } p \in \Omega \tag{2.10}$$

in the viscosity sense.

It is easily seen that $u \in C^2(\Omega)$ is v-convex if it satisfies (2.10) everywhere in $\Omega$. It is known that the h-convexity and v-convexity are equivalent [32]; see also the related results in Carnot groups [39, 24]. Below we also review a well-known result concerning the Lipschitz regularity of h-convex functions.

**Theorem 2.7** (Local Lipschitz regularity of h-convex functions [32, Theorem 3.1])**.** *Suppose that $u : \mathbb{H} \to \mathbb{R}$ is an h-convex (v-convex) function. Then $u$ is locally bounded and Lipschitz. Moreover, the following estimate holds:*

$$\|\nabla_H u\|_{L^\infty(B_R)} \leq \frac{C}{R} \|u\|_{L^\infty(B_{2R})}.$$

*Here $C > 0$ is independent of $u$ and $R > 0$.*

We remark that the original result in [32, Theorem 3.1] is stated for a general domain $\Omega \subset \mathbb{H}$. Here we only consider the special case $\Omega = \mathbb{H}$ for our own purpose. We refer to [32, 12, 3] for details on this result and to [39, 33, 35, 4] for further discussion on general sub-Riemannian manifolds.

*Remark* 2.8. We can also consider a right invariant version of h-convex functions in $\mathbb{H}$ by using the vector field in Remark 2.1. More precisely, we say a function is right invariant h-convex in an open set $\Omega \subset \mathbb{H}$ if for every $p \in \mathbb{H}$ and $h \in \mathbb{H}_0$ such that $[h^{-1} \cdot p, h \cdot p] \subset \Omega$, we get

$$u(h^{-1} \cdot p) + u(h \cdot p) \geq 2u(p).$$

Equivalently, we may also use the viscosity inequality

$$(\tilde{\nabla}^2_H u)^*(p) \geq 0 \quad \text{for all } p \in \Omega.$$

Applying an argument symmetric to the proof of Theorem 2.7, we can show that the right invariant h-convex function is also locally bounded and Lipschitz (with respect to the right invariant metric).

In general, h-convex functions or right invariant h-convex functions are not necessarily convex in the Euclidean sense, as shown in the following example.

*Example* 2.9. Let $u(p) = x^2y^2 + 2z^2$. It is easily verified that $u$ is an h-convex and right invariant h-convex function that is not Euclidean convex.

The above example also clearly indicates that a function that is both left invariant h-convex and right invariant h-convex may not be Euclidean convex.

## 3. Definition of h-convex envelope

In this section we aim to extend the definitions of Euclidean convex envelopes to the Heisenberg group.

To define a horizontal convex envelope of $u \in C(\mathbb{H})$ bounded below, we may follow Perron's method and consider a sub-Riemannian analogue of (1.1) as follows.

**Definition 3.1.** Suppose that $u \in C(\mathbb{H})$ is bounded below. A function $\Gamma u : \mathbb{H} \to \mathbb{R}$ is said to be the h-convex envelope of $u$ if $\Gamma u$ is the greatest h-convex function majorized by $u$, i.e., $\Gamma u$ is given by (1.7).

The function $\Gamma u$ is well-defined, since $u$ is bounded below and any constant is h-convex. It is clear that

$$\inf_{\mathbb{H}} u \leq \Gamma u \leq u \quad \text{in } \mathbb{H}.$$

It is not difficult to show that $\Gamma u$ is locally Lipschitz in $\mathbb{H}$ due to the Lipschitz estimate of h-convex functions in Theorem 2.7. One may also obtain the local Lipchitz continuity of $\Gamma u$ by first showing its h-convexity, as below, and then applying Theorem 2.7.

**Lemma 3.2** (H-convexity of the envelope)**.** *Suppose that $u \in C(\mathbb{H})$ is bounded from below. Let $\Gamma u$ be given as in* (1.7)*. Then $\Gamma u$ is h-convex in $\mathbb{H}$.*

*Proof.* As mentioned above, $\Gamma u$ is locally Lipschitz in $\mathbb{H}$. The proof of the h-convexity of $\Gamma u$ streamlines the argument of Perron's method. More precisely, if there exists $\varphi \in C^2(\mathbb{H})$ and $p_0$ such that $\Gamma u - \varphi$ attains a strict maximum at $p_0$, then we may find $v_j \in C(\mathbb{H})$ h-convex and $p_j \in \mathbb{H}$ with $p_j \to p_0$ as $j \to \infty$ such that $v_j - \varphi$ attains a local maximum at $p_j$. By the h-convexity of $v_j$, we obtain that

$$(\nabla^2_H \varphi)^\star(p_j) \geq 0,$$

from which we deduce that

$$(\nabla^2_H \varphi)^\star(p_0) \geq 0$$

by passing to the limit as $j \to \infty$. □

Motivated by [37], we may consider the following obstacle problem

$$\max\{-\lambda^*[v],\ v-u\}=0 \quad \text{in } \mathbb{H} \tag{3.1}$$

in the viscosity sense, where $\lambda^*[v]$ denotes the least eigenvalue of $(\nabla_H^2 v)^\star$.

**Theorem 3.3** (Characterization by an obstacle problem)**.** *Assume that $u \in C(\mathbb{H})$ is bounded from below. Let $\Gamma u$ be the h-convex envelope defined in* (1.7)*. Then*

$$(\Gamma u)(p) = \sup\{v(p) : v \textit{ is a subsolution of } (3.1)\}.$$

We omit the proof, since it is merely a direct adaptation of [37, Theorem 2] to the sub-Riemannian circumstances based on (1.7).

We shall give several concrete examples in Section 4.4 and discuss an application to convexity of solutions to nonlinear PDEs in Section 5.

## 4. Pointwise Convexification

4.1. **A partially convexifying operator.** We next use the convexification similar to (1.2) to find the horizontal convex envelope. In $\mathbb{H}$, it is natural to consider an operator $S$ as given by (1.8) for any $u \in USC(\mathbb{H})$ bounded from below. In contrast to the Euclidean case (1.2), the main difference here is that $p_i$ are restricted on the horizontal plane $\mathbb{H}_p$ rather than the whole space. It is obvious that

$$\inf_{\mathbb{H}} u \le S[u] \le u \quad \text{in } \mathbb{H}.$$

It is also clear that $S[u]=u$ in $\mathbb{H}$ if and only if $u$ is h-convex. As is explained later, $S[u]$ is not necessarily h-convex for an arbitrary $u \in C(\mathbb{H})$; see Example 4.5.

Let us now verify that $S$ maps $u \in USC(\mathbb{H})$ to $S[u] \in USC(\mathbb{H})$.

**Lemma 4.1** (Upper semicontinuity preserving)**.** *Suppose that $u \in USC(\mathbb{H})$ is bounded from below. Let the operator $S$ be defined as in* (1.8)*. Then $S[u] \in USC(\mathbb{H})$.*

*Proof.* Fix $\overline{p} \in \mathbb{H}$ arbitrarily. In view of (1.8), for any $\varepsilon > 0$, there exist $\overline{c}_i \in [0,1]$ and $\overline{p}_i \in \mathbb{H}_{\overline{p}} (i=1,2,3)$ such that

$$\sum_{i=1,2,3} \overline{c}_i = 1, \qquad \sum_{i=1,2,3} \overline{c}_i \overline{p}_i = \overline{p}$$

and

$$S[u](\overline{p}) \ge \sum_{i=1,2,3} \overline{c}_i u(\overline{p}_i) - \varepsilon. \tag{4.1}$$

Moreover, by the continuity of $p \mapsto \mathbb{H}_p$ and the upper semicontinuity of $u$, for any $p \in \mathbb{H}$ sufficiently close to $\overline{p}$, we can find $p_i \in \mathbb{H}_p$ near $\overline{p}_i$ such that

$$\sum_{i=1,2,3} \overline{c}_i p_i = p$$

and for all $i=1,2,3$

$$u(p_i) \le u(\overline{p}_i) + \varepsilon.$$

It follows that

$$\sum_i \overline{c}_i u(p_i) \le \sum_i \overline{c}_i u(\overline{p}_i) + 3\varepsilon.$$

By (4.1) and (1.8), we obtain that

$$S[u](p) \leq S[u](\overline{p}) + 4\varepsilon,$$

which implies the upper semicontinuity of $S[u]$. $\square$

However, $S$ does not preserve lower semicontinuity in general, which is very different from the Euclidean case [1, Lemma 4]. An example is as follows.

*Example* 4.2 (Loss of lower semicontinuity preserving). Let us construct $u \in C(\mathbb{H})$ bounded below satisfying

$$u(p) = 1 \quad \text{for all } p \in \mathbb{H}_0, \text{ and} \tag{4.2}$$

$$\lim_{x\to\infty} u(q_x) = \lim_{x\to\infty} u\left((q_x)^{-1}\right) = 0, \quad \text{where } q_x = (x, 0, -1/2). \tag{4.3}$$

Take $\varepsilon > 0$ arbitrarily small and consider the point $p_\varepsilon = (\varepsilon, \varepsilon, 0) \in \mathbb{H}$. It is not difficult to verify that

$$p_+, p_- \in \operatorname{span}\{X(p_\varepsilon), Y(p_\varepsilon)\} = \mathbb{H}_{p_\varepsilon},$$

where

$$p_+ = \left(\frac{1}{\varepsilon}, 0, -\frac{1}{2}\right), \quad p_- = \left(-\frac{1}{\varepsilon}, 0, \frac{1}{2}\right).$$

By the definition of $S$ in (1.8), we easily see that

$$S[u](p_\varepsilon) \leq \frac{1}{2}u(p_+) + \frac{1}{2}u(p_-)$$

and therefore by (4.3)

$$\liminf_{\varepsilon\to 0} S[u](p_\varepsilon) \leq 0.$$

On the other hand, by (4.2) we deduce that $S[u](0) = 1$. Hence, we conclude that the lower semicontinuity of $S[u]$ fails to hold at the origin.

The counterexample above would not exist if we could exclude the situation like (4.3). The lower semicontinuity for $S[u]$ can be obtained by assuming that $u$ is coercive, as indicated in the following result.

**Lemma 4.3** (Lower semicontinuity preserving under coercivity)**.** *Suppose that* $u \in LSC(\mathbb{H})$ *satisfies the coercivity condition* (1.10)*. Let the operator* $S$ *be defined as in* (1.8)*. Then* $S[u] \in LSC(\mathbb{H})$*.*

*Proof.* Without loss of generality, we may only show that

$$\liminf_{p\to 0} S[u](p) \geq S[u](0).$$

For any sequence $\{p_j\} \subset \mathbb{H}$ with $p_j \to 0$ as $j \to \infty$, there exist $p_{ij} \in \mathbb{H}_{p_j}$ and $c_{ij} \in [0,1]$ $(i = 1, 2, 3)$ for each $j$ such that

$$\sum_{i=1,2,3} c_{ij} = 1, \quad \sum_{i=1,2,3} c_{ij}p_{ij} = p_j, \quad \text{and} \quad \sum_{i=1,2,3} c_{ij}u(p_{ij}) \leq S[u](p_j) + \frac{1}{j}. \tag{4.4}$$

Thanks to the coercivity condition (1.10), we see that $\{p_{ij}\}_{i=1,2,3}$ are bounded uniformly in $j$. We thus can take a subsequence (still indexed by $j$ for simplicity) such that as $j \to \infty$

$$c_{ij} \to \overline{c}_i \in [0,1], \quad p_{ij} \to \overline{p}_i \in \mathbb{H}.$$

It follows from (4.4) that

$$\sum_i \overline{c}_i = 1, \quad \sum_i \overline{c}_i\overline{p}_i = 0,$$

and, by lower semicontinuity of $u$,

$$\sum_{i=1,2,3} \overline{c}_i u(\overline{p}_i) \leq \liminf_{j\to\infty} S[u](p_j).$$

Moreover, by the locally uniform continuity of the horizontal plane $\mathbb{H}_p$ with respect to $p$, we have $\overline{p}_i \in \mathbb{H}_0$. By definition of $S$, we thus have

$$S[u](0) \leq \liminf_{j\to\infty} S[u](p_j).$$

This concludes the proof, since the converging sequence $\{p_j\}$ is taken arbitrarily. □

The following lemma is thus an immediate consequence.

**Lemma 4.4** (Preservation of coercivity)**.** *Suppose that $u \in C(\mathbb{H})$ satisfies* (1.10)*. Let $S$ be given by* (1.8)*. Then $S[u]$ also satisfies* (1.10)*.*

*Proof.* Note that the condition (1.10) implies that for each $C_1 > 0$, there exist $R > 0$ and $C_2 > 0$ depending on $R$ such that

$$u(p) \geq C_1|p| - C_2$$

for all $p \in \mathbb{H}$. It is clear that $S[u](p) \geq C_1|p| - C_2$ for all $p \in \mathbb{H}$, since the right hand side is convex in the Euclidean sense and therefore h-convex in $H$. It follows that

$$\liminf_{R'\to\infty} \inf_{|p|\geq R'} \frac{S[u](p)}{|p|} \geq C_1,$$

which yields the coercivity of $S[u]$ thanks to the arbitrariness of $C_1 > 0$. □

We finally remark that it is possible to define a right-invariant version of the operator $S$ as below. For any $u \in C(\mathbb{H})$ bounded below, let $\tilde{S}$ be given by (1.15). It is easily seen that the properties of $S$ above hold also for $\tilde{S}$ via analogous arguments above. In particular, $\tilde{S}[u]$ is continuous and coercive in the sense of (1.10) in $\mathbb{H}$ provided that $u$ is continuous and coercive in the same sense.

4.2. **Iterated convexification.** One may expect that $S[u]$ is h-convex in $\mathbb{H}$ for any $u \in C(\mathbb{H})$, but it turns out to be false in general. An example similar to Example 4.2 can be easily built as below.

*Example* 4.5 (Failure of h-convexification)*.* Consider $p = (0,0,0)$ and $h = (1,0,0) \in \mathbb{H}_0$. Let

$$p_1 = (1,1,1/2), \quad p_2 = (1,-1,-1/2),$$
$$p_3 = (-1,-1,1/2), \quad p_4 = (-1,1,-1/2).$$

Note that $p_i \notin \mathbb{H}_0$ for all $i = 1, ..., 4$. We thus can take $u \in C(\mathbb{H})$ such that $u(p_1) = u(p_2) = u(p_3) = u(p_4) = 0$ and $u(p) = 1$ for all $p \in \mathbb{H}_0$.

Since $p_1, p_2 \in \mathbb{H}_h$ and $p_3, p_4 \in \mathbb{H}_{h^{-1}}$, by definition (1.8), we have

$$S[u](h) \leq \frac{1}{2}u(p_1) + \frac{1}{2}u(p_2), \quad S[u](h^{-1}) \leq \frac{1}{2}u(p_3) + \frac{1}{2}u(p_4).$$

On the other hand, since $u \equiv 1$ in $\mathbb{H}_0$, we get

$$S[u](0) \geq 1.$$

It follows that

$$S[u](h) + S[u](h^{-1}) < 2S[u](0),$$

which states that $S[u]$ is not h-convex at the origin.

The function $u$ in this example is not coercive. However, it is not difficult to raise the values of $u$ near space infinity without influencing much the effect of the operator $S$ on $u$ at those points we are interested in. A more explicit example will given in Example 4.15 of Section 4.4.

Although the operator $S$ does not give us the convex envelope immediately, we may iterate it and passing to the limit. In other words, we take

$$U(p) := \lim_{n\to\infty} S^n[u](p) \tag{4.5}$$

for $u \in C(\mathbb{H})$ and any $p \in \mathbb{H}$. It is easily seen that the pointwise limit of $S^n[u]$ actually exists, thanks to the monotonicity $S[u] \leq u$ for $u \in USC(\mathbb{H})$. In addition, we have

$$U \leq u \quad \text{ in } \mathbb{H}. \tag{4.6}$$

**Lemma 4.6** (H-convexity of $U$)**.** *Suppose that $u \in C(\mathbb{H})$ is bounded from below. Let $U$ be given as in* (4.5)*. Then $U$ is h-convex in $\mathbb{H}$. In particular, $U$ is locally Lipschitz with respect to the gauge metric in $\mathbb{H}$.*

*Proof.* Fix $p \in \mathbb{H}$ and $h \in \mathbb{H}_0$. By definition, for any $\varepsilon > 0$, there exists $n > 0$ sufficiently large such that

$$U(p\cdot h) \geq S^n[u](p\cdot h) - \varepsilon, \quad U(p\cdot h^{-1}) \geq S^n[u](p\cdot h^{-1}) - \varepsilon. \tag{4.7}$$

In view of (1.8), we have

$$S^n[u](p\cdot h) + S^n[u](p\cdot h^{-1}) \geq 2S^{n+1}[u](p),$$

which implies by monotonicity of $S$ that

$$S^n[u](p\cdot h) + S^n[u](p\cdot h^{-1}) \geq 2U(p).$$

Combining this relation with (4.7), we deduce that

$$U(p\cdot h) + U(p\cdot h^{-1}) \geq 2U(p) - 2\varepsilon.$$

We conclude that $U$ is h-convex by letting $\varepsilon \to 0$.

The Lipschitz regularity is an immediate consequence of Theorem 2.7. □

*Remark* 4.7. Assuming in addition that $u \in C(\mathbb{H})$ is coercive in the sense of (1.10), we have $S^n[u]$ is continuous in $\mathbb{H}$. Since $S^n[u]$ is a monotone sequence of continuous functions and $U$ is continuous in $\mathbb{H}$, by Dini's theorem, we therefore obtain locally uniform convergence of $S^n[u]$ to $U$ in $\mathbb{H}$ as $n \to \infty$.

We finally present our main theorem of this section, showing that $U$ and $\Gamma u$ are the same for any given $u \in C(\mathbb{H})$.

**Theorem 4.8** (Characterization by iterated convexification)**.** *Suppose that $u \in C(\mathbb{H})$ is bounded from below. Let $\Gamma u$ and $U$ be given by* (1.7) *and* (4.5)*. Then $\Gamma u \equiv U$ in $\mathbb{H}$.*

*Proof.* We now prove the equality $\Gamma u = U$. Since $\Gamma u \leq u$ and $\Gamma u$ is h-convex, by Remark 2.5 we have, for any $p \in \mathbb{H}$,

$$\Gamma u(p) \leq \sum_i c_i \Gamma u(p_i) \leq \sum_i c_i u(p_i)$$

holds for all $c_i \in [0,1]$ and $p_i \in \mathbb{H}_p$ (i=1, 2, 3) with $\sum_i c_i = 1$, $p_i \in \mathbb{H}$ and $\sum_i c_i p_i = p$. It follows that $\Gamma u \leq S[u]$ in $\mathbb{H}$.

Since $S[\Gamma u] = \Gamma u$, iterating the argument above yields that

$$\Gamma u \leq S^n[u],$$

which implies that $\Gamma u \leq U$.

Since $\Gamma u$ is defined to be the largest h-convex function below $u$, the inequality $\Gamma u \geq U$ is an immediate consequence of Lemma 4.6 together with (4.6). □

*Remark* 4.9. Since $\Gamma u$ and $U$ are equivalent, from now on, we also use $\Gamma u$ to denote the limit of $S^n[u]$ for any given function $u \in C(\mathbb{H})$ bounded below. In particular, in the presence of the coercivity (1.10) of $u$, we have $S^n[u] \to U$ locally uniformly as $n \to \infty$ due to Remark 4.7.

4.3. **Symmetry with respect to $z$-axis.** We consider a special case when $u$ satisfies a symmetry condition. We say $u$ is symmetric with respect to the $z$-axis if (1.13) holds. We can show that the operator $S$ preserves this symmetry condition.

**Lemma 4.10** (Preservation of symmetry)**.** *Suppose that $u \in C(\mathbb{H})$ is bounded from below and symmetric with respect to $z$-axis. Then $S[u]$ given by* (1.8) *is also symmetric with respect to $z$-axis. In particular, $\Gamma u$ satisfies the same symmetry condition as well.*

*Proof.* Pick arbitrarily $p_0 = (x_0, y_0, z_0) \in \mathbb{H}$ and $p_0' = (-x_0, -y_0, z_0) \in \mathbb{H}$. By definition of $S[u]$, for any $\delta > 0$ small, there exist $p_i = (x_i, y_i, z_i) \in \mathbb{H}_{p_0}$ with $c_i \in [0, 1]$, $i = 1, 2, 3$, such that

$$\sum_{i=1,2,3} c_i = 1, \quad \sum_{i=1,2,3} c_i p_i = p_0 \tag{4.8}$$

and

$$S[u](p_0) \geq \sum_{i=1,2,3} c_i u(p_i) - \delta. \tag{4.9}$$

Using the plane equation as in (2.1), we can express the relation $p_i = (x_i, y_i, z_i) \in \mathbb{H}_{p_0}$ by

$$y_0 x_i - x_0 y_i + 2z_i - 2z_0 = 0. \tag{4.10}$$

Set $p_i' = (-x_i, -y_i, z_i) \in \mathbb{H}$. It is easily verified that $p_i' \in \mathbb{H}_{p_0'}$ for $i = 1, 2, 3$.

We can apply the symmetry of $u$ to (4.9) to obtain that

$$S[u](p_0) \geq \sum_{i=1,2,3} c_i u(p_i') - \delta. \tag{4.11}$$

By (4.8), it is also clear that

$$\sum_{i=1,2,3} c_i p_i' = p_0'.$$

It thus follows from (4.11) that

$$S[u](p_0) \geq S[u](p_0') - \delta,$$

which implies that

$$S[u](p_0) \geq S[u](p_0')$$

by letting $\delta \to 0$. Exchanging the roles of $p_0$ and $p_0'$, we obtain that $S[u](p_0) = S[u](p_0')$, which means that $S[u]$ is symmetric with respect to $z$-axis. As an immediate consequence of Theorem 4.8, we can also deduce the symmetry of $\Gamma u$. □

The additional symmetry assumption will largely facilitate our study on properties of the h-convex envelope later. A typical advantage with such symmetry is that the left and right invariant convexification actually coincide.

**Theorem 4.11** (Equivalence under symmetry)**.** *Suppose that $u \in C(\mathbb{H})$ is bounded from below and symmetric with respect to $z$-axis. Let $S[u]$ and $\tilde{S}[u]$ be given as in* (1.8) *and* (1.15) *respectively. Then $S[u] = \tilde{S}[u]$ in $\mathbb{H}$.*

*Proof.* Fix $p_0 = (x_0, y_0, z_0) \in \mathbb{H}$ arbitrarily. As in the proof of Lemma 4.10, we can take for any $\delta > 0$ small $p_i = (x_i, y_i, z_i) \in \mathbb{H}_{p_0}$ and $c_i \in [0, 1]$ $(i = 1, 2, 3)$ such that (4.8) and (4.9) hold. We thus still have (4.10).

Let us again take $p_i' = (-x_i, -y_i, z_i)$. Then the symmetry of $u$ and (4.9) yield that

$$S[u](p_0) \geq \sum_{i=1,2,3} c_i u(p_i') - \delta. \tag{4.12}$$

Moreover, a direct calculation with the choices of $p_i'$ enables us to get

$$y_0 x_i' - x_0 y_i' = -y_0 x_i + x_0 y_i,$$

which, in view of (4.10), implies that

$$y_0 x_i' - x_0 y_i' + 2z_i - 2z_0 = 0.$$

This amounts to saying that $p_i' \in \tilde{\mathbb{H}}_{p_0}$. Since we also have

$$\sum_{i=1,2,3} c_i p_i' = p_0,$$

by (1.15) we obtain

$$S[u](p_0) \geq \tilde{S}[u](p_0) - \delta.$$

Sending $\delta \to 0$, we are led to $S[u](p_0) \geq \tilde{S}[u](p_0)$. As a parallel argument yields that $\tilde{S}[u](p_0) \leq S[u](p_0)$, we complete the proof. □

As a result of Theorem 4.11, we immediately obtain the following.

**Proposition 4.12.** *Suppose that $u \in C(\mathbb{H})$ is bounded from below and symmetric with respect to $z$-axis. Then $u$ is h-convex if and only if it is right invariant h-convex.*

The proof is based on the fact that $u = S[u]$ (resp., $u = \tilde{S}[u]$) if and only if $u$ is h-convex (resp. right invariant h-convex).

4.4. **Examples of h-convex envelopes.** Let us give more concrete examples of the h-convex envelope and the operator $S$. We start with a simple example, for which $u$ can be convexified by the operator $S$ in one step.

*Example* 4.13. Let $f \in C(\mathbb{R})$ be given by

$$f(t) = (|t| - 1)^2, \quad t \in \mathbb{R}.$$

Consider

$$u(p) = f\left(x^2 + y^2 + z^2\right), \quad p = (x, y, z) \in \mathbb{H}.$$

It is clear that $u$ is coercive in $\mathbb{H}$. One may guess that

$$\Gamma u(p) = \begin{cases} 0 & \text{if } x^2 + y^2 + z^2 \leq 1, \\ u(p) & \text{if } x^2 + y^2 + z^2 > 1. \end{cases} \tag{4.13}$$

In other words, we expect that the h-convex envelope of $u$ is

$$U(p) = F_E\left(x^2 + y^2 + z^2\right), \quad p = (x, y, z) \in \mathbb{H},$$

where $F_E$ denotes the Euclidean convex envelope of $f$ in $\mathbb{R}$. In fact, this relation does hold. Note first that the right hand side of (4.13) is h-convex in $\mathbb{H}$. Moreover, we have $S[u] = \Gamma u$ in $\mathbb{H}$ in this case. Indeed, since $u$ takes a minimum value 0 and the minimizers of $u$ form a closed surface

$$x^2 + y^2 + z^2 = 1,$$

for any $p = (x, y, z) \in \mathbb{H}$ such that $x^2 + y^2 + z^2 < 1$, the horizontal plane at $p$ and the closed surface must intersect at a closed curve on the plane. One therefore can take on the intersection three points whose convex combination coincides with $p$. This immediately yields that

$$S[u](p) = 0$$

for any $p = (x, y, z) \in \mathbb{H}$ such that $x^2 + y^2 + z^2 < 1$. It is clear that $S[u] = u$ at the rest of the points.

*Example* 4.14. We next give an example, showing that one sometimes needs to apply the operator $S$ twice to find the envelope $\Gamma u$. Suppose that

$$u(p) = (z^2 - 1)^2, \quad p = (x, y, z) \in \mathbb{H}.$$

In this case, we have

$$\Gamma u(x, y, z) = S^2[u](x, y, z) = \begin{cases} 0 & \text{if } |z| \le 1, \\ (z^2 - 1)^2 & \text{if } |z| > 1. \end{cases} \tag{4.14}$$

In fact, using the same argument for Example 4.13, we get

$$S[u](x, y, z) = \begin{cases} 0 & \text{if } |z| \le 1 \text{ and } (x, y) \ne (0, 0), \\ (z^2 - 1)^2 & \text{if } |z| > 1 \text{ or } (x, y) = (0, 0). \end{cases}$$

One can easily apply the operator $S$ again on $S[u]$ to obtain the envelope given in (4.14).

*Example* 4.15. Let $u : \mathbb{R}^3 \to \mathbb{R}$ be defined as

$$u(x, y, z) = (x - y)z + (x - y)^2 z^2 + (x^2 + y^2)^2 + z^2.$$

Clearly, $u$ is bounded from below and coercive. We show that $S[u](p)$ is not h-convex.

Note we have $u(p) \ge 0$ for all $p \in \mathbb{H}_0$ and

$$S[u](0) = 0.$$

Let $t > 0$ and $h_t = (t, t, 0) \in \mathbb{H}_0$. Then the horizontal plane at $h_t$ is the collection of points

$$\mathbb{H}_{h_t} = (t, t, 0) \cdot \mathbb{H}_0 = \left\{ \left( t + x, t + y, \frac{t}{2}(y - x) \right) : x, y \in \mathbb{R} \right\}.$$

Choose $c_1 = c_2 = \frac{1}{2}$ and $p_1, p_2 \in \mathbb{H}_{h_t}$ defined as

$$p_1 = \left( t + t, t - t, \frac{t}{2}(-t - t) \right) = (2t, 0, -t^2)$$

$$p_2 = \left( t - t, t + t, \frac{t}{2}(t + t) \right) = (0, 2t, t^2).$$

Then

$$u(p_1) = -2t^3 + 4t^6 + 16t^4 + t^4,$$

$$u(p_2) = -2t^3 + 4t^6 + 16t^4 + t^4,$$

and

$$\begin{aligned} S[u](h_t) &\leq \frac{1}{2}u(p_1) + \frac{1}{2}u(p_2) \\ &= -2t^3 + 4t^6 + 17t^4. \end{aligned}$$

On the other hand, at $h_t^{-1} = (-t, -t, 0)$, we have

$$0 \leq S[u](h_t^{-1}) \leq u(h_t^{-1}) = 4t^4.$$

For $t > 0$ sufficiently small, $S[u](h_t) < 0$. Thus,

$$S[u](h_t) + S[u](h_t^{-1}) < 0 = 2S[u](0).$$

## 5. Supersolution preserving property for elliptic equations

Let us now apply the notion of h-convex envelope to investigate the so-called supersolution preserving property. In this paper we focus our attention to the elliptic equation (1.6) but similar results can be shown for parabolic problems as well.

Assume that $u$ is a coercive supersolution of (1.6). We aim to understand whether the h-convex envelope $\Gamma u$ is a supersolution of the same equation, since an affirmative answer, combined with a comparison principle, will imply that the unique solution is h-convex.

This method is proposed in the Euclidean space by Alvarez, Lasry and Lions [1]. One can use this method to show that the unique solution of the linear equation

$$u - \Delta u + \langle \zeta, \nabla u \rangle = f(p) \quad \text{in } \mathbb{R}^N$$

is convex for any $\zeta \in \mathbb{R}^N$ provided that $f$ is convex in $\mathbb{R}^N$. We remark that in general we cannot expect that the same result holds in the Heisenberg group, as indicated by Example 1.1.

5.1. **The right invariant envelope.** Our first result concerns the right convexification as given in (1.15). In addition to (A1)(A2), we need the concavity condition on $F$ below.

(A3) $(p, r, \xi, A) \mapsto F(p, r, \xi, A)$ is concave in the sense that for any $k \in \mathbb{N}$,

$$\sum_{i=1}^{k} c_i F(p_i, r_i, \xi_i, A_i) \leq F\left(\sum_{i=1}^{k} c_i p_i,\ \sum_{i=1}^{k} c_i r_i,\ \sum_{i=1}^{k} c_i \xi_i,\ \sum_{i=1}^{k} c_i A_i\right)$$

holds for any $c_i \in [0, 1]$ with $\sum_i c_i = 1$, $p_i \in \mathbb{H}$, $r_i \in \mathbb{R}$, $\xi_i \in \mathbb{R}^2$ and $A_i \in \mathbf{S}^2$ (for all $i = 1, 2, \ldots, k$) satisfying

$$p_i \in \tilde{\mathbb{H}}_{\sum_{i=1}^{k} c_i p_i}. \tag{5.1}$$

Let us define an operator of approximate right convexification as follows. For any $\varepsilon > 0$ small and any $p = (x, y, z) \in \mathbb{H}$, set

$$\tilde{S}_\varepsilon[u](p) := \inf\left\{ \sum_i c_i u(p_i) + \frac{1}{\varepsilon}\tilde{W}(p_1, p_2, p_3) :\ c_i \in [0, 1],\ p_i \in \mathbb{H}\ (i = 1, 2, 3), \right.$$
$$\left. \sum_i c_i = 1,\ \sum_i c_i p_i = p \right\}. \tag{5.2}$$

Here we denote
$$\tilde{W}(p_1,p_2,p_3)=\sum_{i=1,2,3}c_i\tilde{g}_i^2\left(p_1,p_2,p_3\right),$$
where
$$\tilde{g}_i(p_1,p_2,p_3)=\left(\sum_i c_iy_i\right)x_i-\left(\sum_i c_ix_i\right)y_i-2z_i+2\sum_i c_iz_i.$$
Note that, since the right invariant horizontal plane $\tilde{H}_p$ at $p\in\mathbb{H}$ is given by (2.3), the quantity $\tilde{g}_i$ essentially measures how far the point $p_i$ is away from the right invariant horizontal plane passing through $\sum_i c_ip_i$.

**Theorem 5.1** (Supersolution preserving by right invariant convexification)**.** *Assume that (A1), (A2) and (A3) hold. Let $u\in C(\mathbb{H})$ be a supersolution of* (1.6)*. Suppose that $u$ satisfies the coercivity condition* (1.10)*. Let $\tilde{S}_\varepsilon[u]$ be given by* (5.2)*. Then $\tilde{S}_\varepsilon[u]$ is also a supersolution of* (1.6) *for any $\varepsilon>0$ small. Moreover, $\tilde{S}[u]$ given by* (1.15) *is a lower semicontinuous supersolution of* (1.6) *as well.*

*Proof.* Fix $\varepsilon>0$ arbitrarily. Let us first show that $\tilde{S}_\varepsilon[u]$ is a supersolution. Suppose that there is $\varphi\in C^2(\mathbb{H})$ such that $\tilde{S}_\varepsilon[u]-\varphi$ attains a minimum at $p^\varepsilon=(x^\varepsilon,y^\varepsilon,z^\varepsilon)\in\mathbb{H}$. We aim to show that
$$F\left(p^\varepsilon,\tilde{S}_\varepsilon[u](p^\varepsilon),\nabla_H\varphi(p^\varepsilon),(\nabla_H^2\varphi)^\star(p_\varepsilon)\right)\geq 0. \tag{5.3}$$
We may further assume that $\varphi$ is bounded in $\mathbb{H}$. For our use later, we denote
$$\eta_\varepsilon=\nabla\varphi(p^\varepsilon),\quad Q_\varepsilon=\nabla^2\varphi(p^\varepsilon).$$

By definition of $\tilde{S}_\varepsilon[u]$ and the coercivity of $u$, there exist $c_i\in[0,1]$ and $p_i^\varepsilon\in\mathbb{H}$ ($i=1,2,3$) with
$$\sum_i c_ip_i^\varepsilon=p^\varepsilon \tag{5.4}$$
such that
$$\tilde{S}_\varepsilon[u](p^\varepsilon)=\sum_i c_iu(p_i^\varepsilon)+\frac{1}{\varepsilon}\tilde{W}(p_1^\varepsilon,p_2^\varepsilon,p_3^\varepsilon). \tag{5.5}$$
We may assume that $c_i\neq 0$ for every $i=1,2,3$, for otherwise we simply reduce to the situation with fewer terms in the sum above and the whole argument below still works.

It is then clear that
$$\Phi_\varepsilon(p_1,p_2,p_3)=\sum_{i=1,2,3}c_iu(p_i)-\varphi\left(\sum_i c_ip_i\right)+\frac{1}{\varepsilon}\tilde{W}(p_1,p_2,p_3) \tag{5.6}$$
attains a minimum at $(p_1^\varepsilon,p_2^\varepsilon,p_3^\varepsilon)$.

Denote $p_i^\varepsilon=(x_i^\varepsilon,y_i^\varepsilon,z_i^\varepsilon)$ for $i=1,2,3$. In view of the minimality of $\Phi_\varepsilon$ at $(p_1^\varepsilon,p_2^\varepsilon,p_3^\varepsilon)$, we use the Crandall-Ishii lemma [10] to obtain, for any $\sigma>0$, $(\eta_i,Q_i)\in\overline{J}_E^{2,-}u(p_i^\varepsilon)$ satisfying
$$c_i\eta_i=c_i\nabla\varphi(p^\varepsilon)-\frac{1}{\varepsilon}\nabla_i\tilde{W}(p_1^\varepsilon,p_2^\varepsilon,p_3^\varepsilon) \tag{5.7}$$
and
$$\mathbf{Q}\geq\mathbf{A}-\sigma\mathbf{A}^2, \tag{5.8}$$
where
$$\mathbf{Q}=\begin{pmatrix}c_1Q_1&0&0\\0&c_2Q_2&0\\0&0&c_3Q_3\end{pmatrix}$$

and

$$\mathbf{A} = \begin{pmatrix} c_1^2 Q_\varepsilon & c_1c_2Q_\varepsilon & c_1c_3Q_\varepsilon \\ c_1c_2Q_\varepsilon & c_2^2Q_\varepsilon & c_2c_3Q_\varepsilon \\ c_1c_3Q_\varepsilon & c_2c_3Q_\varepsilon & c_3^2Q_\varepsilon \end{pmatrix} - \frac{1}{\varepsilon}\nabla^2 \tilde{W}(p_1^\varepsilon, p_2^\varepsilon, p_3^\varepsilon). \tag{5.9}$$

For later use, we denote by $\mathbf{A}_1$ the first matrix on the right hand side of (5.9).

Let $p_i^\varepsilon = (x_i, y_i, z_i)$ for $i = 1,2,3$. By direct calculations, we have

$$\begin{aligned} \frac{\partial \tilde{W}}{\partial x_i} &= 2c_i \left( \tilde{g}_i \sum_j c_j y_j - \sum_j c_j \tilde{g}_j y_j \right), \\ \frac{\partial \tilde{W}}{\partial y_i} &= 2c_i \left( -\tilde{g}_i \sum_j c_j x_j + \sum_j c_j \tilde{g}_j x_j \right), \\ \frac{\partial \tilde{W}}{\partial z_i} &= 4c_i \left( -\tilde{g}_i + \sum_j c_j \tilde{g}_j \right), \end{aligned}$$

and, moreover,

$$\begin{aligned} \frac{\partial^2 \tilde{W}}{\partial x_i \partial x_j} =& 2\delta_{ij} c_j \left( \sum_j c_j y_j \right)^2 - 2c_ic_j \left( \sum_j c_j y_j \right)(y_i + y_j) + 2c_ic_j \left( \sum_j c_j y_j^2 \right), \\ \frac{\partial^2 \tilde{W}}{\partial y_i \partial y_j} =& 2\delta_{ij} c_j \left( \sum_j c_j x_j \right)^2 - 2c_ic_j \left( \sum_j c_j x_j \right)(x_i + x_j) + 2c_ic_j \left( \sum_j c_j x_j^2 \right), \\ \frac{\partial^2 \tilde{W}}{\partial z_i \partial z_j} =& 8\delta_{ij} c_i - 8c_ic_j, \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \tilde{W}}{\partial x_i \partial y_j} =& -2\delta_{ij} c_i \left( \sum_j c_j x_j \right) \left( \sum_j c_j y_j \right) \\ & + 2c_ic_jx_i \left( \sum_j c_j y_j \right) + 2c_ic_jy_j \left( \sum_j c_j x_j \right) - 2c_ic_j \left( \sum_j c_j x_j y_j \right), \\ \frac{\partial^2 \tilde{W}}{\partial x_i \partial z_j} =& -4\delta_{ij} c_i \left( \sum_j c_j y_j \right) + 4c_ic_jy_j, \\ \frac{\partial^2 \tilde{W}}{\partial y_i \partial z_j} =& 4\delta_{ij} c_i \left( \sum_j c_j x_j \right) - 4c_ic_jx_j \end{aligned}$$

for all $i, j = 1,2,3$. Then at the point $(p_1^\varepsilon, p_2^\varepsilon, p_3^\varepsilon)$, we get

$$\nabla_i \tilde{W} = 2c_i \left( \tilde{g}_i y_\varepsilon - \sum_j c_j \tilde{g}_j y_j, -\tilde{g}_i x_\varepsilon + \sum_j c_j \tilde{g}_j x_j, -2\tilde{g}_i \right)$$

and

$$\begin{aligned}
\frac{\partial^2 \tilde{W}}{\partial x_i \partial x_j} =& 2\delta_{ij} c_j y_\varepsilon^2 - 2c_i c_j y_\varepsilon (y_i + y_j) + 2c_i c_j \left( \sum_j c_j y_j^2 \right) \\
\frac{\partial^2 \tilde{W}}{\partial y_i \partial y_j} =& 2\delta_{ij} c_j x_\varepsilon^2 - 2c_i c_j x_\varepsilon (x_i + x_j) + 2c_i c_j \left( \sum_j c_j x_j^2 \right) \\
\frac{\partial^2 \tilde{W}}{\partial z_i \partial z_j} =& 8\delta_{ij} c_i - 8c_i c_j, \\
\frac{\partial^2 \tilde{W}}{\partial x_i \partial y_j} =& -2\delta_{ij} c_i x_\varepsilon y_\varepsilon + 2c_i c_j x_i y_\varepsilon + 2c_i c_j y_j x_\varepsilon - 2c_i c_j \left( \sum_j c_j x_j y_j \right), \\
\frac{\partial^2 \tilde{W}}{\partial x_i \partial z_j} =& -4\delta_{ij} c_i y_\varepsilon + 4c_i c_j y_j, \\
\frac{\partial^2 \tilde{W}}{\partial y_i \partial z_j} =& 4\delta_{ij} c_i x_\varepsilon - 4c_i c_j x_j,
\end{aligned}$$

where $\delta_{ij}$ is the Kronecker delta. Note that for any $p = (x, y, z) \in \mathbb{H}$, if $(\eta, Q) \in \overline{J}_E^{2,-} u(p)$, then $(\xi, P) \in \overline{J}_H^{2,-} u(p)$, where

$$\xi = M_p \eta, \quad P = M_p Q M_p^T.$$

Here $M_p$ is a $2 \times 3$ matrix given by

$$M_p = \begin{pmatrix} 1 & 0 & -y/2 \\ 0 & 1 & x/2 \end{pmatrix}$$

and $M_p^T$ denotes its transpose. We use these relations to find $(\xi_i, P_i) \in \overline{J}_H^{2,-} u(p_i^\varepsilon)$, that is,

$$\xi_i = M_{p_i^\varepsilon} \eta_i, \quad P_i = M_{p_i^\varepsilon} Q_i M_{p_i^\varepsilon}^T \tag{5.10}$$

for all $i = 1, 2, 3$. Then by (5.7) and (5.10), we obtain

$$c_i \xi_i = c_i M_{p_i^\varepsilon} \nabla \varphi(p^\varepsilon) - \frac{1}{\varepsilon} M_{p_i^\varepsilon} \nabla_i \tilde{W}(p_1^\varepsilon, p_2^\varepsilon, p_3^\varepsilon), \tag{5.11}$$

which, by direct calculations, yields that

$$\sum_i c_i \xi_i = \nabla_H \varphi(p^\varepsilon), \tag{5.12}$$

since

$$\begin{aligned}
&M_{p_i^\varepsilon} \nabla_i \tilde{W}(p_1^\varepsilon, p_2^\varepsilon, p_3^\varepsilon) \\
&= 2c_i \left( \tilde{g}_i \sum_i c_i y_i - \sum_i c_i \tilde{g}_i y_i + \tilde{g}_i y_i,\ -\tilde{g}_i \sum_i c_i x_i + \sum_i c_i \tilde{g}_i x_i - \tilde{g}_i x_i \right)^T.
\end{aligned}$$

More calculations are needed for the second derivatives. For any $(a, b) \in \mathbb{R}^2$, set

$$\ell[a, b] = \left( a, b, -\frac{y_1}{2} a + \frac{x_1}{2} b, a, b, -\frac{y_2}{2} a + \frac{x_2}{2} b, a, b, -\frac{y_3}{2} a + \frac{x_3}{2} b \right)^T \in \mathbb{R}^9.$$

We multiply both sides of (5.8) by $\ell[a, b]$ from the left and by its transpose $\ell[a, b]^T$ from the right. We first have

$$\langle \mathbf{Q} \ell[a, b], \ell[a, b] \rangle = \langle \mathbf{P} k[a, b], k[a, b] \rangle = \left\langle \left( \sum_i c_i P_i \right) (a, b)^T, (a, b)^T \right\rangle, \tag{5.13}$$

where $k[a,b]=(a,b,a,b,a,b)^T\in\mathbb{R}^6$ and

$$\mathbf{P}=\begin{pmatrix} c_1P_1 & 0 & 0\\ 0 & c_2P_2 & 0\\ 0 & 0 & c_3P_3\end{pmatrix}.$$

For the right hand side, straightforward calculations yield

$$\begin{aligned}\langle \mathbf{A}_1\ell[a,b],\ell[a,b]\rangle &= (a,b)\left(\sum_i c_iM_{p_i^\varepsilon}\right)Q_\varepsilon\left(\sum_i c_iM_{p_i^\varepsilon}^T\right)\begin{pmatrix}a\\ b\end{pmatrix}\\ &=\left\langle \left(\nabla_H^2\varphi\right)^\star(p^\varepsilon)(a,b)^T,(a,b)^T\right\rangle.\end{aligned}\tag{5.14}$$

In addition, we can calculate to see that

$$\nabla^2\tilde{W}(p_1^\varepsilon,p_2^\varepsilon,p_3^\varepsilon)\ell[a,b]=0,\tag{5.15}$$

which yields

$$\left\langle\left(\nabla^2\tilde{W}(p_1^\varepsilon,p_2^\varepsilon,p_3^\varepsilon)\right)\ell[a,b],\ell[a,b]\right\rangle=0\tag{5.16}$$

and

$$\left\langle\left(\nabla^2\tilde{W}(p_1^\varepsilon,p_2^\varepsilon,p_3^\varepsilon)\right)^2\ell[a,b],\ell[a,b]\right\rangle=0.\tag{5.17}$$

Combining (5.13)–(5.17) with (5.8), we are led to

$$\sum_i c_iP_i\geq(\nabla_H^2\varphi)^\star(p^\varepsilon)-O(\sigma)\tag{5.18}$$

when $\sigma>0$ is taken small.

Since $u$ is a supersolution of (1.6), we get

$$F\left(p_i^\varepsilon,u(p_i^\varepsilon),\xi_i,P_i\right)\geq 0\quad i=1,2,3.$$

Multiplying the inequality above by $c_i$ and summing them up, we deduce that

$$\sum_i c_iF\left(p_i^\varepsilon,u(p_i^\varepsilon),\xi_i,P_i\right)\geq 0.$$

By the concavity condition (A3), we obtain

$$F\left(\sum_i c_ip_i^\varepsilon,\ \sum_i u(p_i^\varepsilon),\ \sum_i c_i\xi_i,\ \sum_i c_iP_i\right)\geq 0.$$

Note that (5.5) implies that

$$\tilde{S}_\varepsilon[u](p^\varepsilon)\geq\sum_i c_iu(p_i^\varepsilon).\tag{5.19}$$

Adopting (5.4), (5.12), (5.18) and (5.19) as well as (A1) and (A2), we end up with

$$F\left(p^\varepsilon,\tilde{S}_\varepsilon[u](p^\varepsilon),\nabla_H\varphi(p^\varepsilon),(\nabla_H^2\varphi)^\star(p_\varepsilon)-O(\sigma)\right)\geq 0,$$

which yields (5.3) as desired by letting $\sigma\to 0$.

We finally prove that $\tilde{S}[u]$ is a supersolution. The proof is essentially based on the standard stability theory. Suppose that there exist $\hat{p}\in\mathbb{H}$ and $\varphi\in C^2(\mathbb{H})$ such that $\tilde{S}[u]-\varphi$ attains a strict minimum in $\mathbb{H}$ at $\hat{p}$. Since $u$ is coercive, there exist $\hat{p}_i\in\mathbb{H}_{\hat{p}}$ and $c_i\in[0,1]$ $(i=1,2,3)$ satisfying

$$\sum_i c_i=1,\quad \sum_i c_i\hat{p}_i=\hat{p}$$

such that

$$(p_1, p_2, p_3) \mapsto \sum_i c_i u(p_i) - \varphi\left(\sum_i c_i p_i\right)$$

attains a minimum at $(\hat{p}_1, \hat{p}_2, \hat{p}_3)$. It follows that for any $\varepsilon > 0$ there exist $p_i^\varepsilon \in \mathbb{H}$ such that (5.6) attains a minimum at $(p_1^\varepsilon, p_2^\varepsilon, p_3^\varepsilon)$. Let $p^\varepsilon = \sum_i c_i p_i^\varepsilon$. Then by definition (5.6) amounts to saying that $\tilde{S}_\varepsilon[u] - \varphi$ attains a minimum in $\mathbb{H}$ at $p^\varepsilon$.

Moreover, we claim that $p^\varepsilon \to \hat{p}$ as $\varepsilon \to 0$. Indeed, using the coercivity of $u$, we may take a subsequence so that as $\varepsilon \to 0$, $p_i^\varepsilon \to q_i$ for some $q_i \in \mathbb{H}$ $(i = 1, 2, 3)$ and $p^\varepsilon \to q_0 = \sum_i c_i q_i$. Since

$$\Phi_\varepsilon(p_1^\varepsilon, p_2^\varepsilon, p_3^\varepsilon) \leq \Phi_\varepsilon(\hat{p}_1, \hat{p}_2, \hat{p}_3), \tag{5.20}$$

we deduce that

$$\frac{1}{\varepsilon}\tilde{W}(p_1^\varepsilon, p_2^\varepsilon, p_3^\varepsilon) \leq \sum_i c_i u(\hat{p}_i) - \sum_i c_i u(p_i^\varepsilon) - \varphi(\hat{p}) + \varphi(p^\varepsilon).$$

Noticing that the right hand side is bounded above, we get, as $\varepsilon \to 0$,

$$\tilde{W}(p_1^\varepsilon, p_2^\varepsilon, p_3^\varepsilon) \to 0,$$

which yields that $q_i \in \mathbb{H}_{q_0}$. Hence, letting $\varepsilon \to 0$ in (5.20), we obtain

$$\tilde{S}[u](q_0) - \varphi(q_0) \leq \tilde{S}[u](\hat{p}) - \varphi(\hat{p}),$$

which implies that $q_0 = \hat{p}$ due to the strict minimum of $\tilde{S}[u] - \varphi$ at $\hat{p}$. We complete the proof of the claim.

Since we have shown that $\tilde{S}_\varepsilon[u]$ is a supersolution of (1.6), the inequality (5.3) holds. Noticing that $\tilde{S}[u] \geq \tilde{S}_\varepsilon[u]$, by (A2) we get

$$F\left(p^\varepsilon, \tilde{S}[u](p^\varepsilon), \nabla_H \varphi(p^\varepsilon), (\nabla_H^2 \varphi)^\star(p_\varepsilon)\right) \geq 0.$$

Passing to the limit as $\varepsilon \to 0$, we apply the continuity of $F$ and the fact that $p_\varepsilon \to \hat{p}$ to obtain

$$F\left(\hat{p},\ \tilde{S}[u](\hat{p}),\ \nabla_H \varphi(\hat{p}),\ (\nabla_H^2 \varphi)^\star(\hat{p})\right) \geq 0,$$

as desired. □

*Remark* 5.2. The key to the proof of Theorem 5.1 lies at the relations (5.12) and (5.15). One may wonder whether one can use the same method to show $S[u]$ is a supersolution by replacing $\tilde{W}$ with $W$ given by

$$W(p_1, p_2, p_3) = \sum_i c_i g_i^2(p_1, p_2, p_3), \quad p_i \in \mathbb{H}, \text{ i=1, 2, 3},$$

where

$$g_i(p_1, p_2, p_3) = \left(\sum_i c_i y_i\right) x_i - \left(\sum_i c_i x_i\right) y_i + 2z_i - 2\sum_i c_i z_i$$

for $p_i = (x_i, y_i, z_i)$, $i = 1, 2, 3$. We however are not able to obtain the desired conclusion in this case. Note that the horizontal gradient with respect to the variable $p_i$ is calculated to be

$$\nabla_{H,i} W = \left(2c_i g_i \sum_i c_i y_i - 2c_i g_i y_i - 2c_i \sum_i c_i g_i y_i, -2c_i g_i \sum_i c_i x_i + 2c_i g_i x_i + 2c_i \sum_i c_i g_i x_i\right).$$

We therefore have

$$\sum_i \nabla_{H,i} W = 4\left(-\sum_i c_i g_i y_i,\ \sum_i c_i g_i x_i\right),$$

which fails to vanish in general. In other words, one cannot obtain (5.12) in this case. Similarly, (5.18) cannot be expected either.

*Remark* 5.3. It is implied by (A3) that $p \mapsto -F(p, r, \xi, A)$ is right invariant h-convex. This condition is necessary for the result in Theorem 5.1. In fact, in the trivial case when

$$F(p, r, \xi, A) = r - f(p)$$

for $(p, r, \xi, A) \in \mathbb{H} \times \mathbb{R} \times \mathbb{R}^2 \times \mathbf{S}^2$, $u = f$ is clearly the unique solution to this entirely degenerate equation. The assumption (A3) reduces to the right invariant h-convexity of $f$, which certainly implies that $\tilde{S}[u] = u = f$ in $\mathbb{H}$.

5.2. **The left invariant envelope with symmetry.** Under the symmetry of $u$ with respect to $z$-axis, Theorem 5.1 enables us to show that $\Gamma u$ is a supersolution of (1.6) if $u$ itself is a supersolution.

**Theorem 5.4** (Symmetric supersolution preserving)**.** *Assume that (A1)–(A3) hold. Let $u$ be a lower semicontinuous supersolution of* (1.6)*. Suppose that $u$ satisfies the coercivity condition* (1.10)*. Assume in addition that $u$ is symmetric with respect to $z$-axis. Let $S[u]$ be given by* (1.8)*. Then $S[u]$ is also a supersolution of* (1.6)*. Moreover, the convex envelope $\Gamma u$ is a supersolution of* (1.6) *as well.*

*Proof.* By Theorem 5.1, we see that $\tilde{S}[u]$ is a supersolution of (1.6). Due to the symmetry of $u$, by Theorem 4.11 we have $\tilde{S}[u] = S[u]$ in $\mathbb{H}$, which implies that $S[u]$ is also a supersolution. Moreover, $S[u]$ is also symmetric thanks to Lemma 4.10. In addition, by Lemma 4.4, $S[u]$ also satisfies the coercivity condition (1.10).

We thus can iterate the argument to show that $S^n[u]$ is a supersolution of (1.6) for any $n = 1, 2, \ldots$ Since $S^n[u] \to \Gamma u$ in $\mathbb{H}$ locally uniformly as $n \to \infty$ (see Remark 4.7), we conclude that $\Gamma u$ is a supersolution by the standard stability theory of viscosity solutions. □

The convexity preserving property of the h-convex envelope enables us to show the h-convexity of a symmetric solution of (1.6) provided that a comparison principle for (1.6) is available.

**Corollary 5.5** (H-convexity of symmetric solutions)**.** *Assume that (A1)–(A3) hold. Let $u$ be a continuous solution of* (1.6) *satisfying* (1.10)*. Assume in addition that $u$ is symmetric with respect to $z$-axis. If the comparison principle for* (1.6) *holds, then $u$ is h-convex in $\mathbb{H}$.*

*Proof.* Since $\Gamma u$ is a supersolution by Theorem 5.4, we apply the comparison principle to deduce that $u \leq \Gamma u$. Noticing that $\Gamma u \leq u$ by definition, we conclude that $u = \Gamma u$ in $\mathbb{H}$, which yields the h-convexity of $u$. □

In the proof above, it in fact suffices to show that $u = S[u]$ hold in $\mathbb{H}$ by using the same argument.

It is natural to ask when the solution $u$ is symmetric with respect to $z$-axis. It turns out to be sufficient to have the following two ingredients at hand.

- A comparison principle that allows coercive solutions as mentioned in Remark 2.3 is needed.
- The following symmetric assumption (A4) on $F$ is additionally imposed.

(A4) $(p, \xi) \mapsto F(p, r, \xi, A)$ is symmetric in the sense that

$$F(p, r, \xi, A) = F(p', r, -\xi, A)$$

for any $p = (x, y, z) \in \mathbb{H}$ with $p' = (-x, -y, z) \in \mathbb{H}$, $r \in \mathbb{R}$, $\xi \in \mathbb{R}^2$ and $A \in \mathbf{S}^2$.

**Proposition 5.6** (Symmetry of solutions)**.** *Assume that (A4) holds. If $u$ is a subsolution (resp., supersolution, solutions) of* (1.6)*, then $v(x, y, z) = u(-x, -y, z)$ is also a subsolution (resp., supersolution, solutions) of* (1.6)*. In particular, if $u$ is the unique solution of* (1.6)*, then $u$ is symmetric with respect to $z$-axis.*

*Proof.* Let us only verify the subsolution part. Suppose that there exist $p_0 = (x_0, y_0, z_0) \in \mathbb{H}$ and $\phi \in C^2(\mathbb{H})$ such that $v - \phi$ attains a maximum at $p_0$. We aim to show that

$$F(p_0, v(p_0), \nabla_H \phi(p_0), (\nabla_H^2 \phi)^\star(p_0)) \leq 0. \tag{5.21}$$

It is clear that $u - \psi$ attains a maximum at $p_0' = (-x_0, -y_0, z_0)$, where $\psi(x, y, z) = \phi(-x, -y, z)$. Since $u$ is a subsolution of (1.6), we have

$$F(p_0', u(p_0'), \nabla_H \psi(p_0'), (\nabla_H^2 \psi)^\star(p_0')) \leq 0. \tag{5.22}$$

In order to use (5.22) to show (5.21), we make the following straightforward calculations: for any $(x, y, z) \in \mathbb{H}$,

$$\begin{aligned}
(X\psi)(x, y, z) &= \left(-\frac{\partial \phi}{\partial x} - \frac{y}{2}\frac{\partial \phi}{\partial z}\right)(-x, -y, z) = -(X\phi)(-x, -y, z);\\
(Y\psi)(x, y, z) &= \left(-\frac{\partial \phi}{\partial y} + \frac{x}{2}\frac{\partial \phi}{\partial z}\right)(-x, -y, z) = -(Y\phi)(-x, -y, z);\\
(X^2\psi)(x, y, z) &= \left(\frac{\partial^2 \phi}{\partial x^2} + y\frac{\partial^2 \phi}{\partial x \partial z} + \frac{y^2}{4}\frac{\partial^2 \phi}{\partial z^2}\right)(-x, -y, z) = (X^2\phi)(-x, -y, z);\\
(Y^2\psi)(x, y, z) &= \left(\frac{\partial^2 \phi}{\partial y^2} - x\frac{\partial^2 \phi}{\partial y \partial z} + \frac{x^2}{4}\frac{\partial^2 \phi}{\partial z^2}\right)(-x, -y, z) = (Y^2\phi)(-x, -y, z);\\
\frac{1}{2}\left(XY\psi + YX\psi\right)(x, y, z) &= \left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{x}{2}\frac{\partial^2 \phi}{\partial x \partial z} + \frac{y}{2}\frac{\partial^2 \phi}{\partial y \partial z} - \frac{xy}{4}\frac{\partial^2 \phi}{\partial z^2}\right)(-x, -y, z)\\
&= \frac{1}{2}\left(XY\phi + YX\phi\right)(-x, -y, z).
\end{aligned}$$

It then follows that

$$\nabla_H \psi(p_0') = -\nabla_H \phi(p_0), \quad (\nabla_H^2 \psi)^\star(p_0') = (\nabla_H^2 \phi)^\star(p_0).$$

Plugging these into (5.22), we are led to

$$F(p_0', v(p_0), -\nabla_H \phi(p_0), (\nabla_H^2 \phi)^\star(p_0)) \leq 0.$$

By the symmetry condition (A4), we get (5.21) immediately. □

Theorem 5.4 and Proposition 5.6 thus imply the following.

**Theorem 5.7** (H-convexity of solutions)**.** *Assume that (A1)–(A4) hold. Assume that the comparison principle for* (1.6) *holds. Let $u$ be the unique continuous solution of* (1.6) *satisfying* (1.10)*. Let $S[u]$ be given by* (1.8)*. Then $S[u] = u$ in $\mathbb{H}$. In particular, $u$ is h-convex in $\mathbb{H}$.*

*Remark* 5.8. As pointed out in Proposition 4.12, under (A4), one can replace (5.1) in the concavity assumption (A3) by

$$p_i \in \mathbb{H}_{\sum_{i=1}^k c_i p_i}.$$

In other words, we can assume that $p \mapsto F(p, r, \xi, X)$ satifies h-concavity instead of the right invariant h-concavity.

5.3. **Examples in a special case.** Since a comparison theorem is available in [22] when $F$ is of the form (2.4), our convexity result in particular applies to the semilinear equation (1.14), as shown in Theorem 1.2. Note that the symmetry condition of $f$ enables us to assume h-convexity of $f$ rather than its right invariant h-convexity, since they are equivalent, as mentioned in Proposition 4.12 and Remark 5.8.

We here skip discussion on existence of solutions to (1.14) and the more general equation (1.6), since it can be obtained by Perron's method. But we provide a concrete example below in order to avoid possible triviality of our results.

*Example* 5.9. Consider (1.14) with $0 \leq \alpha < 1/3$, $\beta = 0$ and

$$f(x, y, z) = (1 - 3\alpha)(x^2 + y^2) + x^2y^2 + 2z^2 - 4\alpha$$

for $(x, y, z) \in \mathbb{H}$. It is clear that $f$ is coercive and symmetric with respect to $z$-axis. Moreover, $f$ is h-convex in $\mathbb{H}$, since

$$(\nabla_H^2 f)^\star(x, y, z) = \begin{pmatrix} 2(1 - 3\alpha) + 3y^2 & 3xy \\ 3xy & 2(1 - 3\alpha) + 3x^2 \end{pmatrix} \tag{5.23}$$

is nonnegative for all $(x, y, z) \in \mathbb{H}$. We remark that $f$ is not convex in the Euclidean sense.

By direct calculations, one can verify that the unique solution in this case is

$$u(x, y, z) = x^2 + y^2 + x^2y^2 + 2z^2.$$

for $(x, y, z) \in \mathbb{H}$. It is also easily seen that $u$ is h-convex in $\mathbb{H}$ (but not convex in $\mathbb{R}^3$), since it happens to coincide with $f$ with $\alpha = 0$.

Although in Theorem 5.7 $u$ is assumed to be coercive in the sense of (1.10), it is worth stressing that for (1.14) (with $\alpha \geq 0, \beta \geq 0$), no coercivity conditions on $u$ or on $f$ are essentially needed. The coercivity of $f$, together with its h-convexity, does assist one to show the coercivity of the solution $u$ by the comparison theorem; in fact, $u = f$ is clearly a subsolution of (1.14).

On the other hand, if $u$ is not coercive, one can replace $f$ by, for example,

$$f_\varepsilon(x, y, z) = f(x, y, z) + \varepsilon(x^2 + y^2 + z^2)$$

with $\varepsilon > 0$ small so that the corresponding solution $u^\varepsilon$ is also coercive. Since $f$ is still h-convex and symmetric, we can then apply Theorem 5.7 to deduce the h-convexity of $u^\varepsilon$. Letting $\varepsilon \to 0$, we have $u^\varepsilon \to u$ locally uniformly by adopting the standard stability theory of viscosity solutions. The h-convexity of $u$ follows immediately.

As Theorem 5.7 holds for (1.14) without coercivity assumptions on $u$, the following trivial example is thus covered.

*Example* 5.10. Pick $\alpha, \beta \geq 0$ arbitrarily. Let $f \equiv C$ in $\mathbb{H}$ for any $C \in \mathbb{R}$. It is then clear that $u = C$ is the unique solution of (1.14), which is obviously h-convex.

For the equation (1.14), (A4) is satisfied if $f$ is symmetric with respect to the $z$-axis. On the other hand, if the symmetry condition on $f$ is dropped, then in general the unique solution $u$ of (1.14) will not be symmetric and the h-convexity of $u$ may fail to hold, as shown in Example 1.1.

Besides the symmetry condition, we also have the following example suggesting that the concavity $\xi \mapsto F(p, r, \xi, A)$ seems to be necessary.

*Example* 5.11. Consider the equation

$$u + |\nabla_H u|^2 = f(p)$$

with $f(x, y, z) = (4\varepsilon^2 + 1)(x^2 + y^2) + 2z$ being h-convex and right invariant h-convex. However, it is easily seen that

$$u(x, y, z) = -\varepsilon x^2 - \varepsilon y^2 + 2z$$

is a solution but it is neither h-convex nor right invariant h-convex. The drawback of this example is that $u$ is not coercive. It would be interesting to construct better examples satisfying coercivity.

5.4. **The Euclidean envelope.** We conclude this part by providing a result on the Euclidean convexity of solutions to (1.6), although it is not our main concern in this paper. In this case, we need to strengthen the assumption (A3):

(A3)' $(p, r, \xi, A) \mapsto F(p, r, \xi, A)$ is concave in the Euclidean sense, namely, for any $k \in \mathbb{N}$,

$$\sum_{i=1}^{k} c_i F(p_i, r_i, \xi_i, A_i) \leq F\left(\sum_{i=1}^{k} c_i p_i,\ \sum_{i=1}^{k} c_i r_i,\ \sum_{i=1}^{k} c_i \xi_i,\ \sum_{i=1}^{k} c_i A_i\right)$$

holds for any $c_i \in [0, 1]$, $p_i \in \mathbb{H}$, $r_i \in \mathbb{R}$, $\xi_i \in \mathbb{R}^2$ and $A_i \in \mathbf{S}^2$ $(i = 1, 2, \ldots, k)$.

**Theorem 5.12** (Supersolution preserving of the Euclidean envelope)**.** *Assume that (A1), (A2) and (A3)' hold. Let $u$ be a lower semicontinuous supersolution of* (1.6)*. Suppose that $u$ satisfies the coercivity condition* (1.10)*. Let $\Gamma_E u$ be the Euclidean convex envelope of $u$ given by* (1.2) *with $n = 3$. Then $\Gamma_E u$ is also a supersolution of* (1.6)*. Moreover, if the comparison principle for* (1.6) *holds, then the unique solution of* (1.6) *is convex in $\mathbb{R}^3$ (in the Euclidean sense).*

*Proof.* The proof is based on slight modification of that of Theorem 5.1. Since we consider the Euclidean convex combination, we can simply set the penalty term $\tilde{W} \equiv 0$ in the proof of Theorem 5.1 and the whole argument still works. Indeed, if we let $\varphi \in C^2(\mathbb{H})$ denote the test function of $\Gamma_E u$ at a point $p_0 \in \mathbb{H}$ and let $(\hat{p}_1, \hat{p}_2, \hat{p}_3)$ denote the minimizer of

$$(p_1, p_2, p_3) \mapsto \sum_{i=1,2,3} c_i u(p_i) - \varphi\left(\sum_i c_i p_i\right),$$

then for any $\sigma > 0$ we have $(\eta_i, Q_i) \in \overline{J}^{2,-} u(p_i)$ such that

$$\eta_i = \nabla\varphi(p_0)$$

and

$$\begin{pmatrix} c_1 Q_1 & 0 & 0 \\ 0 & c_2 Q_2 & 0 \\ 0 & 0 & c_3 Q_3 \end{pmatrix} \geq \begin{pmatrix} c_1^2 Q_0 & c_1 c_2 Q_0 & c_1 c_3 Q_0 \\ c_1 c_2 Q_0 & c_2^2 Q_0 & c_2 c_3 Q_0 \\ c_1 c_3 Q_0 & c_2 c_3 Q_0 & c_3^2 Q_0 \end{pmatrix} - \sigma \begin{pmatrix} c_1^2 Q_0 & c_1 c_2 Q_0 & c_1 c_3 Q_0 \\ c_1 c_2 Q_0 & c_2^2 Q_0 & c_2 c_3 Q_0 \\ c_1 c_3 Q_0 & c_2 c_3 Q_0 & c_3^2 Q_0 \end{pmatrix}^2,$$

where $Q_0 = \nabla^2\varphi(p_0)$. We next turn to the horizontal jets $(\xi_i, P_i) \in \overline{J}_H^{2,-} u(p_i)$. By the calculations similar to the proof of Theorem 5.1, we get the following variants of (5.12) and (5.18):

$$\sum_i c_i\xi_i = \nabla_H\varphi(p_0);$$

$$\sum_i c_i P_i \geq (\nabla_H^2\varphi)^\star(p_0) - O(\sigma).$$

Applying the supersolution property of $u$ at $\hat{p}_i$ and taking the weighted average, we obtain

$$\sum_i c_i F\left(\hat{p}_i, u(\hat{p}_i), \xi_i, P_i\right) \geq 0,$$

which, by (A3)' and the ellipticity of $F$, yields

$$F\left(p_0, \Gamma_E u(p_0), \nabla_H\varphi(p_0), (\nabla_H^2\varphi)^\star(p_0) - O(\sigma)\right) \geq 0.$$

We complete the proof by letting $\sigma \to 0$.

If the comparison principle for (1.6) holds, then the unique solution $u$ satisfies $u \leq \Gamma_E u$. Since the reverse inequality clearly holds, the convexity of $u$ follows immediately. □

This result is closely related to the vast existing literature on the convexity of solutions to a large variety of elliptic and parabolic equations in the Euclidean space; see [28, 25, 26, 15, 18, 1, 23, 30] etc. In fact, Theorem 5.12 gives a clearer and more direct answer for equations like (1.6).

As the Euclidean convexity implies h-convexity, Theorem 5.12 helps us understand h-convexity of solutions to a large class of nonlinear elliptic equations including (1.14). A trivial example for Theorem 5.12 is in Example 5.10. We give a less trivial one below.

*Example* 5.13. Let us revisit (1.14) with $\alpha = 1$, $\beta = 0$ and

$$f(x, y, z) = x + 2z^2 + \varepsilon(x^2 + y^2)$$

for $\varepsilon > 0$ and $(x, y, z) \in \mathbb{H}$. We are unable to apply Theorem 5.7 to obtain h-convexity of the unique solution $u$, since $f$ is not symmetric with respect to $z$-axis and therefore (A4) fails to hold. However, we can use Theorem 5.12 to conclude that $u$ is convex in the Euclidean sense. In fact, the unique solution in this case can be explicitly written as

$$u(x, y, z) = (1 + \varepsilon)(x^2 + y^2 + 4) + x + 2z^2$$

for $(x, y, z) \in \mathbb{H}$.

On the other hand, when $f$ is not convex but only h-convex like Example 5.9, we can only show the h-convexity of $u$ under the symmetry condition of $f$.

It is worth stressing that even if we want prove the Euclidean convexity of solutions to sub-elliptic PDEs, we probably still need the strong concavity of $F$ in all arguments, as shown in Example 5.11.

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Qing Liu, Department of Applied Mathematics, Fukuoka University, Fukuoka 814-0180, Japan, qingliu@fukuoka-u.ac.jp

Xiaodan Zhou, Analysis on Metric Spaces Unit, Okinawa Institute of Science and Technology Graduate University, Okinawa 904-0495, Japan, xiaodan.zhou@oist.jp