



Endpoint L^1 estimates for Hodge systems

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Abstract

Let $d \geq 2$. In this paper we give a simple proof of the endpoint Besov-Lorentz estimate

$$\|I_\alpha F\|_{\dot{B}_{d/(d-\alpha),1}^{0,1}(\mathbb{R}^d;\mathbb{R}^k)} \leq C \|F\|_{L^1(\mathbb{R}^d;\mathbb{R}^k)}$$

for all $F \in L^1(\mathbb{R}^d; \mathbb{R}^k)$ which satisfy a first order cocancelling differential constraint, where $\alpha \in (0, d)$ and I_α is a Riesz potential. We show how this implies endpoint Besov–Lorentz estimates for Hodge systems with L^1 data via fractional integration for exterior derivatives.

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1 Introduction

In the L^1 theory for linear elliptic systems it is quite difficult to obtain better than weak-type bounds. A program in this direction was pioneered in the seminal work of J. Bourgain and H. Brezis [5] (see also [6, 29]) and received remarkable contributions from Lanzani and Stein [14] and Van Schaftingen [30–32], while endpoint fine parameter improvements on the Lorentz [10, 25] and Besov-Lorentz [28] scales have only recently been obtained.

The purpose of this paper is to give a simple proof of the Besov-Lorentz estimates obtained in [28] for a restricted class of operators and to show how this estimate can be used to resolve several open questions in the theory, in particular estimates for Hodge systems [31, Open Problems 1 & 2] and the endpoint extension of [32, Propositions 8.8 & 8.10] in the case of first order operators. Our starting place is an estimate the first and third named authors proved in [10], that for $d \geq 2$ and $\alpha \in (0, d)$ there exists a constant $C > 0$ for which one has the inequality

$$\|I_\alpha F\|_{L^{d/(d-\alpha),1}(\mathbb{R}^d;\mathbb{R}^d)} \leq C \|F\|_{L^1(\mathbb{R}^d;\mathbb{R}^d)} \tag{1.1}$$

for all $F \in L^1(\mathbb{R}^d; \mathbb{R}^d)$ such that $\operatorname{div} F = 0$. Here $L^{d/(d-\alpha),1}(\mathbb{R}^d; \mathbb{R}^d)$ is a Lorentz space (see Sect. 2 for a precise definition) and I_α denotes the Riesz potential of order $\alpha \in (0, d)$, defined for $F \in L^1(\mathbb{R}^d; \mathbb{R}^k)$ by

$$I_\alpha F(x) := \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty t^{\alpha/2-1} p_t * F(x) dt \equiv \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^d} \frac{F(y)}{|x-y|^{d-\alpha}} dy, \tag{1.2}$$

where $p_t(x) := (4\pi t)^{-d/2} \exp(-|x|^2/4t)$ is the heat kernel in \mathbb{R}^d and

$$\gamma(\alpha) = \frac{\pi^{d/2} 2^\alpha \Gamma(\alpha/2)}{\Gamma\left(\frac{d}{2} - \frac{\alpha}{2}\right)}$$

is a normalization constant (see, e.g. [26, p. 117]).

The estimate (1.1) is a partial replacement for the failure of the Hardy–Littlewood–Sobolev embedding in the L^1 endpoint, cf. [26, p. 119], while a comprehensive resolution of the question of a replacement has been given by D. Stolyarov, who in [28] (see also [3, Conjecture 2] where such an inequality was conjectured to hold) establishes the sharper inequality

$$\|I_\alpha F\|_{\dot{B}_{d/(d-\alpha),1}^{0,1}(\mathbb{R}^d;\mathbb{R}^k)} \leq C \|F\|_{L^1(\mathbb{R}^d;\mathbb{R}^k)} \tag{1.3}$$

for a very general class of subspaces of $L^1(\mathbb{R}^d; \mathbb{R}^k)$ that includes the kernels of J. Van Schaftingen’s class of cocancelling operators [32] (see Definition 2.5 below where we recall this class). The argument in [28] is quite involved, and it is there commented by Stolyarov that whether the inequality (1.3) admits a simpler proof if one only seeks its validity for the more restrictive class of divergence free measures is unknown. We

will shortly give such a proof, which benefited from several insights from his paper and the series of lectures¹ he gave on the topic.

To this end, let us recall the approach to (1.1) in [10]: For the space of divergence free measures one finds appropriate atoms, one demonstrates the sufficiency of an estimate on an atom, and one establishes the estimate for a single atom. The atoms in this case are oriented piecewise- C^1 loops which satisfy the uniform ball-growth condition: To any oriented piecewise- C^1 loop $\Gamma \subset \mathbb{R}^d$ one associates the measure which is given by integration along the curve

$$\int \Phi \cdot d\mu_\Gamma := \int_0^{|\Gamma|} \Phi(\gamma(t)) \cdot \dot{\gamma}(t) dt, \tag{1.4}$$

for $\Phi \in C_0(\mathbb{R}^d; \mathbb{R}^d)$, where $\gamma : [0, |\Gamma|] \rightarrow \mathbb{R}^d$ is the parametrization of Γ by arclength. The atoms are then such piecewise- C^1 closed curves for which

$$\|\mu_\Gamma\|_{\mathcal{M}^1(\mathbb{R}^d)} := \sup_{x \in \mathbb{R}^d, r > 0} \frac{\|\mu_\Gamma\|(B(x, r))}{r} \leq \bar{C} \tag{1.5}$$

for some universal constant $\bar{C} > 0$, where $\|\mu_\Gamma\|$ is the total variation measure of μ_Γ . The sufficiency of an estimate on these atoms follows in two steps. First, by Smirnov’s integral decomposition of divergence free measures [21] one has an approximation of such objects in the strict topology by convex combinations of oriented C^1 closed loops. Second, a surgery on such loops shows how any oriented C^1 closed loop Γ admits a further decomposition into oriented piecewise- C^1 closed loops $\{\Gamma_i\}_{i=1}^{N(\Gamma)}$ which satisfy (1.5) with some universal constant and whose total length is bounded by a constant times the length of this loop. This approximation/decomposition and the triangle inequality then yields that it suffices to prove the estimate for a single loop which satisfies the ball growth condition (1.5). Finally, the estimate (1.1) for a single loop was argued in [10] by a hands on interpolation that utilizes several pointwise estimates for Riesz potentials and bounds for various maximal functions.

While the argument of (1.1) in [10] for a single loop with a ball growth condition involves only estimates for various maximal functions, in this paper we observe that it can be further simplified by the consideration of a very natural *stronger* quantity that arises in Stolyarov’s estimates:

$$\int_0^\infty t^{\alpha/2-1} \|p_t * F\|_{L^{d/(d-\alpha), 1}(\mathbb{R}^d; \mathbb{R}^k)} dt. \tag{1.6}$$

In particular, in [28], Stolyarov shows how if one controls a discrete analogue of (1.6) this implies (1.3). As we will see below in Sect. 2, the continuous version (1.6) also controls the Besov-Lorentz norm and therefore, taking into account the reduction to atoms established in [10], for the demonstration of the Besov–Lorentz inequality for divergence free functions it suffices to prove the inequality

¹ We are indebted to D. Stolyarov for the efforts he put into giving these lectures, which can be found at “<https://vimeo.com/497090776>”.

$$\int_0^\infty t^{\alpha/2-1} \|p_t * \mu_\Gamma\|_{L^{d/(d-\alpha),1}(\mathbb{R}^d;\mathbb{R}^d)} dt \leq C \|\mu_\Gamma\|(\mathbb{R}^d) \tag{1.7}$$

for all oriented piecewise- C^1 closed loops Γ which satisfy (1.5). Let us remark that it is not difficult to see that (1.6) controls the Lorentz norm of the Riesz potential of a function, since this follows directly from the representation (1.2) and Minkowski's inequality for integrals. The argument for the Besov-Lorentz case is only slightly more complicated because of the more technical definition of the space.

We therefore proceed to argue the validity of the inequality (1.7). We claim this follows easily from the estimates

$$\|p_t * \mu_\Gamma\|_{L^1(\mathbb{R}^d;\mathbb{R}^d)} \leq \|p_t\|_{L^1(\mathbb{R}^d)} \|\mu_\Gamma\|(\mathbb{R}^d) = \|\mu_\Gamma\|(\mathbb{R}^d), \tag{1.8}$$

$$\|p_t * \mu_\Gamma\|_{L^\infty(\mathbb{R}^d;\mathbb{R}^d)} \leq \|p_t\|_{L^\infty(\mathbb{R}^d)} \|\mu_\Gamma\|(\mathbb{R}^d) = \frac{c}{t^{d/2}} \|\mu_\Gamma\|(\mathbb{R}^d), \tag{1.9}$$

$$\|p_t * \mu_\Gamma\|_{L^1(\mathbb{R}^d;\mathbb{R}^d)} \leq C_1 \frac{\|\mu_\Gamma\|(\mathbb{R}^d)^2}{t^{1/2}} \tag{1.10}$$

$$\|p_t * \mu_\Gamma\|_{L^\infty(\mathbb{R}^d;\mathbb{R}^d)} \leq \frac{C_2}{t^{(d-1)/2}} \|\mu_\Gamma\|_{\mathcal{M}^1(\mathbb{R}^d)}. \tag{1.11}$$

The former two are standard (linear) convolution inequalities for L^1 functions, while the latter two are *nonlinear* and only hold because we consider closed loops oriented by their tangent. Indeed, (1.10) follows from the fact such objects admit a generalized minimal surface spanning Γ , while (1.11) utilizes the fact that we work with curves (and we later make use of the fact that they satisfy (1.5)). Note that if we only utilized (1.8) and (1.9) it would not be sufficient for our purposes, since for any $1 \leq p \leq +\infty$ interpolation would yield the estimate

$$\|p_t * \mu_\Gamma\|_{L^{p,1}(\mathbb{R}^d;\mathbb{R}^d)} \leq \|p_t * \mu_\Gamma\|_{L^1(\mathbb{R}^d;\mathbb{R}^d)}^\theta \|p_t * \mu_\Gamma\|_{L^\infty(\mathbb{R}^d;\mathbb{R}^d)}^{1-\theta},$$

where $\theta = 1/p$. In particular, when $p = \frac{d}{d-\alpha}$, using (1.8) and (1.9) we find

$$\|p_t * \mu_\Gamma\|_{L^{d/(d-\alpha),1}(\mathbb{R}^d;\mathbb{R}^d)} \leq C \frac{\|\mu_\Gamma\|(\mathbb{R}^d)}{t^{\alpha/2}},$$

which is not good enough to get a finite upper bound, as if utilized to estimate the quantity (1.6) gives a logarithmic divergence at both zero and infinity and therefore cannot yield the inequality (1.7).

The subtlety is to notice that the combination of (1.8) and (1.11) gives an estimate with slightly better behavior at zero, while the combination of (1.9) and (1.10) gives an estimate with slightly better behavior at infinity, the inequalities

$$\|p_t * \mu_\Gamma\|_{L^{d/(d-\alpha),1}(\mathbb{R}^d;\mathbb{R}^d)} \leq C'_1 \frac{\|\mu_\Gamma\|(\mathbb{R}^d)^{(d-\alpha)/d}}{t^{\alpha(d-1)/2d}}, \tag{1.12}$$

$$\|p_t * \mu_\Gamma\|_{L^{d/(d-\alpha),1}(\mathbb{R}^d;\mathbb{R}^d)} \leq C'_2 \frac{\|\mu_\Gamma\|(\mathbb{R}^d)^{1+(d-\alpha)/d}}{t^{\alpha/2+(d-\alpha)/2d}}. \tag{1.13}$$

Indeed,

$$\begin{aligned} \alpha(d - 1)/2d &< \alpha/2, \\ \alpha/2 + (d - \alpha)/2d &> \alpha/2, \end{aligned}$$

and therefore it remains to divide the integral so as to linearize the estimate, which follows from dividing at $|\Gamma|^2$ (alternatively, one may first reduce to the case $|\Gamma| = 1$ by dilation, though we here avoided this argument because the nonlinearity of the estimates (1.10) and (1.11) becomes less clear).

We postpone further details until Sect. 2, including the proof of the slightly more technical Besov-Lorentz inequality, so that we can continue to a second purpose of this paper, which is to catalog some implications of the inequality (1.3) in the divergence free case. Indeed, a fundamental contribution of J. Van Schaftingen’s paper [32] is that divergence free vector fields are generic in the class of vector fields which admit a first order cocanceling annihilator. In particular, following his argument we establish

Theorem 1.1 *Let $d \geq 2$, $\alpha \in (0, d)$, and suppose $L(D)$ is a first order homogeneous linear partial differential operator acting on vector fields $F: \mathbb{R}^d \rightarrow \mathbb{R}^k$. Then the estimate*

$$\|I_\alpha F\|_{\dot{B}_{d/(d-\alpha),1}^{0,1}(\mathbb{R}^d;\mathbb{R}^k)} \leq C\|F\|_{L^1(\mathbb{R}^d;\mathbb{R}^k)} \text{ for } L(D)F = 0$$

holds if and only if $L(D)$ is cocanceling, see Definition 2.5.

We recall that the cocanceling assumption is *very* mild: As was observed in [9, 18, 32], failure of this assumption is equivalent to the existence of an unconstrained subspace of $L(D)$ -free fields.

Beyond an intrinsic interest in the mapping properties of fractional integrals, the inequality given in Theorem 1.1 has implications for PDEs. For example, in [10] it was demonstrated how (1.1) implies a Lorentz space sharpening of an estimate of Bourgain and Brezis [4, 5]: If $F \in L^1(\mathbb{R}^3; \mathbb{R}^3)$ is divergence free, the solution of the Div-Curl system

$$\begin{aligned} \operatorname{curl} Z &= F \\ \operatorname{div} Z &= 0 \end{aligned}$$

admits the estimate

$$\|Z\|_{L^{3/2,1}(\mathbb{R}^3;\mathbb{R}^3)} \leq C\|F\|_{L^1(\mathbb{R}^3;\mathbb{R}^3)}$$

for some $C > 0$.

Theorem 1.1 of course implies a similar improvement to this inequality, though in this form is useful for more general applications. For example, we immediately obtain

Corollary 1.2 *Let $d \geq 2$, $\alpha \in (0, d)$, and $k \in \mathbb{N} \cap [0, d]$. There exists a constant $C = C(\alpha, d) > 0$ such that for $k \leq d - 2$*

$$\|I_\alpha du\|_{\dot{B}_{d/(d-\alpha),1}^{0,1}(\mathbb{R}^d; \Lambda^{k+1}\mathbb{R}^d)} \leq C \|du\|_{L^1(\mathbb{R}^d; \Lambda^{k+1}\mathbb{R}^d)}, \tag{1.14}$$

while for $k \geq 2$

$$\|I_\alpha d^*u\|_{\dot{B}_{d/(d-\alpha),1}^{0,1}(\mathbb{R}^d; \Lambda^{k-1}\mathbb{R}^d)} \leq C \|d^*u\|_{L^1(\mathbb{R}^d; \Lambda^{k-1}\mathbb{R}^d)}, \tag{1.15}$$

for all $u \in C_c^\infty(\mathbb{R}^d; \Lambda^k\mathbb{R}^d)$.

Here, for $k \in \mathbb{N} \cap [0, d]$, $\Lambda^k\mathbb{R}^d$ denotes the vector space of k -forms, $C^\infty(\mathbb{R}^d; \Lambda^k\mathbb{R}^d)$ denotes the space of functions from \mathbb{R}^d to the space of k -forms with smooth coefficients,

$$\begin{aligned} d &: C^\infty(\mathbb{R}^d; \Lambda^k\mathbb{R}^d) \rightarrow C^\infty(\mathbb{R}^d; \Lambda^{k+1}\mathbb{R}^d) \\ d^* &: C^\infty(\mathbb{R}^d; \Lambda^k\mathbb{R}^d) \rightarrow C^\infty(\mathbb{R}^d; \Lambda^{k-1}\mathbb{R}^d) \end{aligned}$$

are the exterior differential and exterior co-differential, respectively, and, with an overloading of notation, $I_\alpha du, I_\alpha d^*u$ denote the Riesz potential acting on the $k + 1, k - 1$ -forms du, d^*u . Precisely, for any l -form $Y \in L^1(\mathbb{R}^d; \Lambda^l\mathbb{R}^d)$, one can express Y in global coordinates as

$$Y = \sum_{|I|=l} Y_I dx_I$$

where $Y_I \in L^1(\mathbb{R}^d; \mathbb{R})$, cf. [11, p. 237]. Then the Riesz potential of such a Y is given by the formula

$$I_\alpha Y = \sum_{|I|=l} I_\alpha Y_I dx_I, \tag{1.16}$$

where $I_\alpha Y_I$ is as defined in (1.2). From this one sees $I_\alpha Y$ is well-defined for $Y \in L^1(\mathbb{R}^d; \Lambda^l\mathbb{R}^d)$ for any $l = 1, \dots, d$, and in particular that all such Y are in the domain of $I_2 \equiv (-\Delta)^{-1}$ for $d \geq 3$.

From Corollary 1.2 one not only obtains improvements to the left-hand-side of estimates for the Div-Curl system, but more generally the Hodge systems considered by Bourgain and Brezis in their paper [5] (see also Lanzani and Stein [14] for a slicing argument in the spirit of Van Schaftingen’s simplification [32] of the original argument of Bourgain and Brezis). In particular, we give an affirmative answer to [31, Open Problems 1 & 2], the following

Theorem 1.3 *Let $d \geq 3$ and $k \in \mathbb{N} \cap [1, d - 1]$. If $F \in L^1(\mathbb{R}^d; \Lambda^{k-1}\mathbb{R}^d)$ and $G \in L^1(\mathbb{R}^d; \Lambda^{k+1}\mathbb{R}^d)$ satisfy the compatibility conditions*

$$d^*F = dG = 0,$$

then the function $Z = d(-\Delta)^{-1}F + d^*(-\Delta)^{-1}G$ satisfies

$$\begin{aligned} d^*Z &= F, \\ dZ &= G, \end{aligned}$$

and there exists a constant $C > 0$ such that

$$\|Z\|_{\dot{B}_{d/(d-1),1}^{0,1}(\mathbb{R}^d; \Lambda^k \mathbb{R}^d)} \leq C \left(\|F\|_{L^1(\mathbb{R}^d; \Lambda^{k-1} \mathbb{R}^d)} + \|G\|_{L^1(\mathbb{R}^d; \Lambda^{k+1} \mathbb{R}^d)} \right),$$

where we additionally require $F \equiv 0$ in the case $k = 1$ or $G \equiv 0$ in the case $k = d - 1$.

Note that the conditions $d^*F = dG = 0$ are necessitated by properties of the exterior differential and exterior co-differential, $d \circ d = 0$ and $d^* \circ d^* = 0$, while the fact that the expression $Z = d(-\Delta)^{-1}F + d^*(-\Delta)^{-1}G$ is well-defined follows from the assumptions $F \in L^1(\mathbb{R}^d; \Lambda^{k-1} \mathbb{R}^d)$, $G \in L^1(\mathbb{R}^d; \Lambda^{k+1} \mathbb{R}^d)$ and the formula (1.16).

Finally, let us record the following duality estimates, which extend [32, Propositions 8.8 & 8.10] to the endpoint $q = \infty$.

Proposition 1.4 *Let $d \geq 2$, $\alpha \in (0, d)$, and suppose $L(D)$ is a first order cocanceling operator on \mathbb{R}^d . Then the estimates for vector fields*

$$\begin{aligned} \int_{\mathbb{R}^d} F \cdot \varphi \, dx &\leq C \|F\|_{L^1(\mathbb{R}^d; \mathbb{R}^k)} \|D\varphi\|_{L^{d,\infty}(\mathbb{R}^d; \mathbb{R}^{k \times d})}, \\ \int_{\mathbb{R}^d} F \cdot \varphi \, dx &\leq C \|F\|_{L^1(\mathbb{R}^d; \mathbb{R}^k)} \|\varphi\|_{\dot{B}_{d/\alpha}^{\alpha,\infty}(\mathbb{R}^d; \mathbb{R}^k)} \end{aligned}$$

hold if $L(D)F = 0$.

Here we use a slightly unusual notation for the Besov spaces $\dot{B}_p^{\alpha,q}$, which is consistent with our earlier notation. In other words, $\dot{B}_p^{\alpha,q} = \dot{B}_{p,p}^{\alpha,q}$.

The plan of the paper is as follows. In Sect. 2, we first recall the definition of the Lorentz spaces and several results concerning them before we prove Theorem 1.1. In Sect. 3 we prove Corollary 1.2, Theorem 1.3, and Proposition 1.4. In Sect. 4 we address an implicit claim in [23] that the estimate in the curl free case was optimal on the Lorentz scale. In particular, we here give a proof of this claim, which in turn, by J. Van Schaftingen’s argument implies optimality of the result of the first and third named authors in [10] on the Lorentz scale. It is likely these results are optimal on the Besov-Lorentz scale, though we do not have an example which confirms this. Finally, in Sect. 5, we give direct proofs of several of the results for $F \in L^1(\mathbb{R}^d; \mathbb{R}^d)$ such that $\text{curl } F = 0$. Of course, this is not as general as the divergence free setting, though notably it does not require the surgery construction from [10] and therefore provides a streamlined proof for the Lorentz inequality that does not require anything beyond the coarea formula and basic interpolation of Lebesgue or Lorentz spaces.

2 Lorentz and Besov–Lorentz estimates

We begin by recalling some results concerning the Lorentz spaces $L^{q,r}(\mathbb{R}^d)$, where we follow the development of R. O’Neil in [15].

Definition 2.1 For f a measurable function on \mathbb{R}^d , we define

$$m(f, y) := |\{|f| > y\}|.$$

As this is a non-increasing function of y , it admits a left-continuous inverse, called the non-negative rearrangement of f , and which we denote $f^*(x)$. Further, for $x > 0$ we define

$$f^{**}(x) := \frac{1}{x} \int_0^x f^*(t) dt.$$

We can now give a definition of the Lorentz spaces $L^{q,r}(\mathbb{R}^d)$.

Definition 2.2 Let $1 < q < +\infty$ and $1 \leq r < +\infty$. We define

$$\|f\|_{L^{q,r}(\mathbb{R}^d)} := \left(\int_0^\infty \left[t^{1/q} f^{**}(t) \right]^r \frac{dt}{t} \right)^{1/r},$$

and for $1 \leq q \leq +\infty$ and $r = +\infty$

$$\|f\|_{L^{q,\infty}(\mathbb{R}^d)} := \sup_{t>0} t^{1/q} f^{**}(t).$$

The Lorentz space $L^{q,r}(\mathbb{R}^d)$ is defined as

$$L^{q,r}(\mathbb{R}^d) := \{ f \text{ measurable} : \|f\|_{L^{q,r}(\mathbb{R}^d)} < +\infty \}.$$

For such parameters q, r , these functionals can be shown to be norms and the associated spaces $L^{q,r}(\mathbb{R}^d)$ Banach spaces (see, e.g., [27,Chapter V]). Concerning estimates involving the norm for functions in these spaces, a simpler quantity for our purposes is a quasi-norm which does not involve rearrangements:

$$\| \|f\| \|_{L^{q,r}(\mathbb{R}^d)} \equiv q^{1/r} \left(\int_0^\infty \left(t |\{|f| > t\}|^{1/q} \right)^r \frac{dt}{t} \right)^{1/r}. \tag{2.1}$$

In particular, one can show this is equivalent to the norm on $\|f\|_{L^{q,r}(\mathbb{R}^d)}$ (see, e.g. [27,Theorem 3.21 on p. 204]):

Proposition 2.3 Let $1 < q < +\infty$ and $1 \leq r \leq +\infty$. Then

$$\| \|f\| \|_{L^{q,r}(\mathbb{R}^d)} \leq \|f\|_{L^{q,r}(\mathbb{R}^d)} \leq q' \| \|f\| \|_{L^{q,r}(\mathbb{R}^d)}.$$

Finally, we recall that these spaces support an analogue of Young’s convolution inequality, see [15, Theorem 3.1]:

Theorem 2.4 *Let $f \in L^{q_1, r_1}(\mathbb{R}^d)$ and $g \in L^{q_2, r_2}(\mathbb{R}^d)$, and suppose $1 < q < +\infty$ and $1 \leq r \leq +\infty$ satisfy*

$$\begin{aligned} \frac{1}{q_1} + \frac{1}{q_2} - 1 &= \frac{1}{q} \\ \frac{1}{r_1} + \frac{1}{r_2} &\geq \frac{1}{r}. \end{aligned}$$

Then

$$\|f * g\|_{L^{q, r}(\mathbb{R}^d)} \leq 3q \|f\|_{L^{q_1, r_1}(\mathbb{R}^d)} \|g\|_{L^{q_2, r_2}(\mathbb{R}^d)}.$$

We next give a few more details of the argument of Theorem 1.1 in the base case $L(D) = \text{div}$. Toward the inequality (1.1), as discussed in the introduction, the reduction argument given in [10] implies that it suffices to prove (1.7) for every oriented piecewise C^1 closed loop Γ that satisfies (1.5). This inequality, in turn, will follow if we can establish the convolution inequalities (1.8), (1.9), (1.10), and (1.11).

For such curves, (1.8) and (1.9) follow from standard convolution inequalities, while we now explain in more detail the inequalities (1.10) and (1.11). The inequality (1.10) follows from the fact that in Euclidean space oriented piecewise C^1 closed loops can be identified with integral currents which admit spanning surfaces. In particular, by [7.4.2.10], given $T = \mu_\Gamma$, there exists a (generalized) surface S which satisfies (in a generalized sense)

$$\partial S = T \tag{2.2}$$

$$\|S\|(\mathbb{R}^d) \leq c \|T\|(\mathbb{R}^d)^2. \tag{2.3}$$

From this, one easily argues the estimate (1.10) by the computation

$$\begin{aligned} \|p_t * \mu_\Gamma\|_{L^1(\mathbb{R}^d; \mathbb{R}^d)} &= \|p_t * T\|_{L^1(\mathbb{R}^d)} \\ &= \|p_t * \partial S\|_{L^1(\mathbb{R}^d)} \\ &\leq \frac{\|t^{1/2} |\nabla p_t|\|_{L^1(\mathbb{R}^d)} \|S\|(\mathbb{R}^d)}{t^{1/2}}, \end{aligned}$$

the identity

$$\|t^{1/2} |\nabla p_t|\|_{L^1(\mathbb{R}^d)} = c'$$

and the isoperimetric inequality (2.3).

Concerning the estimate (1.11), it can be argued even simpler than the $\mathcal{H}^1 - BMO$ duality utilized to estimate an analogous quantity in [10], as it follows from a simple expansion of the convolution on dyadic annuli, using (1.5): In particular,

$$\begin{aligned}
 |p_t * \mu_\Gamma(x)| &\leq \int_{\mathbb{R}^d} p_t(x-y) d\|\mu_\Gamma\|(y) \\
 &= \sum_{n \in \mathbb{Z}} \int_{B(x, 2^n \sqrt{t}) \setminus B(x, 2^{n-1} \sqrt{t})} p_t(x-y) d\|\mu_\Gamma\|(y) \\
 &\leq \sum_{n \in \mathbb{Z}} \frac{1}{(4\pi t)^{d/2}} \exp(-2^{2n-2}/4) \int_{B(x, 2^n \sqrt{t}) \setminus B(x, 2^{n-1} \sqrt{t})} d\|\mu_\Gamma\|(y) \\
 &\leq \sum_{n \in \mathbb{Z}} \frac{1}{(4\pi t)^{d/2}} \exp(-2^{2n-2}/4) \|\mu_\Gamma\|(B(x, 2^n \sqrt{t})) \\
 &\leq \sum_{n \in \mathbb{Z}} \frac{C}{(4\pi t)^{d/2}} 2^n \sqrt{t} \exp(-2^{2n-2}/4) \\
 &= \frac{C'}{t^{(d-1)/2}},
 \end{aligned}$$

where C is as in (1.5) and

$$C' := \sum_{n \in \mathbb{Z}} \frac{C}{(4\pi)^{d/2}} 2^n \exp(-2^{2n-2}/4).$$

This and the argument of the introduction completes the proof of the Lorentz inequality in the case $L(D) = \text{div}$.

Concerning the Besov-Lorentz inequality, we follow the work of Stolyarov [28] with a definition of the space $\dot{B}_{d/(d-\alpha), 1}^{0,1}(\mathbb{R}^d; \mathbb{R}^k)$ through a minor modification of that for Besov spaces:

$$\|F\|_{\dot{B}_{d/(d-\alpha), 1}^{0,1}(\mathbb{R}^d; \mathbb{R}^k)} := \sum_{n \in \mathbb{Z}} \|F * (\psi_{2^{n+1}} - \psi_{2^n})\|_{L^{d/(d-\alpha), 1}(\mathbb{R}^d; \mathbb{R}^k)},$$

where

$$\psi_r(x) = r^d \psi(rx) \tag{2.4}$$

are dilates of some function $\psi \in \mathcal{S}(\mathbb{R}^d)$ which satisfies

$$\begin{aligned}
 \text{supp } \widehat{\psi} &\subset B(0, 1), \\
 \widehat{\psi} &\equiv 1 \text{ on } B(0, 1/2).
 \end{aligned}$$

One familiar with Besov spaces [1, Definition 4.1.2] observes that Besov-Lorentz spaces are defined analogously, only with the replacement of Lebesgue norms in the definition with Lorentz norms. These spaces arise in the real interpolation of Besov spaces [16], and have also been called Lorentz-Besov spaces in the monograph of J. Peetre [17] (see Example 6 on p. 57 as well as p. 106, 232). While these references are classical, a systematic treatment of these spaces as well various relationships with Triebel-Lizorkin analogues seems to be a recent development: In [20, equation (1)

on p. 1018], A. Seeger and W. Trebels define an inhomogeneous version denoted by $B_1^0[L^{d/(d-\alpha),1}]$, while it is remarked in the comments at the end of the introduction there that the results in the paper hold for their homogeneous counterpart, which is denoted by $\dot{B}_1^0[L^{d/(d-\alpha),1}]$. One can check that the only difference between the definition in [20] and ours is the choice of Littlewood-Paley decomposition, provided one utilizes the same norm on the Lorentz space $L^{d/(d-\alpha),1}(\mathbb{R}^d; \mathbb{R}^k)$.

When presented with the Lorentz embedding proved in [10] and the Besov-Lorentz embedding proved here, a natural question is whether one can deduce one from the other. From the results in [20] one understands that the latter is indeed stronger: First, one has the embedding

$$\dot{B}_{d/(d-\alpha),1}^{0,1}(\mathbb{R}^d; \mathbb{R}^k) \hookrightarrow L^{d/(d-\alpha),1}(\mathbb{R}^d; \mathbb{R}^k). \tag{2.5}$$

Indeed, with $s = 0$ in equation (3) on p. 1018 one finds the identification

$$\dot{F}_2^0[L^{d/(d-\alpha),1}] \equiv L^{d/(d-\alpha),1},$$

so that the claim follows by an application of the inhomogeneous variant of Theorem 1.1 (iv) with the choices

$$\begin{aligned} s_0 &= s_1 = 0, \\ q_0 &= 1, \\ q_1 &= 2, \\ p_0 &= p_1 = \frac{d}{d-\alpha}, \\ r_0 &= r_1 = 1, \end{aligned}$$

as one can check that they satisfy

$$s_0 = s_1, \quad p_0 = p_1 \neq q_1, \quad r_0 \leq r_1, \quad q_0 \leq \min\{p_1, q_1, r_1\}.$$

Second, since $s_0 = s_1$ and $p_0 = p_1 \neq q_0$, Theorem 1.2 (iv) asserts the reverse inclusion can only hold if

$$q_1 = 1 \geq \max\{p_0 = \frac{d}{d-\alpha}, q_0 = 2, r_0 = 1\},$$

which is not valid (and note here the subscripts are opposite those immediately preceding in the invocation of Theorem 1.1 (iv)).

With the definition we have introduced above, we find that we must estimate

$$\|I_\alpha F\|_{\dot{B}_{d/(d-\alpha),1}^{0,1}(\mathbb{R}^d; \mathbb{R}^k)} = \sum_{n \in \mathbb{Z}} \|I_\alpha F * (\psi_{2^{n+1}} - \psi_{2^n})\|_{L^{d/(d-\alpha),1}(\mathbb{R}^d; \mathbb{R}^k)}.$$

In this form, an observation analogous to that of Stolyarov is that if we define the multiplier

$$\widehat{m}(\xi) := \frac{\widehat{\psi}(\xi) - \widehat{\psi}(2^{-1}\xi)}{(2\pi|\xi|)^\alpha \exp(-4\pi^2|\xi|^2)},$$

then, with the use of the notation for scaling introduced in (2.4), one has

$$I_\alpha F * (\psi_{2^{n+1}} - \psi_{2^n}) = 2^{-n\alpha} p_{2^{-2n}} * F * m_{2^n}.$$

Here we use the fact that \widehat{m} is a Schwartz function to write the expression as a convolution. In particular, the fact that $m \in L^1(\mathbb{R}^d)$ and the invariance of the space $L^1(\mathbb{R}^d)$ with respect to the scaling (2.4) implies $m_{2^n} \in L^1(\mathbb{R}^d)$ with

$$\|m_{2^n}\|_{L^1(\mathbb{R}^d)} = \|m\|_{L^1(\mathbb{R}^d)} =: c.$$

By Young’s inequality on the Lorentz scale we obtain the bound

$$\|I_\alpha F * (\psi_{2^{n+1}} - \psi_{2^n})\|_{L^{d/(d-\alpha),1}(\mathbb{R}^d; \mathbb{R}^k)} \leq 3 \frac{d}{d-\alpha} c 2^{-n\alpha} \|p_{2^{-2n}} * F\|_{L^{d/(d-\alpha),1}(\mathbb{R}^d; \mathbb{R}^k)},$$

so that summation over $n \in \mathbb{Z}$ gives the inequality

$$\|I_\alpha F\|_{\dot{B}_{d/(d-\alpha),1}^{0,1}(\mathbb{R}^d; \mathbb{R}^k)} \leq \frac{3cd}{d-\alpha} \sum_{n \in \mathbb{Z}} 2^{-n\alpha} \|p_{2^{-2n}} * F\|_{L^{d/(d-\alpha),1}(\mathbb{R}^d; \mathbb{R}^k)}. \tag{2.6}$$

The right-hand-side of this inequality is (a constant multiple of) the discrete quantity that Stolyarov obtains an upper bound for in his paper to prove the Besov-Lorentz bound for the general class of subspaces. To pass to the continuous version, we use the semi-group property of the heat kernel and another application of Young’s inequality on the Lorentz scale: For each $n \in \mathbb{Z}$ and all $s \in (2^{-2n-2}, 2^{-2n})$,

$$\begin{aligned} \|p_{2^{-2n}} * F\|_{L^{d/(d-\alpha),1}(\mathbb{R}^d; \mathbb{R}^k)} &= \|p_{2^{-2n-s}} * p_s * F\|_{L^{d/(d-\alpha),1}(\mathbb{R}^d; \mathbb{R}^k)} \\ &\leq \|p_{2^{-2n-s}} * |p_s * F|\|_{L^{d/(d-\alpha),1}(\mathbb{R}^d; \mathbb{R}^k)} \\ &\leq \frac{3d}{d-\alpha} \|p_s * F\|_{L^{d/(d-\alpha),1}(\mathbb{R}^d; \mathbb{R}^k)}. \end{aligned}$$

In particular, integration from $s = 2^{-2n-2}$ to 2^{-2n} with respect to the measure ds/s gives the inequality

$$\|p_{2^{-2n}} * F\|_{L^{d/(d-\alpha),1}(\mathbb{R}^d; \mathbb{R}^k)} \leq \frac{1}{\ln(4)} \frac{3d}{d-\alpha} \int_{2^{-2n-2}}^{2^{-2n}} \|p_s * F\|_{L^{d/(d-\alpha),1}(\mathbb{R}^d; \mathbb{R}^k)} \frac{ds}{s},$$

which in combination with (2.6) yields

$$\|I_\alpha F\|_{\dot{B}_{d/(d-\alpha),1}^{0,1}(\mathbb{R}^d;\mathbb{R}^k)} \leq \left(\frac{3d}{d-\alpha}\right)^2 \frac{c}{\ln(4)} \sum_{n \in \mathbb{Z}} 2^{-n\alpha} \int_{2^{-2n-2}}^{2^{-2n}} \|p_s * F\|_{L^{d/(d-\alpha),1}(\mathbb{R}^d;\mathbb{R}^k)} \frac{ds}{s}.$$

By further manipulation we obtain

$$\begin{aligned} & \sum_{n \in \mathbb{Z}} 2^{-n\alpha} \int_{2^{-2n-2}}^{2^{-2n}} \|p_s * F\|_{L^{d/(d-\alpha),1}(\mathbb{R}^d;\mathbb{R}^k)} \frac{ds}{s} \\ & \leq \sum_{n \in \mathbb{Z}} 2^\alpha \int_{2^{-2n-2}}^{2^{-2n}} s^{\alpha/2-1} \|p_s * F\|_{L^{d/(d-\alpha),1}(\mathbb{R}^d;\mathbb{R}^k)} ds, \end{aligned}$$

and thus

$$\|I_\alpha F\|_{\dot{B}_{d/(d-\alpha),1}^{0,1}(\mathbb{R}^d;\mathbb{R}^k)} \leq \bar{c} \int_0^\infty s^{\alpha/2-1} \|p_s * F\|_{L^{d/(d-\alpha),1}(\mathbb{R}^d;\mathbb{R}^k)} ds \tag{2.7}$$

for

$$\bar{c} := 2^\alpha \left(\frac{3d}{d-\alpha}\right)^2 \frac{c}{\ln(4)}.$$

The inequality (2.7) is exactly the control of the Besov-Lorentz norm by the quantity (1.6) claimed in the introduction. In particular, by the argument of the introduction and that preceding in this Section, we have established the estimate claimed in Theorem 1.1 for $F \in L^1(\mathbb{R}^d; \mathbb{R}^d)$ such that $\operatorname{div} F = 0$.

To conclude the proof of Theorem 1.1, we follow the argument of J. Van Schaftingen in [32] that the general case follows by an algebraic reduction. To do this, we first recall a few facts on differential operators. We will work with first order homogeneous linear differential operators with constant coefficients, which can be written as

$$L(D)F = \sum_{i=1}^d L_i \partial_i F = \sum_{i=1}^d \partial_i (L_i F),$$

where $L_i \in \operatorname{Lin}(\mathbb{R}^k; \mathbb{R}^l) \simeq \mathbb{R}^{l \times k}$. Due to the usefulness of Fourier transform for linear equations, it is natural to look at the symbol map

$$L(\xi) = \sum_{i=1}^d \xi_i L_i \in \operatorname{Lin}(\mathbb{R}^k; \mathbb{R}^l) \quad \text{for } \xi \in \mathbb{R}^d.$$

We make the simple observation that we can write

$$L(D)F = \operatorname{div}(TF), \quad \text{where } TF = (L_1 F | L_2 F | \dots | L_d F),$$

so $T \in \text{Lin}(\mathbb{R}^k, \mathbb{R}^{l \times d})$. The divergence of a matrix field is considered row wise.

We recall the definition of cocancellation:

Definition 2.5 An operator $L(D)$ as above is said to be *cocanceling* if and only if

$$\bigcap_{\xi \in \mathbb{R}^d} \ker L(\xi) = \{0\}.$$

We will show that cocancellation is equivalent with injectivity of the map T defined above. The following lemma should be compared with [32, Prop. 2.5] and the proof of [9, lem. 3.11].

Lemma 2.6 *We have that*

$$\bigcap_{\xi \in \mathbb{R}^d} \ker L(\xi) = \ker T.$$

To prove this, note that a vector $F \in \mathbb{R}^k$ lies in the left hand side if and only if

$$(TF)\xi = 0 \text{ for all } \xi \in \mathbb{R}^d \iff TF = 0,$$

which yields the conclusion.

In particular, $L(D)$ is cocanceling if and only if T is left invertible. If this is the case, we can write an explicit left inverse in terms of the adjoint T^* of T ,

$$T^\dagger = (T^*T)^{-1}T^*.$$

We can thus proceed with the proof of the main result.

Conclusion of the proof of Theorem 1.1 The necessity of cocancellation follows from by plugging in a Dirac mass in the estimate and noting that $I_\alpha \notin L^{d/(d-\alpha)}$.

Conversely, we already proved the desired estimate for $L(D) = \text{div}$. We note that if $L(D)F = 0$, we can write $\text{div}(TF) = 0$ and $F = T^\dagger TF$, so that

$$I_\alpha F = I_\alpha(T^\dagger TF) = T^\dagger I_\alpha(TF),$$

and using the inequality for divergence free measures and the fact that T^\dagger, T are bounded maps on finite dimensional spaces we obtain

$$\begin{aligned} \|I_\alpha F\|_{\dot{B}_{d/(d-\alpha),1}^{0,1}(\mathbb{R}^d; \mathbb{R}^k)} &= \|T^\dagger I_\alpha(TF)\|_{\dot{B}_{d/(d-\alpha),1}^{0,1}(\mathbb{R}^d; \mathbb{R}^k)} \\ &\leq C \|I_\alpha(TF)\|_{\dot{B}_{d/(d-\alpha),1}^{0,1}(\mathbb{R}^d; \mathbb{R}^{l \times d})} \leq C \|TF\|_{L^1(\mathbb{R}^d; \mathbb{R}^{l \times d})} \\ &\leq C \|F\|_{L^1(\mathbb{R}^d; \mathbb{R}^k)}, \end{aligned}$$

which completes the proof. □

3 Hodge systems and duality estimates

We first note that Corollary 1.2 follows from Theorem 1.1 since the L^1 vector fields du and d^*u satisfy the first order conditions $d(du) = 0$ and $d^*(d^*u) = 0$, which are cocanceling for the claimed ranges of k . This algebraic fact is elementary to check, see also [32, Prop. 3.3] where the cocancellation of d on ℓ -forms is proved, $\ell \leq d - 1$. The claim for d^* follows by duality.

Proof of Theorem 1.3 We first infer from Corollary 1.2 with $\alpha = 1$ that

$$\|I_1 F\|_{\dot{B}_{d/(d-1),1}^{0,1}(\mathbb{R}^d; \Lambda^{k-1}\mathbb{R}^d)} \leq C \|F\|_{L^1(\mathbb{R}^d; \Lambda^{k-1}\mathbb{R}^d)}, \tag{3.1}$$

$$\|I_1 G\|_{\dot{B}_{d/(d-1),1}^{0,1}(\mathbb{R}^d; \Lambda^{k+1}\mathbb{R}^d)} \leq C \|G\|_{L^1(\mathbb{R}^d; \Lambda^{k+1}\mathbb{R}^d)}, \tag{3.2}$$

where $F \equiv 0$ if $k = 1$ and $G \equiv 0$ if $k = d - 1$. Next, we note that since the Hodge Laplacian coincides with the real variable Laplacian, we can express

$$Z = dI_2 F + d^*I_2 G = (dI_1)I_1 F + (d^*I_1)I_1 G, \tag{3.3}$$

where we used $(-\Delta)^{-1} = I_2$ and the semigroup property of Riesz potentials. Note that dI_1, d^*I_1 give rise to a zero-homogeneous Fourier multiplier, hence can be represented as Calderón–Zygmund operators, and are therefore bounded on the Besov–Lorentz spaces (Here one should be careful to note that these operators are mappings from one exterior algebra into another.). It follows from (3.1), (3.2), and (3.3) that (and for the convenience of display we remove the notation in the norm which details the images of each map)

$$\begin{aligned} \|Z\|_{\dot{B}_{d/(d-1),1}^{0,1}(\mathbb{R}^d)} &\leq C \left(\|(dI_1)I_1 F\|_{\dot{B}_{d/(d-1),1}^{0,1}(\mathbb{R}^d)} + \|(d^*I_1)I_1 G\|_{\dot{B}_{d/(d-1),1}^{0,1}(\mathbb{R}^d)} \right) \\ &\leq C \left(\|I_1 F\|_{\dot{B}_{d/(d-1),1}^{0,1}(\mathbb{R}^d)} + \|I_1 G\|_{\dot{B}_{d/(d-1),1}^{0,1}(\mathbb{R}^d)} \right) \\ &\leq C \left(\|F\|_{L^1(\mathbb{R}^d)} + \|G\|_{L^1(\mathbb{R}^d)} \right), \end{aligned}$$

which completes the proof. □

Proof of Proposition 1.4 Both inequalities follow from our main result, Theorem 1.1. To prove the first estimate, we observe that the semi-group property of the Riesz potentials and Hölder’s inequality on the Lorentz scale implies

$$\begin{aligned} \int_{\mathbb{R}^d} F \cdot \varphi \, dx &= \int_{\mathbb{R}^d} I_1 F \cdot (-\Delta)^{1/2} \varphi \, dx \\ &\leq C \|I_1 F\|_{L^{d/(d-1),1}(\mathbb{R}^d; \mathbb{R}^k)} \|R^* D\varphi\|_{L^{d,\infty}(\mathbb{R}^d; \mathbb{R}^k)}, \end{aligned}$$

where we denote by R^* the adjoint of the Riesz transforms, $R^* = -\operatorname{div} I_1$, which satisfies the identity $R^* D\varphi = (-\Delta)^{1/2} \varphi$. This inequality, Theorem 1.1, and the bound

$$\|R^* D\varphi\|_{L^{d,\infty}(\mathbb{R}^d; \mathbb{R}^k)} \leq C \|D\varphi\|_{L^{d,\infty}(\mathbb{R}^d; \mathbb{R}^{k \times d})}$$

then implies the desired result, the last inequality following from the fact that R^* is bounded on the Lorentz spaces.

In a similar manner we argue the second inequality of the Proposition. First, by duality we have

$$\int_{\mathbb{R}^d} F \cdot \varphi \, dx \leq C \|F\|_{\dot{B}_{d/(d-\alpha)}^{-\alpha,1}(\mathbb{R}^d; \mathbb{R}^k)} \|\varphi\|_{\dot{B}_{d/\alpha}^{\alpha,\infty}(\mathbb{R}^d; \mathbb{R}^k)}.$$

Next, we observe that the definition of $\dot{B}_{d/(d-\alpha)}^{-\alpha,1}(\mathbb{R}^d; \mathbb{R}^k)$, in analogy with that of the Besov-Lorentz space utilized in Sect. 2, is

$$\|F\|_{\dot{B}_{d/(d-\alpha)}^{-\alpha,1}(\mathbb{R}^d; \mathbb{R}^k)} = \sum_{n \in \mathbb{Z}} 2^{-\alpha n} \|F * (\psi_{2^{n+1}} - \psi_{2^n})\|_{L^{d/(d-\alpha)}(\mathbb{R}^d; \mathbb{R}^k)}.$$

Therefore a slight modification of the argument of Theorem 1.1 in Sect. 2 leads to the estimate

$$\|F\|_{\dot{B}_{d/(d-\alpha)}^{-\alpha,1}(\mathbb{R}^d; \mathbb{R}^k)} \leq C' \|F\|_{L^1(\mathbb{R}^d; \mathbb{R}^k)},$$

which completes the proof of our second claim. □

4 Optimality on the Lorentz scale

In a now classical paper on Sobolev embeddings, Alvino [2] proved (with sharp constant) that one has

$$\|u\|_{L^{d/(d-1),1}(\mathbb{R}^d)} \leq C \|\nabla u\|_{L^1(\mathbb{R}^d; \mathbb{R}^d)}.$$

Such an inequality extends to the case Du is a Radon measure, that is, $Du \in M_b(\mathbb{R}^d; \mathbb{R}^d)$ by approximation.

The first result of this Section is a construction which will show the optimality of Alvino’s result. Here we should clarify our meaning of optimality. This is the endpoint of where the Lorentz spaces are normable, so that on the scale of normable spaces it is clear this is optimal. We will now show that the result cannot hold for a smaller choice of second parameter in the quasi-norm introduced in (2.1), which is what we intend by the phrase optimality.

Lemma 4.1 *For every $q < 1$, there exists a sequence $\{u_N\}_{N \in \mathbb{N}} \subset BV(\mathbb{R}^d)$ with*

$$\|Du_N\|(\mathbb{R}^d) \leq C$$

independent of $N \in \mathbb{N}$ and

$$\lim_{N \rightarrow \infty} \|u_N\|_{L^{d/(d-1),q}(\mathbb{R}^d)} = +\infty.$$

Proof Define

$$u_N = \sum_{i=1}^N h_i \chi_{B(0,r_i)}(x) \tag{4.1}$$

where $h_i \geq 0$ and $r_i \searrow 0$ will be chosen later such that

$$\|Du_N\|(\mathbb{R}^d) = \sum_{i=1}^N h_i \omega_d r_i^{d-1} \leq C$$

independent of N and

$$\lim_{N \rightarrow \infty} \|u_N\|_{L^{d/(d-1),q}(\mathbb{R}^d)} = +\infty.$$

To this end, we define $H_0 := 0$,

$$H_i := \sum_{j=1}^i h_j$$

and compute

$$\begin{aligned} \|u_N\|_{L^{d/(d-1),q}(\mathbb{R}^d)}^q &= \frac{d}{d-1} \int_0^{H_N} (t|\{|u| > t\}|^{(d-1)/d})^q \frac{dt}{t} \\ &= \frac{d}{d-1} \sum_{i=0}^{N-1} \int_{H_i}^{H_{i+1}} (t\omega_d^{(d-1)/d} r_i^{d-1})^q \frac{dt}{t} \\ &= \frac{d}{d-1} \omega_d^{q(d-1)/d} \sum_{i=0}^{N-1} r_i^{(d-1)q} \int_{H_i}^{H_{i+1}} t^{q-1} dt. \end{aligned}$$

Since $q < 1$, $t \mapsto t^{q-1}$ is decreasing and therefore

$$\int_{H_i}^{H_{i+1}} t^{q-1} dt \geq H_{i+1}^{q-1} (H_{i+1} - H_i) = H_{i+1}^{q-1} h_{i+1}.$$

In particular, with a shift of indices we find

$$\|u_N\|_{L^{d/(d-1),q}(\mathbb{R}^d)}^q \geq \frac{d}{d-1} \omega_d^{q(d-1)/d} \sum_{i=1}^N r_i^{(d-1)q} H_i^{q-1} h_i.$$

Therefore it remains to choose h_i, r_i such that

$$\sum_{i=1}^{\infty} r_i^{(d-1)q} H_i^q \frac{h_i}{H_i} = +\infty$$

and recall we must do so in a way the ensures

$$\sum_{i=1}^{\infty} h_i \omega_d r_i^{d-1} \leq C.$$

Choose $h_i = 2^i$, so that $H_i = 2^{i+1}$. Thus we now are left to choose r_i such that

$$\sum_{i=1}^{\infty} r_i^{(d-1)q} H_i^q \frac{h_i}{H_i} = 2^{q-1} \sum_{i=1}^{\infty} \left(2^i r_i^{(d-1)}\right)^q = +\infty$$

and

$$\sum_{i=1}^{\infty} 2^i r_i^{(d-1)} < +\infty.$$

But then the choice $2^i r_i^{(d-1)} = \frac{1}{i^{1/q}}$ is sufficient, as $q < 1$. □

Observe that Lemma 4.1 implies the optimality on the Lorentz scale of Alvino’s result [2], that the second parameter in the Lorentz estimate cannot be taken less than 1. In fact, taking $N \rightarrow \infty$ in the previous proof, we can prove the non inclusion of BV in subcritical Lorentz spaces:

Corollary 4.2 *For every $q < 1$, there exists $u \in BV(\mathbb{R}^d) \setminus L^{d/(d-1),q}(\mathbb{R}^d)$.*

Proof Let u_N be the sequence defined in (4.1). Note that

$$u_N \nearrow u = \sum_{i=1}^{\infty} h_i \chi_{B(0,r_i)} \quad \text{a.e.}$$

This immediately implies

$$\|u\|_{L^{d/(d-1),q}(\mathbb{R}^d)} \geq \|u_N\|_{L^{d/(d-1),q}(\mathbb{R}^d)} \nearrow \infty,$$

so $u \notin L^{d/(d-1),q}(\mathbb{R}^d)$.

Note that since $\|Du_N\|(\mathbb{R}^d) \leq C$, weak-* compactness in BV implies that, on a subsequence, $u_N \overset{*}{\rightharpoonup} u$ in $BV(\mathbb{R}^d)$. It follows that $u \in BV(\mathbb{R}^d)$, which completes the proof. □

For our purposes here it will be useful to observe another consequence of this construction, the following

Lemma 4.3 *For every $r < 1$, there exists a sequence $\{u_N\}_{N \in \mathbb{N}} \subset BV(\mathbb{R}^d)$ with*

$$\|Du_N\|(\mathbb{R}^d) \leq C$$

independent of $N \in \mathbb{N}$ and

$$\lim_{N \rightarrow \infty} \|I_\alpha Du_N\|_{L^{d/(d-\alpha),r}(\mathbb{R}^n;\mathbb{R}^n)} = +\infty.$$

Proof We begin with an inequality for u in terms of potentials and its gradient

$$|u(x)| \leq c I_{1-\alpha} |I_\alpha Du(x)|.$$

We will estimate u by a standard potential estimate for $I_{1-\alpha}$. To this end, we recall an estimate of Hedberg, see e.g. [1, Proposition 3.1.2 (a)] which asserts

$$|I_\beta f| \leq \mathcal{M}(f)^{1-\beta p/d} \|f\|_{L^p(\mathbb{R}^d)}^{\beta p/d}.$$

The choice $f = |I_\alpha Du(x)|$, $\beta = 1 - \alpha$, and $p = d/(d - \alpha)$ yields

$$I_{1-\alpha} |I_\alpha Du(x)| \leq \mathcal{M}(|I_\alpha Du(x)|)^{(n-1)/(n-\alpha)} \|I_\alpha Du\|_{L^{d/(d-\alpha)}(\mathbb{R}^d)}^{1-(d-1)/(d-\alpha)}.$$

By the boundedness of the maximal function (see [8, Theorem 1.4.19] for the case $q(d - 1)/(d - \alpha) < 1$) and properties of the Lorentz spaces, this shows that

$$\|u\|_{L^{d/(d-1),q}(\mathbb{R}^d)} \leq \|I_\alpha Du\|_{L^{d/(d-\alpha),q(d-1)/(d-\alpha)}(\mathbb{R}^n)} \|I_\alpha Du\|_{L^{d/(d-\alpha)}(\mathbb{R}^d)}^{1-(d-1)/(d-\alpha)}.$$

By the embedding proved in [19, 23], this implies

$$\|u\|_{L^{d/(d-1),q}(\mathbb{R}^d)} \leq \|I_\alpha Du\|_{L^{d/(d-\alpha),q(d-1)/(d-\alpha)}(\mathbb{R}^d;\mathbb{R}^d)} \|Du\|_{(\mathbb{R}^d)^{1-(d-1)/(d-\alpha)}}.$$

But then for any $r < 1$ we may choose

$$q = r \times \frac{d - \alpha}{d - 1} < r < 1,$$

and the construction from Lemma 4.1 with this choice of q yields the desired sequence. \square

From this we obtain the optimality of Theorem 1.1 in [23]. Here we remark that while compactness properties of bounded sequences in BV again allows one to write down the limit

$$u = \sum_{i=1}^{\infty} h_i \chi_{B(0,r_i)}(x) \in BV(\mathbb{R}^d),$$

the fact that $I_\alpha Du \notin L^{d/(d-\alpha),r}(\mathbb{R}^d; \mathbb{R}^d)$ is not obvious in this case.

5 Simplifications in the curl free case

That one has Lorentz and Besov-Lorentz embeddings for $F \in L^1(\mathbb{R}^d; \mathbb{R}^d)$ such that $\text{curl } F = 0$ is contained in Theorem 1.1. We here will give an even more direct proof, which improves upon those given in [12, 13, 22, 23] and in this less general setting even simplifies some of the argument from Sect. 2 above. In particular, the goal of this Section is to establish

Theorem 5.1 *Let $d \geq 2$ and $\alpha \in (0, d)$. There exists a constant $C = C(\alpha, d) > 0$ such that*

$$\|F\|_{\dot{B}_{d/(d-\alpha),1}^{-\alpha,1}(\mathbb{R}^d;\mathbb{R}^d)} \leq C \|F\|_{L^1(\mathbb{R}^d;\mathbb{R}^d)}$$

for all $F \in L^1(\mathbb{R}^d; \mathbb{R}^d)$ such that $\text{curl } F = 0$.

Theorem 5.1 has an interesting history, as it was pointed out to us by D. Stolyarov that one can deduce it from Theorem 4 of V.I. Kolyada’s paper [12, Theorem 4]. The third named author was not aware of this during the writing of [23], which gives a different proof of what is unfortunately a slightly weaker result. We can here rectify this in giving a simpler proof of the same result.

Proof of Theorem 5.1 First we claim that it suffices to prove the estimate

$$\int_0^\infty t^{\alpha/2-1} \|p_t * D\chi_E\|_{L^{d/(d-\alpha),1}(\mathbb{R}^d;\mathbb{R}^d)} dt \leq C \|D\chi_E\|(\mathbb{R}^d)$$

for all $\chi_E \in BV(\mathbb{R}^d)$, the space of functions of bounded variation. Indeed, in analogy with the argument in [23], for general $u \in BV(\mathbb{R}^d)$ one begins with the representation

$$Du = \int_{-\infty}^\infty D\chi_{E_s} ds$$

where $E_s := \{u > s\}$. With this representation, an application of Minkowski’s inequality for integrals and Fubini’s theorem yields

$$\begin{aligned} & \int_0^\infty t^{\alpha/2-1} \|p_t * Du\|_{L^{d/(d-\alpha),1}(\mathbb{R}^d;\mathbb{R}^d)} dt \\ & \leq \int_{-\infty}^\infty \int_0^\infty t^{\alpha/2-1} \|p_t * D\chi_{E_s}\|_{L^{d/(d-\alpha),1}(\mathbb{R}^d;\mathbb{R}^d)} dt ds. \end{aligned}$$

Therefore, if one has established the desired inequality for $\chi_E \in BV(\mathbb{R}^d)$, the result for general $u \in BV(\mathbb{R}^d)$ follows from this chain of inequalities and the coarea formula

$$\|Du\|(\mathbb{R}^d) = \int_{-\infty}^\infty \|D\chi_{E_s}\|(\mathbb{R}^d) ds.$$

Notice that if one only wants to prove the Lorentz inequality (1.1), or the weaker Lebesgue inequality, the essential ingredients at the point are only Minkowski's inequality for integrals and the coarea formula.

Toward establishing the desired inequality for sets of finite perimeter, we recall again the classical convolution inequalities

$$\|p_t * D\chi_E\|_{L^1(\mathbb{R}^d; \mathbb{R}^d)} \leq \|p_t\|_{L^1(\mathbb{R}^d)} |D\chi_E|(\mathbb{R}^d), \tag{5.1}$$

$$\|p_t * D\chi_E\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)} \leq \|p_t\|_{L^\infty(\mathbb{R}^d)} |D\chi_E|(\mathbb{R}^d). \tag{5.2}$$

Interpolation of these inequalities alone would not suffice, and so we require two additional inequalities which are special to characteristic functions of sets. It is here that the curl free case is much simpler than the divergence free case, as these two inequalities follow immediately from integration by parts and classical convolution estimates:

$$\|p_t * D\chi_E\|_{L^1(\mathbb{R}^d; \mathbb{R}^d)} \leq \|Dp_t\|_{L^1(\mathbb{R}^d; \mathbb{R}^d)} \|\chi_E\|_{L^1(\mathbb{R}^d)}, \tag{5.3}$$

$$\|p_t * D\chi_E\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)} \leq \|Dp_t\|_{L^1(\mathbb{R}^d; \mathbb{R}^d)} \|\chi_E\|_{L^\infty(\mathbb{R}^d)}. \tag{5.4}$$

In particular, one does not need to perform surgery, consider a ball growth condition, maximal function estimates, or generalized minimal surfaces.

From here, one interpolates (5.1) and (5.4) to obtain

$$\|p_t * D\chi_E\|_{L^{p'}(\mathbb{R}^d; \mathbb{R}^d)} \leq \frac{C_3}{t^{1/2p'}} |D\chi_E|(\mathbb{R}^d)^{1/p}, \tag{5.5}$$

where we have used

$$\|p_t\|_{L^1(\mathbb{R}^d)} = 1, \quad \|\chi_E\|_{L^\infty(\mathbb{R}^d)} = 1, \quad \text{and} \quad \|Dp_t\|_{L^1(\mathbb{R}^d; \mathbb{R}^d)} = \frac{c}{t^{1/2}}.$$

In a similar manner, interpolation of (5.3) and (5.2) yields

$$\|p_t * D\chi_E\|_{L^{p'}(\mathbb{R}^d; \mathbb{R}^d)} \leq \frac{C_4}{t^{1/2p+d/2p'}} \|\chi_E\|_{L^1(\mathbb{R}^d)}^{1/p} |D\chi_E|(\mathbb{R}^d)^{1-1/p} \tag{5.6}$$

where we have used

$$\|p_t\|_{L^\infty(\mathbb{R}^d)} = \frac{1}{(4\pi t)^{d/2}} \quad \text{and} \quad \|Dp_t\|_{L^1(\mathbb{R}^d; \mathbb{R}^d)} = \frac{c}{t^{1/2}}.$$

As before, we use the fact that in interpolation of Lebesgue spaces, one can improve the second parameter in the interpolation on the Lorentz scale. One can do even better here than one, though below one the estimate is no longer linear and so the result for the curl free case does not follow from the inequality for characteristics functions of sets. Notice that if one only wants the Lebesgue scale inequality, the interpolation is an exercise in Real Analysis.

Finally, we can make the estimate by splitting the integral in two pieces

$$\begin{aligned} & \int_0^\infty t^{\alpha/2-1} \|p_t * D\chi_E\|_{L^{d/(d-\alpha),1}(\mathbb{R}^d; \mathbb{R}^d)} dt \\ &= \int_0^r t^{\alpha/2-1} \|p_t * D\chi_E\|_{L^{d/(d-\alpha),1}(\mathbb{R}^d; \mathbb{R}^d)} dt \\ &+ \int_r^\infty t^{\alpha/2-1} \|p_t * D\chi_E\|_{L^{d/(d-\alpha),1}(\mathbb{R}^d; \mathbb{R}^d)} dt \\ &=: I + II. \end{aligned}$$

For I , we use the interpolated inequality (5.5) with the choice of $p = \frac{d}{d-\alpha}$ to obtain

$$\begin{aligned} I &\leq \frac{C_3}{\Gamma\left(\frac{\alpha}{2}\right)} |D\chi_E|(\mathbb{R}^d)^{(d-\alpha)/d} \int_0^r t^{\alpha/2-1-\alpha/2d} dt \\ &= C'_3 |D\chi_E|(\mathbb{R}^d)^{(d-\alpha)/d} r^{\alpha/2-\alpha/2d}, \end{aligned}$$

while for II we use the interpolated inequality (5.6), with the same choice of p to obtain

$$\begin{aligned} II &\leq \frac{C_4}{\Gamma\left(\frac{\alpha}{2}\right)} \|\chi_E\|_{L^1(\mathbb{R}^d)}^{(d-\alpha)/d} |D\chi_E|(\mathbb{R}^d)^{\alpha/d} \int_r^\infty t^{\alpha/2-1-1/2p-d/2p'} dt \\ &= C'_4 \|\chi_E\|_{M_b(\mathbb{R}^d)}^{(d-\alpha)/d} |D\chi_E|(\mathbb{R}^d)^{\alpha/d} r^{-1/2+\alpha/2d}. \end{aligned}$$

The desired inequality then follows from optimizing in r and the isoperimetric inequality

$$|E|^{1-1/d} \leq c_d |D\chi_E|(\mathbb{R}^d).$$

This concludes the proof, the Section, and the paper. □

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