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ORIGINAL PAPER

The fractional variation and the precise representative of $BV^{\alpha,p}$ functions

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Abstract
We continue the study of the fractional variation following the distributional approach developed in the previous works Bruè et al. (2021), Comi and Stefani (2019), Comi and Stefani (2019). We provide a general analysis of the distributional space $BV^{\alpha,p}(\mathbb{R}^n)$ of L^p functions, with $p \in [1, +\infty]$, possessing finite fractional variation of order $\alpha \in (0, 1)$. Our two main results deal with the absolute continuity property of the fractional variation with respect to the Hausdorff measure and the existence of the precise representative of a $BV^{\alpha,p}$ function.

Keywords Fractional gradient · Fractional divergence · Fractional variation · Hausdorff measure · Fractional capacity · Precise representative

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1 Introduction

1.1 The fractional variation

For a parameter $\alpha \in (0, 1)$ and an exponent $p \in [1, +\infty]$, the *space of L^p functions with bounded fractional variation* is

$$BV^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in L^p(\mathbb{R}^n) : |D^\alpha f|(\mathbb{R}^n) < +\infty \right\}, \tag{1.1}$$

where

$$|D^\alpha f|(\mathbb{R}^n) = \sup \left\{ \int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx : \varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n), \ \|\varphi\|_{L^\infty(\mathbb{R}^n;\, \mathbb{R}^n)} \le 1 \right\} \tag{1.2}$$

is the (total) *fractional variation* of the function $f \in L^p(\mathbb{R}^n)$. Here and in the following, for sufficiently smooth functions and vector-fields, we let

$$\nabla^\alpha f(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(y - x)(f(y) - f(x))}{|y - x|^{n+\alpha+1}} \, dy, \quad x \in \mathbb{R}^n,$$

and

$$\operatorname{div}^\alpha \varphi(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(y - x) \cdot (\varphi(y) - \varphi(x))}{|y - x|^{n+\alpha+1}} \, dy, \quad x \in \mathbb{R}^n,$$

be the *fractional gradient* and the *fractional divergence* operators respectively, where $\mu_{n,\alpha}$ is a suitable renormalizing constant depending on n and α only. The above fractional operators are *dual*, in the sense that

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx = - \int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha f \, dx. \tag{1.3}$$

The fractional variation was considered by the first and the third authors in the work [7] in the *geometric* framework $p = 1$, also in relation with the naturally associated notion of *fractional Caccioppoli perimeter*. The fractional variation of an L^p function for an arbitrary exponent $p \in [1, +\infty]$ was then studied by the same authors in the subsequent paper [8], in connection with some embedding-type results arising from some optimal inequalities proved by the second author [31, 32].

Since the first appearance of the fractional gradient [16], the literature around ∇^α and $\operatorname{div}^\alpha$ has been rapidly growing in various directions, such as the study of PDEs [25, 26, 28, 29] and of functionals [4, 5, 17] involving these fractional operators, the discovery of new optimal embedding estimates [27, 31, 32] and the development of a distributional and asymptotic analysis in this fractional framework [6–8, 30]. We also refer the reader to the survey [33] and to the monograph [24].

At the present stage of the theory, the fine properties of functions having finite fractional variation are not completely understood and, to our knowledge, only some results [7] in the *geometric regime* $p = 1$ are available in the literature.

Besides providing a general treatment of the space $BV^{\alpha,p}(\mathbb{R}^n)$, in the present paper we aim to develop the existing theory in this direction. On the one side, we study the relation between the fractional variation and the Hausdorff measure. On the other side, we establish the existence of the precise representative of a $BV^{\alpha,p}$ function.

1.2 The Hausdorff dimension of the fractional variation

The natural idea behind the definition of the space $BV^{\alpha,p}(\mathbb{R}^n)$ is that a function $f \in L^p(\mathbb{R}^n)$ belongs to $BV^{\alpha,p}(\mathbb{R}^n)$ if and only if there exists a finite vector-valued Radon measure $D^\alpha f \in \mathscr{M}(\mathbb{R}^n;\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx = -\int_{\mathbb{R}^n} \varphi \cdot dD^\alpha f$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, generalizing the integration-by-parts formula (1.3).

In the classical integer case $\alpha = 1$, the variation of a function $f \in BV(\mathbb{R}^n)$ is known to satisfy

$$|Df| \ll \mathscr{H}^{n-1}, \tag{1.4}$$

where $\mathscr{H}^s$ is the s-dimensional Hausdorff measure. If $f = \chi_E$ for some measurable set $E \subset \mathbb{R}^n$, then it actually holds that

$$|D\chi_E| = \mathscr{H}^{n-1} \llcorner \mathscr{F}E, \tag{1.5}$$

where $\mathscr{F}E$ is the De Giorgi *reduced boundary* of E, see the monographs [3, 20].

Roughly speaking, formulas (1.4) and (1.5) mean that the variation measure of a BV function in $\mathbb{R}^n$ lives on sets with Hausdorff dimension $n-1$ at least. By the analogy between the integer and the fractional settings, one may expect that a similar phenomenon should occur also for the fractional variation of order $\alpha \in (0,1)$ on a set of Hausdorff dimension $n-\alpha$ at least. In [7,Corollary 5.4], the first and the third authors confirmed this parallelism by showing that, for a measurable set $E \subset \mathbb{R}^n$ such that $\chi_E \in BV^\alpha(\mathbb{R}^n)$ (or, more generally, for any measurable set having locally finite *fractional Caccioppoli perimeter*, see [7,Definition 4.1]), it holds that

$$|D^\alpha \chi_E| \le c_{n,\alpha}\, \mathscr{H}^{n-\alpha} \llcorner \mathscr{F}^\alpha E, \tag{1.6}$$

where $c_{n,\alpha} > 0$ depends on n and α only and $\mathscr{F}^\alpha E$ is the fractional analogue of the De Giorgi *reduced boundary* (1.5), the so-called *fractional reduced boundary* of E, see [7,Definition 4.7]. However, as shown in [7,Lemma 3.28] by the same authors, if $f \in BV^\alpha(\mathbb{R}^n)$ then the function $u = I_{1-\alpha} f$ (where I_s is the *Riesz potential* of order $s \in (0,n)$, see below for the precise definition) does satisfy $|Du|(\mathbb{R}^n) < +\infty$, with

$$Du = D^\alpha f \quad \text{in } \mathscr{M}(\mathbb{R}^n;\mathbb{R}^n). \tag{1.7}$$

In particular, by combining (1.4) with the above (1.7), we immediately get that

$$|D^\alpha f| \ll \mathscr{H}^{n-1} \tag{1.8}$$

for all $f \in BV^{\alpha}(\mathbb{R}^n)$, thus ruling out the existence of a *coarea formula* in this fractional setting, see [7,Corollary 5.6].

Equations (1.6) and (1.8) illustrate the richness arising from the innocent-looking definition (1.2) and lead to the idea that the behavior of the fractional variation of a function $f \in L^p(\mathbb{R}^n)$ may depend on its integrability exponent $p \in [1, +\infty]$. Our first main result provides a rigorous formulation of this intuitive idea and can be stated as follows.

Theorem 1 **(Absolute continuity properties of the fractional variation)** *Let* $\alpha \in (0, 1)$, $p \in [1, +\infty]$ *and assume that* $f \in BV^{\alpha,p}(\mathbb{R}^n)$. *We have the following cases:*

(i) if $p \in \left[1, \frac{n}{1-\alpha}\right)$, *then* $|D^\alpha f| \ll \mathscr{H}^{n-1}$;

(ii) if $p \in \left[\frac{n}{1-\alpha}, +\infty\right]$, *then* $|D^\alpha f| \ll \mathscr{H}^{n-\alpha-\frac{n}{p}}$.

As shown by Theorem 1, the fractional variation in the *subcritical regime* $p < \frac{n}{1-\alpha}$ is comparable with the Hausdorff measure of dimension $n-1$, in accordance with (1.8). In fact, we can actually prove a deeper property, in analogy with the relation (1.7). Precisely, the Riesz potential operator

$$I_{1-\alpha} \colon BV^{\alpha,p}(\mathbb{R}^n) \to BV^{1,\frac{np}{n-(1-\alpha)p}}(\mathbb{R}^n)$$

is continuous whenever $p < \frac{n}{1-\alpha}$ (see Proposition 4(i) below for the detailed statement), from which item (i) in Theorem 1 immediately follows. Here and in the following, for any $p \in [1, +\infty]$, we let

$$BV^{1,p}(\mathbb{R}^n) = \left\{ f \in L^p(\mathbb{R}^n) : |Df|(\mathbb{R}^n) < +\infty \right\}$$

be the space of L^p functions having finite variation, extending the definition in (1.1) to the integer case $\alpha = 1$.

In the *supercritical regime* $p \geq \frac{n}{1-\alpha}$ instead, the fractional variation is comparable with the Hausdorff measure of dimension $n - \alpha - \frac{n}{p}$, thus recovering (1.6) in the case $p = +\infty$. The proof of item (ii) of Theorem 1 is more delicate and requires a finer analysis. The overall idea is to adapt the strategy developed in [7,Section 5] for sets with (locally) finite *fractional Caccioppoli perimeter* to the present more general L^p framework. The key role in this approach is played by the following *decay estimate* for the fractional variation of a function $f \in BV^{\alpha,p}(\mathbb{R}^n)$ with $p \geq \frac{n}{n-\alpha}$,

$$|D^\alpha f|(B_r(x)) \leq c_{n,\alpha,p} \|f\|_{L^p(\mathbb{R}^n)}\, r^{n-\alpha-\frac{n}{p}}, \tag{1.9}$$

valid for $|D^\alpha f|$-a.e. $x \in \mathbb{R}^n$ and all $r > 0$ sufficiently small, where $c_{n,\alpha,p} > 0$ is a constant depending on n, α, and p only (see Theorem 10 below for the precise statement). The validity of (1.9) is suggested by the following heuristic argument, valid for all $f \in BV^{\alpha,p}(\mathbb{R}^n)$ such that

$$(D^\alpha f)_j \geq 0 \quad \text{for all } j \in \{1, \ldots, n\}. \tag{1.10}$$

If $\varphi \in C_c^\infty(B_2)$ is such that $\varphi \geq 0$ and $\varphi \equiv 1$ on B_1, then

$$\begin{aligned}(D^\alpha f)_j(B_r(x)) &\leq \int_{\mathbb{R}^n} \varphi\left(\frac{y-x}{r}\right) d(D^\alpha f)_j(y) \\ &= -r^{-\alpha} \int_{\mathbb{R}^n} f(y)\,(\nabla^\alpha \varphi)_j\left(\frac{y-x}{r}\right) dy,\end{aligned}$$

thanks to (1.3) and the α-homogeneity of the fractional gradient ([30,Theorem 4.3]), so that

$$\begin{aligned}(D^\alpha f)_j(B_r(x)) &\leq \|f\|_{L^p(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |\nabla^\alpha \varphi(y)|^{\frac{p}{p-1}} r^n\, dy\right)^{1-\frac{1}{p}} r^{-\alpha} \\ &= \|f\|_{L^p(\mathbb{R}^n)} \|\nabla^\alpha \varphi\|_{L^{\frac{p}{p-1}}(\mathbb{R}^n;\mathbb{R}^n)} r^{n-\alpha-\frac{n}{p}},\end{aligned}$$

which gives (1.9). Without (1.10), the decay estimate (1.9) is a consequence of some new integrability properties in Lorentz spaces of the fractional gradient and of an integration-by-parts formula of $BV^{\alpha,p}$ functions on balls which may be of some independent interest (see Theorems 8 and 9, respectively).

We note that Theorem 1 still holds even in the limit as $\alpha \to 1^-$. Indeed, for all $p \in [1, +\infty]$ and $f \in BV^{1,p}(\mathbb{R}^n)$ we get that $|Df| \ll \mathscr{H}^{n-1}$, since point (i) now applies to all $p \in [1, +\infty)$, while point (ii) refers only to $p = +\infty$, for which we have $n - 1 - \frac{n}{p} = n - 1$. This is in fact a well-known result for functions in $BV_{\mathrm{loc}}(\mathbb{R}^n)$, see [3,Lemma 3.76] for instance. On the contrary, Theorem 1 is not optimal in the limit as $\alpha \to 0^+$. Indeed, in virtue of [6,Theorem 3.3 and Remark A.3], if $p \in [1, +\infty)$ and $f \in BV^{0,p}(\mathbb{R}^n)$, then $|D^0 f| \ll \mathscr{L}^n$ (where the space $BV^{0,p}(\mathbb{R}^n)$ is defined as in (1.1) with $\alpha = 0$, see [6] for a more detailed presentation).

1.3 The precise representative of a $BV^{\alpha,p}$ function

Formulas (1.4) and (1.5) suggest that the set of discontinuity points (in the measure-theoretical sense) of a BV function should have Hausdorff dimension $n - 1$. In more precise terms, if $f \in BV(\mathbb{R}^n)$, then the limit

$$f^\star(x) = \lim_{r \to 0^+} \fint_{B_r(x)} f(y)\, dy \tag{1.11}$$

exists for $\mathscr{H}^{n-1}$-a.e. $x \in \mathbb{R}^n$. In fact, the limit in (1.11) can be strengthened as

$$\lim_{r \to 0^+} \fint_{B_r(x)} |f(y) - f^\star(x)|^{\frac{n}{n-1}}\, dy = 0$$

for $\mathscr{H}^{n-1}$-a.e. $x \in \mathbb{R}^n \setminus J_f$, see [11,Section 5.9] for example, where $J_f \subset \mathbb{R}^n$ is the so-called *jump set* of the function $f \in BV(\mathbb{R}^n)$ (if $f \in W^{1,1}(\mathbb{R}^n)$, then J_f is empty).

The function $f^\star$ defined by (1.11) is the so-called *precise representative* of the function f (by convention, we set $f^*(x) = 0$ if the limit in (1.11) does not exist). The well-posedness of the precise representative (1.11) of a $BV^{\alpha,p}$ function is not known at the present moment. Our second main result moves in this direction and can be briefly stated as follows (for a more precise statement, we refer the reader to Corollary 5 below).

Theorem 2 (The precise representative of a $BV^{\alpha,p}$ function) *Let $\alpha \in (0,1)$, $p \in [1,+\infty]$ and $\varepsilon > 0$. If $f \in BV^{\alpha,p}(\mathbb{R}^n)$, then the limit $f^\star(x)$ exists for $\mathscr{H}^{n-\alpha+\varepsilon}$-a.e. $x \in \mathbb{R}$. Moreover, for any such point $x \in \mathbb{R}^n$, it holds that*

$$\lim_{r\to 0^+} \fint_{B_r(x)} |f(y) - f^\star(x)|^q \, dy = 0$$

for any $q \in [1, \bar{q}_\varepsilon]$, where $\bar{q}_\varepsilon \in \left[1, \frac{n}{n-\alpha}\right)$ is such that $\lim_{\varepsilon\to 0^+} \bar{q}_\varepsilon = \frac{n}{n-\alpha}$.

The idea behind the proof of Theorem 2 is very simple and relies on three ingredients naturally arising from our general investigation of the $BV^{\alpha,p}(\mathbb{R}^n)$ space. First, we show that C_c^∞ functions are dense in energy in $BV^{\alpha,p}(\mathbb{R}^n)$ provided that $p \in \left[1, \frac{n}{n-\alpha}\right)$, extending the approximation [7,Theorem 3.8] already proved by the first and the third author in the *geometric regime* $p = 1$. Second, by combining this approximation with an optimal embedding inequality [32] due to the second author, we establish a fractional analogue of the Gagliardo–Nirenberg–Sobolev inequality, that is, $BV^{\alpha,p}(\mathbb{R}^n) \subset L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$ with continuous inclusion. Third, we exploit this fractional embedding inequality to prove the continuous inclusion of $BV^{\alpha,p}(\mathbb{R}^n)$ into some *Bessel potential space* of suitable fractional order. At this point, the existence of the precise representative of a $BV^{\alpha,p}$ function for $p < \frac{n}{n-\alpha}$ can be inferred from the known theory of *Bessel potential spaces*, see [1,Section 6.1] for example. The remaining exponents $p \geq \frac{n}{n-\alpha}$ can be recovered from the previous analysis by a simple cut-off argument that may be of some separate interest (see Lemma 1 for the detailed statement).

1.4 Future developments

Generally speaking, the precise representative of a function turns out to be the correct object when dealing with the product between the function itself and a sufficiently well-behaved measure.

For example, the precise representative allows to state the general Leibniz rule for the product of two BV functions. Precisely, if $f, g \in BV(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then $fg \in BV(\mathbb{R}^n)$ with

$$D(fg) = g^\star Df + f^\star Dg \quad \text{in } \mathscr{M}(\mathbb{R}^n; \mathbb{R}^n). \tag{1.12}$$

Note that the two products appearing in right-hand side of (1.12) are well posed thanks to the combination of the absolute continuity property of the variation (1.4) and the existence of the precise representative (1.11).

With Theorems 1 and 2 at hand, the analysis developed in the present work naturally leads to study the interactions between the fractional variation measure and the precise representative of $BV^{\alpha,p}$ functions, aiming at a more general formulation of the Leibniz rule and of the Gauss–Green formula in this fractional setting. These results are the main topic of the subsequent paper [9].

1.5 Organization of the paper

The paper is organized as follows.

In Sect. 2, we quickly set up the notation used throughout the entire work and recall the elementary features of the fractional operators involved.

In Sect. 3, we carry out the general analysis of the $BV^{\alpha,p}(\mathbb{R}^n)$ space. On the one side, we deal with the approximation in energy by smooth functions and the consequent embedding theorems in Lebesgue and Bessel potential spaces, preparing the ground for the proof of Theorem 2. On the other side, we treat some integration-by-parts formulas of $BV^{\alpha,p}$ functions against rough test vector-fields and on balls, developing the tools needed for the proof of the decay estimate (1.9) and thus of Theorem 1.

In Sect. 4, we prove our first main result Theorem 1. We divide the proof into two parts, dealing with the *subcritical regime* (i) and the *supercritical regime* (ii) separately, see Proposition 4(i) and Corollary 3 respectively. At the end of this section, we provide two examples to show the sharpness of our result in the one-dimensional case $n = 1$.

In Sect. 5, after having recalled some known properties of the fractional capacity in Bessel potential spaces and having proved a localization lemma for $BV^{\alpha,p}$ functions, we end our paper with the proof of our second main result Theorem 2.

2 Preliminaries

2.1 General notation

We start with a brief description of the main notation used in this paper. In order to keep the exposition the most reader-friendly as possible, we retain the same notation adopted in the previous works [6–8].

Given an open set $\Omega \subset \mathbb{R}^n$, we say that a set E is compactly contained in Ω, and we write $E \Subset \Omega$, if the $\overline{E}$ is compact and contained in Ω. We let $\mathscr{L}^n$ and $\mathscr{H}^\alpha$ be the n-dimensional Lebesgue measure and the α-dimensional Hausdorff measure on $\mathbb{R}^n$, respectively, with $\alpha \in [0, n]$. Unless otherwise stated, a measurable set is a $\mathscr{L}^n$-measurable set. We also use the notation $|E| = \mathscr{L}^n(E)$. All functions we consider in this paper are Lebesgue measurable, unless otherwise stated. We denote by $B_r(x)$ the standard open Euclidean ball with center $x \in \mathbb{R}^n$ and radius $r > 0$. We let $B_r = B_r(0)$. For all $\beta > 0$, we set $\omega_\beta = \pi^{\frac{\beta}{2}} / \Gamma\left(\frac{\beta+2}{2}\right)$, where Γ is Euler's *Gamma function*, and we recall that $|B_1| = \omega_n$ and $\mathscr{H}^{n-1}(\partial B_1) = n\omega_n$.

For $k \in \mathbb{N}_0 \cup \{+\infty\}$ and $m \in \mathbb{N}$, we let $C_c^k(\Omega; \mathbb{R}^m)$ and $\mathrm{Lip}_c(\Omega; \mathbb{R}^m)$ be the spaces of C^k-regular and, respectively, Lipschitz-regular, m-vector-valued functions defined on $\mathbb{R}^n$ with compact support in the open set $\Omega \subset \mathbb{R}^n$.

For $m \in \mathbb{N}$, the total variation on Ω of the m-vector-valued Radon measure μ is defined as

$$|\mu|(\Omega) = \sup\left\{\int_\Omega \varphi \cdot d\mu : \varphi \in C_c^\infty(\Omega; \mathbb{R}^m),\ \|\varphi\|_{L^\infty(\Omega;\, \mathbb{R}^m)} \le 1\right\}.$$

We thus let $\mathscr{M}(\Omega; \mathbb{R}^m)$ be the space of m-vector-valued Radon measure with finite total variation on Ω. We say that $(\mu_k)_{k\in\mathbb{N}} \subset \mathscr{M}(\Omega; \mathbb{R}^m)$ *weakly converges* to $\mu \in \mathscr{M}(\Omega; \mathbb{R}^m)$, and we write $\mu_k \rightharpoonup \mu$ in $\mathscr{M}(\Omega; \mathbb{R}^m)$ as $k \to +\infty$, if

$$\lim_{k\to+\infty} \int_\Omega \varphi \cdot d\mu_k = \int_\Omega \varphi \cdot d\mu \tag{2.1}$$

for all $\varphi \in C_c^0(\Omega; \mathbb{R}^m)$. Note that we make a little abuse of terminology, since the limit in (2.1) actually defines the *weak*-convergence* in $\mathscr{M}(\Omega; \mathbb{R}^m)$.

For any exponent $p \in [1, +\infty]$, we let $L^p(\Omega; \mathbb{R}^m)$ be the space of m-vector-valued Lebesgue p-integrable functions on Ω.

We let

$$W^{1,p}(\Omega; \mathbb{R}^m) = \left\{u \in L^p(\Omega; \mathbb{R}^m) : [u]_{W^{1,p}(\Omega;\, \mathbb{R}^m)} = \|\nabla u\|_{L^p(\Omega;\, \mathbb{R}^{n+m})} < +\infty\right\}$$

be the space of m-vector-valued Sobolev functions on Ω, see for instance [18,Chapter 11] for its precise definition and main properties. We also let

$$w^{1,p}(\Omega; \mathbb{R}^m) = \left\{u \in L^p_{\mathrm{loc}}(\Omega; \mathbb{R}^m) : [u]_{W^{1,p}(\Omega;\, \mathbb{R}^m)} < +\infty\right\}.$$

We let

$$BV(\Omega; \mathbb{R}^m) = \left\{u \in L^1(\Omega; \mathbb{R}^m) : [u]_{BV(\Omega;\, \mathbb{R}^m)} = |Du|(\Omega) < +\infty\right\}$$

be the space of m-vector-valued functions of bounded variation on Ω, see for instance [3,Chapter 3] or [11,Chapter 5] for its precise definition and main properties. We also let

$$bv(\Omega; \mathbb{R}^m) = \left\{u \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^m) : [u]_{BV(\Omega;\, \mathbb{R}^m)} < +\infty\right\}.$$

For $\alpha \in (0, 1)$ and $p \in [1, +\infty)$, we let

$$W^{\alpha,p}(\Omega; \mathbb{R}^m) = \left\{u \in L^p(\Omega; \mathbb{R}^m) : [u]_{W^{\alpha,p}(\Omega;\, \mathbb{R}^m)} < +\infty\right\},$$

where

$$[u]_{W^{\alpha,p}(\Omega;\, \mathbb{R}^m)} = \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+p\alpha}}\, dx\, dy\right)^{\frac{1}{p}},$$

be the space of m-vector-valued fractional Sobolev functions on Ω, see [10] for its precise definition and main properties. We also let

$$w^{\alpha,p}(\Omega;\mathbb{R}^m)=\left\{u\in L^p_{\mathrm{loc}}(\Omega;\mathbb{R}^m):[u]_{W^{\alpha,p}(\Omega;\mathbb{R}^m)}<+\infty\right\}.$$

For $\alpha\in(0,1)$ and $p=+\infty$, we simply let

$$W^{\alpha,\infty}(\Omega;\mathbb{R}^m)=\left\{u\in L^\infty(\Omega;\mathbb{R}^m):\sup_{x,y\in\Omega,\,x\neq y}\frac{|u(x)-u(y)|}{|x-y|^\alpha}<+\infty\right\},$$

so that $W^{\alpha,\infty}(\Omega;\mathbb{R}^m)=C^{0,\alpha}_b(\Omega;\mathbb{R}^m)$, the space of m-vector-valued bounded α-Hölder continuous functions on Ω.

In order to avoid heavy notation, if the elements of a function space $F(\Omega;\mathbb{R}^m)$ are real-valued (i.e., $m=1$), then we will drop the target space and simply write $F(\Omega)$.

Given $\alpha\in(0,n)$, we let

$$I_\alpha f(x)=2^{-\alpha}\pi^{-\frac{n}{2}}\frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}\,dy,\quad x\in\mathbb{R}^n, \tag{2.2}$$

be the Riesz potential of order α of $f\in C^\infty_c(\mathbb{R}^n;\mathbb{R}^m)$. We recall that, if $\alpha,\beta\in(0,n)$ satisfy $\alpha+\beta<n$, then we have the following *semigroup property*

$$I_\alpha(I_\beta f)=I_{\alpha+\beta}f \tag{2.3}$$

for all $f\in C^\infty_c(\mathbb{R}^n;\mathbb{R}^m)$. In addition, if $1<p<q<+\infty$ satisfy $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, then there exists a constant $C_{n,\alpha,p}>0$ such that the operator in (2.2) satisfies

$$\|I_\alpha f\|_{L^q(\mathbb{R}^n;\,\mathbb{R}^m)}\le C_{n,\alpha,p}\|f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^m)} \tag{2.4}$$

for all $f\in C^\infty_c(\mathbb{R}^n;\,\mathbb{R}^m)$. As a consequence, the operator in (2.2) extends to a linear continuous operator from $L^p(\mathbb{R}^n;\mathbb{R}^m)$ to $L^q(\mathbb{R}^n;\mathbb{R}^m)$, for which we retain the same notation. For a proof of (2.3) and (2.4), we refer the reader to [34,Chapter V, Section 1] and to [14,Section 1.2.1].

Given $\alpha\in(0,1)$, we also let

$$(-\Delta)^{\frac{\alpha}{2}}f(x)=\nu_{n,\alpha}\int_{\mathbb{R}^n}\frac{f(x+y)-f(x)}{|y|^{n+\alpha}}\,dy,\quad x\in\mathbb{R}^n, \tag{2.5}$$

be the fractional Laplacian (of order α) of $f\in \mathrm{Lip}_b(\mathbb{R}^n;\mathbb{R}^m)$, where

$$\nu_{n,\alpha}=2^\alpha\pi^{-\frac{n}{2}}\frac{\Gamma\left(\frac{n+\alpha}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right)},\quad \alpha\in(0,1).$$

Finally, we let

$$Rf(x) = \pi^{-\frac{n+1}{2}}\,\Gamma\left(\tfrac{n+1}{2}\right) \lim_{\varepsilon\to 0^+} \int_{\{|y|>\varepsilon\}} \frac{y\, f(x+y)}{|y|^{n+1}}\, dy, \quad x \in \mathbb{R}^n, \tag{2.6}$$

be the (vector-valued) *Riesz transform* of a (sufficiently regular) function f. We refer the reader to [14,Sections 2.1 and 2.4.4], [34,Chapter III, Section 1] and [35,Chapter III] for a more detailed exposition. We warn the reader that the definition in (2.6) agrees with the one in [35] and differs from the one in [14, 34] for a minus sign, so that $R = \nabla I_1$ on $C_c^\infty(\mathbb{R}^n)$ in particular. The Riesz transform (2.6) is a singular integral of convolution type, thus in particular it defines a continuous operator $R\colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n;\mathbb{R}^n)$ for any given $p \in (1,+\infty)$, see [13,Corollary 5.2.8]. We also recall that its components R_i satisfy

$$\sum_{i=1}^n R_i^2 = -\mathrm{Id} \quad \text{on } L^2(\mathbb{R}^n),$$

see [13,Proposition 5.1.16].

2.2 The operators ∇^α and div^α

We briefly recall the definitions and the essential features of the non-local operators ∇^α and div^α, see [6–8, 31] and [24,Section 15.2].

Let $\alpha \in (0,1)$ and set

$$\mu_{n,\alpha} = 2^\alpha\, \pi^{-\frac{n}{2}}\, \frac{\Gamma\left(\frac{n+\alpha+1}{2}\right)}{\Gamma\left(\frac{1-\alpha}{2}\right)}.$$

We let

$$\nabla^\alpha f(x) = \mu_{n,\alpha} \lim_{\varepsilon\to 0^+} \int_{\{|y|>\varepsilon\}} \frac{y\, f(x+y)}{|y|^{n+\alpha+1}}\, dy, \quad x \in \mathbb{R}^n,$$

be the *fractional α-gradient* of $f \in \mathrm{Lip}_c(\mathbb{R}^n)$ and, similarly, we let

$$\mathrm{div}^\alpha \varphi(x) = \mu_{n,\alpha} \lim_{\varepsilon\to 0^+} \int_{\{|y|>\varepsilon\}} \frac{y \cdot \varphi(x+y)}{|y|^{n+\alpha+1}}\, dy, \quad x \in \mathbb{R}^n,$$

be the *fractional α-divergence* of $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$. The non-local operators ∇^α and div^α are well defined in the sense that the involved integrals converge and the limits exist. Moreover, since

$$\int_{\{|z|>\varepsilon\}} \frac{z}{|z|^{n+\alpha+1}}\, dz = 0, \quad \forall \varepsilon > 0,$$

it is immediate to check that $\nabla^\alpha c = 0$ for all $c \in \mathbb{R}$ and

$$\nabla^\alpha f(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(y-x)(f(y)-f(x))}{|y-x|^{n+\alpha+1}} \, dy, \quad x \in \mathbb{R}^n,$$

for all $f \in \mathrm{Lip}_c(\mathbb{R}^n)$. Analogously, we have

$$\operatorname{div}^\alpha \varphi(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(y-x) \cdot (\varphi(y)-\varphi(x))}{|y-x|^{n+\alpha+1}} \, dy, \quad x \in \mathbb{R}^n,$$

for all $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n)$. From the above expressions, it is not difficult to recognize that, given $f \in \mathrm{Lip}_c(\mathbb{R}^n)$ and $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n; \mathbb{R}^n)$, it holds that

$$\nabla^\alpha f \in L^p(\mathbb{R}^n; \mathbb{R}^n) \quad \text{and} \quad \operatorname{div}^\alpha \varphi \in L^p(\mathbb{R}^n)$$

for all $p \in [1, +\infty]$, see [7,Corollary 2.3]. Finally, the fractional operators ∇^α and $\operatorname{div}^\alpha$ are *dual*, in the sense that

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx = -\int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha f \, dx$$

for all $f \in \mathrm{Lip}_c(\mathbb{R}^n)$ and $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n; \mathbb{R}^n)$, as proved in [30,Section 6] and [7,Lemma 2.5].

With a slight abuse of notation, in the following we let ∇^1 and div^1 be the usual (local) gradient and divergence. Note that this notation is coherent with the asymptotic behavior of the fractional operators ∇^α and $\operatorname{div}^\alpha$ when $\alpha \to 1^-$ for sufficiently regular functions, see the analysis made in [8].

3 The $BV^{\alpha,p}(\mathbb{R}^n)$ space

In this section we study the main properties of the $BV^{\alpha,p}$ functions, following the strategy adopted in [7,Section 3].

3.1 Definition of $BV^{\alpha,p}(\mathbb{R}^n)$

Let $\alpha \in (0, 1]$ and $p \in [1, +\infty]$. We say that a function $f \in L^p(\mathbb{R}^n)$ belongs to the space $BV^{\alpha,p}(\mathbb{R}^n)$ if $|D^\alpha f|(\mathbb{R}^n) < +\infty$, where

$$|D^\alpha f|(\mathbb{R}^n) = \sup\left\{\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx : \varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n), \|\varphi\|_{L^\infty(\mathbb{R}^n; \mathbb{R}^n)} \le 1\right\}, \tag{3.1}$$

see [7,Section 3] for the case $p = 1$ and the discussion in [8,Section 3.3] for the case $p \in (1, +\infty]$. In the case $p = 1$, we simply write $BV^{\alpha,1}(\mathbb{R}^n) = BV^\alpha(\mathbb{R}^n)$. The resulting linear space

$$BV^{\alpha,p}(\mathbb{R}^n) = \left\{f \in L^p(\mathbb{R}^n) : |D^\alpha f|(\mathbb{R}^n) < +\infty\right\}$$

endowed with the norm

$$\|f\|_{BV^{\alpha,p}(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)} + |D^\alpha f|(\mathbb{R}^n), \quad f \in BV^{\alpha,p}(\mathbb{R}^n),$$

is a Banach space and that the fractional variation defined in (3.1) is lower semicontinuous with respect to the L^p-convergence. Similarly as it was proved in the case $p = 1$ in [7,Theorem 3.2], it is possible to show the following result relating non-local distributional gradients of $BV^{\alpha,p}$ functions to vector valued Radon measures.

Theorem 3 **(Structure Theorem for $BV^{\alpha,p}$ functions)** *Let $\alpha \in (0,1)$, $p \in [1,+\infty]$ and $f \in L^p(\mathbb{R}^n)$. Then, $f \in BV^{\alpha,p}(\mathbb{R}^n)$ if and only if there exists a finite vector valued Radon measure $D^\alpha f \in \mathscr{M}(\mathbb{R}^n;\mathbb{R}^n)$ such that*

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx = - \int_{\mathbb{R}^n} \varphi \cdot dD^\alpha f \tag{3.2}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$. In addition, for any open set $U \subset \mathbb{R}^n$ it holds

$$|D^\alpha f|(U) = \sup\left\{ \int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx : \varphi \in C_c^\infty(U;\mathbb{R}^n),\ \|\varphi\|_{L^\infty(U;\mathbb{R}^n)} \le 1 \right\}.$$

3.2 Approximation by smooth functions

Here and in the rest of the paper, we let $(\varrho_\varepsilon) \subset C_c^\infty(\mathbb{R}^n)$ be a family of standard mollifiers as in [7,Section 3.3]. The following approximation theorem is the extension to $BV^{\alpha,p}$ functions of [7,Lemma 3.5 and Theorem 3.7]. We leave its proof to the interested reader.

Theorem 4 **(Approximation by $C^\infty \cap BV^{\alpha,p}$ functions)** *Let $\alpha \in (0,1]$ and $p \in [1,+\infty]$. Let $f \in BV^{\alpha,p}(\mathbb{R}^n)$ and define $f_\varepsilon = f * \varrho_\varepsilon$ for all $\varepsilon > 0$. Then $(f_\varepsilon)_{\varepsilon>0} \subset BV^{\alpha,p}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ with $D^\alpha f_\varepsilon = (\varrho_\varepsilon * D^\alpha f)\mathscr{L}^n$ for all $\varepsilon > 0$. Moreover, the following properties hold.*

(i) If $p < +\infty$, then $f_\varepsilon \to f$ in $L^p(\mathbb{R}^n)$ as $\varepsilon \to 0^+$; if $p = +\infty$, then $f_\varepsilon \to f$ in $L^q_{\mathrm{loc}}(\mathbb{R}^n)$ as $\varepsilon \to 0^+$ for all $q \in [1,+\infty)$;
(ii) $D^\alpha f_\varepsilon \rightharpoonup D^\alpha f$ in $\mathscr{M}(\mathbb{R}^n;\mathbb{R}^n)$ and $|D^\alpha f_\varepsilon|(\mathbb{R}^n) \to |D^\alpha f|(\mathbb{R}^n)$ as $\varepsilon \to 0^+$.

The following result extends the approximation by test functions given in [7,Theorem 3.8] to functions in $BV^{\alpha,p}(\mathbb{R}^n)$ for $\alpha \in (0,1)$ and all exponents $p \in \left[1, \frac{n}{n-\alpha}\right)$. In the proof below and in the following, we let

$$\mathcal{D}^\alpha f(x) = \int_{\mathbb{R}^n} \frac{|f(x+h) - f(x)|}{|h|^{n+\alpha}} \, dh, \quad x \in \mathbb{R}^n, \tag{3.3}$$

for any $f \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^m)$, $m \in \mathbb{N}$. Note that $|\nabla^\alpha f(x)| \le \mu_{n,\alpha} \mathcal{D}^\alpha f(x)$ for all $x \in \mathbb{R}^n$ and that $\mathcal{D}^\alpha f \in L^p(\mathbb{R}^n)$ for all $p \in [1,+\infty]$.

Theorem 5 **(Approximation by C_c^∞ functions)** *Let $\alpha \in (0,1)$ and $p \in \left[1, \frac{n}{n-\alpha}\right)$. If $f \in BV^{\alpha,p}(\mathbb{R}^n)$, then there exists $(f_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ such that*

(i) $f_k \to f$ in $L^p(\mathbb{R}^n)$ as $k \to +\infty$;
(ii) $|D^\alpha f_k|(\mathbb{R}^n) \to |D^\alpha f|(\mathbb{R}^n)$ as $k \to +\infty$.

Proof Let $(\eta_R)_{R>0} \subset C_c^\infty(\mathbb{R}^n)$ be a family of cut-off functions such that

$$0 \le \eta_R \le 1, \quad \eta_R = 1 \text{ on } B_R, \quad \operatorname{supp}(\eta_R) \subset \overline{B}_{2R}, \quad \operatorname{Lip}(\eta_R) \le \frac{2}{R}.$$

We can also assume that $\eta_R(x) = \eta_1(\frac{x}{R})$ for all $x \in \mathbb{R}^n$ and $R > 0$. The proof now goes as the one of [7,Theorem 3.8] with minor modifications. We simply have to check that

$$\lim_{R\to+\infty} \int_{\mathbb{R}^n} |f(x)| \int_{\mathbb{R}^n} \frac{|\eta_R(y) - \eta_R(x)|}{|x-y|^{n+\alpha}}\, dy\, dx = 0. \tag{3.4}$$

Indeed, by Hölder's inequality, we have

$$\int_{\mathbb{R}^n} |f(x)| \int_{\mathbb{R}^n} \frac{|\eta_R(y) - \eta_R(x)|}{|x-y|^{n+\alpha}}\, dy\, dx \le \|f\|_{L^p(\mathbb{R}^n)} \|\mathcal{D}^\alpha \eta_R\|_{L^q(\mathbb{R}^n)},$$

where $\frac{1}{p} + \frac{1}{q} = 1$, and a simple change of variables shows that

$$\|\mathcal{D}^\alpha \eta_R(x)\|_{L^q(\mathbb{R}^n)} = R^{\frac{n}{q}-\alpha} \|\mathcal{D}^\alpha \eta_1\|_{L^q(\mathbb{R}^n)}$$

for all $R > 0$. The claim in (3.4) thus follows provided that $\frac{n}{q} - \alpha < 0$, which is equivalent to $p \in \left[1, \frac{n}{n-\alpha}\right)$, and the proof is complete. □

For the sake of completeness, we also treat the case $\alpha = 1$ of the previous result.

Proposition 1 *Let $n \in \mathbb{N}$ and $p \in [1, +\infty)$ be such that $p \le \frac{n}{n-1}$ for $n \ge 2$. If $f \in BV^{1,p}(\mathbb{R}^n)$, then there exists $(f_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ such that*

(i) $f_k \to f$ in $L^p(\mathbb{R}^n)$ as $k \to +\infty$;
(ii) $|Df_k|(\mathbb{R}^n) \to |Df|(\mathbb{R}^n)$ as $k \to +\infty$.

Proof Thanks to Theorem 4, we can assume $f \in C^\infty(\mathbb{R}^n) \cap BV^{1,p}(\mathbb{R}^n)$ without loss of generality. Now let $(\eta_R)_{R>0} \subset C_c^\infty(\mathbb{R}^n)$ be a family of cut-off functions as in the proof of Theorem 5. Clearly, $\eta_R f \to f$ in $L^p(\mathbb{R}^n)$ as $R \to +\infty$. Moreover, since $\nabla(\eta_R f) = \eta_R \nabla f + f \nabla \eta_R$, we thus just need to check that $\|f\, \nabla\eta_R\|_{L^1(\mathbb{R}^n;\, \mathbb{R}^n)} \to 0^+$ as $R \to +\infty$. Indeed, by Hölder's inequality, we can estimate

$$\begin{aligned}
\|f\, \nabla\eta_R\|_{L^1(\mathbb{R}^n;\, \mathbb{R}^n)} &= \int_{B_{2R}\setminus B_R} |f|\, |\nabla \eta_R|\, dx \le \frac{2}{R} \int_{B_{2R}\setminus B_R} |f|\, dx \\
&\le \frac{2}{R} |B_{2R} \setminus B_R|^{1-\frac{1}{p}} \|f\|_{L^p(B_{2R}\setminus B_R)} \\
&\le 2\, |B_2 \setminus B_1|^{1-\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^n\setminus B_R)}\, R^{n-1-\frac{n}{p}}
\end{aligned}$$

and the conclusion immediately follows. □

3.3 Gagliardo–Nirenberg–Sobolev inequality

Thanks to the approximation by test functions given by Theorem 5, we can extend [7,Theorem 3.9] and prove the analogue of the Gagliardo–Nirenberg–Sobolev inequality for the space $BV^{\alpha,p}(\mathbb{R}^n)$ whenever $p \in \left[1, \frac{n}{n-\alpha}\right)$.

Theorem 6 **(Gagliardo–Nirenberg–Sobolev inequality)** *Let $\alpha \in (0,1)$ and let $p \in \left[1, \frac{n}{n-\alpha}\right)$. There exists a constant $c_{n,\alpha} > 0$, depending on n and α only, such that*

$$\|f\|_{L^{\frac{n}{n-\alpha},r}(\mathbb{R}^n)} \le c_{n,\alpha} |D^\alpha f|(\mathbb{R}^n)$$

for all $f \in BV^{\alpha,p}(\mathbb{R}^n)$, where $r = +\infty$ if $n = 1$ and $r = 1$ if $n \ge 2$. As a consequence, $BV^{\alpha,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ continuously for all $q \in \left[p, \frac{n}{n-\alpha}\right)$, with also $q = \frac{n}{n-\alpha}$ if $n \ge 2$.

Proof Assume that $f \in C_c^\infty(\mathbb{R}^n)$ to start. Arguing as in the proof of [8,Theorem 3.8], we can estimate $|f| \le c_{n,\alpha}\, I_\alpha |\nabla^\alpha f|$ for some constant $c_{n,\alpha} > 0$ depending only on n and α (possibly varying from line to line). Thanks to the Hardy–Littlewood–Sobolev inequality, we immediately deduce that

$$\|f\|_{L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n)} \le c_{n,\alpha}\, \|\nabla^\alpha f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)}$$

for all $f \in C_c^\infty(\mathbb{R}^n)$. Moreover, if $n \ge 2$, then we can apply [32,Theorem 1.1] to the vector field $F = \nabla^\alpha f$ in order to get that

$$\|I_\alpha \nabla^\alpha f\|_{L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n;\mathbb{R}^n)} \le c_{n,\alpha}\, \|\nabla^\alpha f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)}$$

for all $f \in C_c^\infty(\mathbb{R}^n)$. Since $I_\alpha \nabla^\alpha f = Rf$ for all $f \in C_c^\infty(\mathbb{R}^n)$ and

$$R \colon L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n) \to L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n;\mathbb{R}^n)$$

strongly (recall the definition in (2.6) and the properties of the Riesz transform), we immediately deduce that

$$\|f\|_{L^{\frac{n}{n-\alpha},1}(\mathbb{R}^n)} \le c_{n,\alpha}\, \|\nabla^\alpha f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)}$$

for all $f \in C_c^\infty(\mathbb{R}^n)$, with $n \ge 2$. The conclusion then follows by combining a standard approximation argument exploiting Theorem 5 with Fatou's Lemma. □

For $\alpha = 1$, the previous result can be stated as follows.

Proposition 2 **(Alvino's inequality)** *Let $n \in \mathbb{N}$ and $p \in [1, +\infty)$. If $n \geq 2$ and $p \leq \frac{n}{n-1}$, then there exists a dimensional constant $c_n > 0$ such that*

$$\|f\|_{L^{\frac{n}{n-1},1}(\mathbb{R}^n)} \leq c_n \, |Df|(\mathbb{R}^n)$$

for all $f \in BV^{1,p}(\mathbb{R}^n)$. If $n = 1$, then $\|f\|_{L^\infty(\mathbb{R})} \leq |Df|(\mathbb{R})$ for all $f \in BV^{1,p}(\mathbb{R})$.

Proof While the case $n = 1$ is a well-known property of functions having bounded variation, the case $n \geq 2$ follows from Alvino's inequality [2] for functions in $BV(\mathbb{R}^n)$ (also see [33,Section 5]) in combination with Proposition 1. We leave the details to the interested reader. □

3.4 The space $S^{\alpha,p}(\mathbb{R}^n)$ and the embedding $BV^{\alpha,p} \subset S^{\beta,q}$

Let $\alpha \in (0, 1)$ and $p \in [1, +\infty]$. We define the *weak fractional α-gradient* of a function $f \in L^p(\mathbb{R}^n)$ as the function $\nabla^\alpha f \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$ satisfying

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx = -\int_{\mathbb{R}^n} \nabla^\alpha f \cdot \varphi \, dx$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$. We hence let the linear space

$$S^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in L^p(\mathbb{R}^n) : \exists \, \nabla^\alpha f \in L^p(\mathbb{R}^n; \mathbb{R}^n) \right\}$$

endowed with the norm

$$\|f\|_{S^{\alpha,p}(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)} + \|\nabla^\alpha f\|_{L^p(\mathbb{R}^n;\, \mathbb{R}^n)}, \quad f \in S^{\alpha,p}(\mathbb{R}^n),$$

be the *distributional fractional Sobolev space*.

As shown in [7,Proposition 3.20], $(S^{\alpha,p}(\mathbb{R}^n), \| \cdot \|_{S^{\alpha,p}(\mathbb{R}^n)})$ is a Banach space for all $p \in [1, +\infty]$. Thanks to [7,Theorem 3.23] for $p = 1$ and to [6,Theorem A.1] for $p \in (1, +\infty)$ (we refer the reader also to [17,Theorem 2.7] for a simpler proof), the set $C_c^\infty(\mathbb{R}^n)$ is dense in $S^{\alpha,p}(\mathbb{R}^n)$. As a consequence, for $p \in (1, +\infty)$ it is possible to identify $S^{\alpha,p}(\mathbb{R}^n)$ with *the fractional Bessel potential space* $L^{\alpha,p}(\mathbb{R}^n)$, see [6,Corollary 2.1] and the discussion therein.

We now want to provide a rigorous formulation of the naïve intuition that

if the order of differentiability decreases,
then the order of integrability increases,

that is to say, if $\nabla^\alpha f \in L^p(\mathbb{R}^n; \mathbb{R}^n)$ for some $\alpha \in (0, 1)$ and $p \in [1, +\infty)$, then $\nabla^\beta f \in L^q(\mathbb{R}^n; \mathbb{R}^n)$ for some *lower* fractional differentiation order $\beta < \alpha$ and some *higher* integrability exponent $q > p$ (depending on β).

For $p > 1$, the above principle is a simple consequence of the known embedding theorems between the fractional Bessel potential spaces, thanks to the aforementioned identification $S^{\alpha,p}(\mathbb{R}^n) = L^{\alpha,p}(\mathbb{R}^n)$.

The more delicate case $p = 1$ is covered in Theorem 7 below. We refer the reader also to [7,Theorem 3.32] and to [8,Propositions 3.2(i), 3.3 and 3.12] for similar results in this direction.

Theorem 7 ($BV^{\alpha,p} \subset S^{\beta,q}$ **for** $p < \frac{n}{n-\alpha}$) *Let* $\alpha, \beta \in (0,1]$, *with* $\beta < \alpha$, *and let* $p, q \in [1, +\infty]$ *be such that* $p \le q < \frac{n}{n+\beta-\alpha}$. *Then* $BV^{\alpha,p}(\mathbb{R}^n) \subset S^{\beta,q}(\mathbb{R}^n)$ *continuously.*

Proof Assume that $f \in C_c^\infty(\mathbb{R}^n)$ and let $R > 0$. Arguing as in the proof of [8,Proposition 3.12], we can estimate

$$|\nabla^\beta f|(x) \le \frac{\mu_{n,1+\beta-\alpha}}{n+\beta-\alpha}\left(\int_{|h|<R} \frac{|\nabla^\alpha f|(x+h)}{|h|^{n+\beta-\alpha}}\,dh + \left|\int_{|h|\ge R} \frac{\nabla^\alpha f(x+h)}{|h|^{n+\beta-\alpha}}\,dh\right|\right)$$

for all $x \in \mathbb{R}^n$. On the one side, we can write

$$\int_{|h|<R} \frac{|\nabla^\alpha f|(x+h)}{|h|^{n+\beta-\alpha}}\,dh = \left(\frac{\chi_{B_R}}{|\cdot|^{n+\beta-\alpha}} * |\nabla^\alpha f|\right)(x)$$

for all $x \in \mathbb{R}^n$, so that

$$\begin{aligned}\left\|\int_{|h|<R} \frac{|\nabla^\alpha f|(x+h)}{|h|^{n+\beta-\alpha}}\,dh\right\|_{L^q(\mathbb{R}^n)} &= \left\|\frac{\chi_{B_R}}{|\cdot|^{n+\beta-\alpha}} * |\nabla^\alpha f|\right\|_{L^q(\mathbb{R}^n)} \\ &\le \left(\frac{n\omega_n}{n-(n+\beta-\alpha)q}\right)^{1/q} \frac{\|\nabla^\alpha f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)}}{R^{n+\beta-\alpha-\frac{n}{q}}}\end{aligned}$$

by Young's inequality. On the other side, arguing as in the proof of [8,Proposition 3.12], we can write

$$\begin{aligned}\int_{|h|\ge R} \frac{\nabla^\alpha f(x+h)}{|h|^{n+\beta-\alpha}}\,dh &= \int_{\mathbb{R}^n} \frac{f(x+Ry)}{R^\beta}\,dD^\alpha\chi_{B_1}(y) \\ &\quad - (n+\beta-\alpha)\int_R^{+\infty} \frac{f(x+ry)}{r^{\beta+1}}\,dr\,dD^\alpha\chi_{B_1}(y)\end{aligned}$$

for all $x \in \mathbb{R}^n$, so that

$$\left\|\int_{|h|\ge R} \frac{\nabla^\alpha f(\cdot+h)}{|h|^{n+\beta-\alpha}}\,dh\right\|_{L^q(\mathbb{R}^n;\,\mathbb{R}^n)} \le c_{n,\alpha,\beta}\,R^{-\beta}\,\|f\|_{L^q(\mathbb{R}^n)}$$

by Minkowski's integral inequality, for some constant $c_{n,\alpha,\beta} > 0$ depending only on n, α and β. Hence we get that

$$\|\nabla^\beta f\|_{L^q(\mathbb{R}^n;\,\mathbb{R}^n)} \le c_{n,\alpha,\beta,q}\left(R^{\frac{n}{q}-n-\beta+\alpha}\,\|\nabla^\alpha f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} + R^{-\beta}\,\|f\|_{L^q(\mathbb{R}^n)}\right)$$

whenever $R > 0$, for some constant $c_{n,\alpha,\beta,q} > 0$ depending only on n, α, β and q. Choosing $R = \left(\frac{\|f\|_{L^q(\mathbb{R}^n)}}{\|\nabla^\alpha f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)}}\right)^{\frac{1}{\frac{n}{q}-n+\alpha}}$, we get that

$$\|\nabla^\beta f\|_{L^q(\mathbb{R}^n;\,\mathbb{R}^n)} \le c_{n,\alpha,\beta,q}\, \|f\|_{L^q(\mathbb{R}^n)}^{1-\frac{\beta q}{n-q(n-\alpha)}}\, \|\nabla^\alpha f\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)}^{\frac{\beta q}{n-q(n-\alpha)}}$$

for all $f \in C_c^\infty(\mathbb{R}^n)$. The conclusion thus follows from Theorems 5 and 6 (Propositions 1 and 2 in the case $\alpha = 1$) via a routine approximation argument, since clearly $p < \frac{n}{n-\alpha}$. □

3.5 Generalized integration-by-parts formula for $BV^{\alpha,p}$ functions

The following result is a generalization of the fractional integration-by-parts formula (3.2) (the case $p = 1$ was actually already analyzed in [8,Proposition 2.7]). This result will be useful for integrating by parts $BV^{\alpha,p}$ functions on balls, see Theorem 9 below.

Proposition 3 (**$W^{1,q} \cap C_b$-regular test**) *Let $\alpha \in (0,1)$ and let $p, q \in [1,+\infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in BV^{\alpha,p}(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx = -\int_{\mathbb{R}^n} \varphi \cdot dD^\alpha f \tag{3.5}$$

for all $\varphi \in W^{1,q}(\mathbb{R}^n;\mathbb{R}^n) \cap C_b(\mathbb{R}^n;\mathbb{R}^n)$ if $q > 1$, and for all $\varphi \in BV(\mathbb{R}^n;\mathbb{R}^n) \cap C_b(\mathbb{R}^n;\mathbb{R}^n)$ if $q = 1$.

Proof We divide the proof into two steps and adopt the same strategy of [7,Theorem 3.8] and [8,Proposition 2.7] with minor modifications.

Step 1. Assume $\varphi \in W^{1,q}(\mathbb{R}^n;\mathbb{R}^n) \cap \mathrm{Lip}_b(\mathbb{R}^n;\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ and let $(\eta_R)_{R>0} \subset C_c^\infty(\mathbb{R}^n)$ be a family of cut-off functions as in [7,Section 3.3]. On the one hand, since

$$\left|\int_{\mathbb{R}^n} f\eta_R \operatorname{div}^\alpha \varphi \, dx - \int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx\right| \le \|\operatorname{div}^\alpha \varphi\|_{L^q(\mathbb{R}^n)} \|f(1-\eta_R)\|_{L^p(\mathbb{R}^n)}$$

for all $R > 0$, by Lebesgue's Dominated Convergence Theorem we have

$$\lim_{R\to+\infty} \int_{\mathbb{R}^n} f\eta_R \operatorname{div}^\alpha \varphi \, dx = \int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx.$$

On the other hand, by [8,Lemmas 2.2 and 2.5] we can write

$$\begin{aligned}\int_{\mathbb{R}^n} f\eta_R \operatorname{div}^\alpha \varphi \, dx &= \int_{\mathbb{R}^n} f \operatorname{div}^\alpha(\eta_R\varphi)\, dx - \int_{\mathbb{R}^n} f\,\varphi \cdot \nabla^\alpha \eta_R \, dx \\ &\quad - \int_{\mathbb{R}^n} f \operatorname{div}^\alpha_{\mathrm{NL}}(\eta_R, \varphi)\, dx\end{aligned}$$

for all $R > 0$. Since $\varphi\eta_R \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$, (3.5) implies that

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha(\eta_R\varphi)\,dx = -\int_{\mathbb{R}^n} \eta_R\varphi \cdot dD^\alpha f$$

for all $R > 0$. Since

$$\left|\int_{\mathbb{R}^n} \eta_R\varphi \cdot dD^\alpha f - \int_{\mathbb{R}^n} \varphi \cdot dD^\alpha f\right| \le \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \int_{\mathbb{R}^n} (1-\eta_R)\,d|D^\alpha f|$$

for all $R > 0$, by Lebesgue's Dominated Convergence Theorem (with respect to the finite measure $|D^\alpha f|$) we have

$$\lim_{R\to+\infty} \int_{\mathbb{R}^n} \eta_R\,\varphi \cdot dD^\alpha f = \int_{\mathbb{R}^n} \varphi \cdot dD^\alpha f.$$

Finally, on the one side we can estimate

$$\left|\int_{\mathbb{R}^n} f\,\varphi \cdot \nabla^\alpha \eta_R\,dx\right| \le \mu_{n,\alpha} \int_{\mathbb{R}^n} |f(x)|\,|\varphi(x)| \int_{\mathbb{R}^n} \frac{|\eta_R(y)-\eta_R(x)|}{|y-x|^{n+\alpha}}\,dy\,dx$$

for all $R > 0$, while, on the other side,

$$\left|\int_{\mathbb{R}^n} f \operatorname{div}^\alpha_{\mathrm{NL}}(\eta_R,\varphi)\,dx\right| \le \mu_{n,\alpha} \int_{\mathbb{R}^n} |f(x)| \int_{\mathbb{R}^n} |\eta_R(y)-\eta_R(x)| \frac{|\varphi(y)-\varphi(x)|}{|y-x|^{n+\alpha}}\,dy\,dx$$

for all $R > 0$. We claim that

$$\lim_{R\to+\infty} \int_{\mathbb{R}^n} |f(x)|\,|\varphi(x)| \int_{\mathbb{R}^n} \frac{|\eta_R(y)-\eta_R(x)|}{|y-x|^{n+\alpha}}\,dy\,dx = 0. \tag{3.6}$$

Indeed, $f\varphi \in L^1(\mathbb{R}^n; \mathbb{R}^n)$ and (3.6) follows by Lebesgue's Dominated Convergence Theorem. We also claim that

$$\lim_{R\to+\infty} \int_{\mathbb{R}^n} |f(x)| \int_{\mathbb{R}^n} |\eta_R(y)-\eta_R(x)|\,\frac{|\varphi(y)-\varphi(x)|}{|y-x|^{n+\alpha}}\,dy\,dx = 0. \tag{3.7}$$

Indeed, since $\varphi \in \mathrm{Lip}_b(\mathbb{R}^n; \mathbb{R}^n)$ and $\|\eta_R\|_{L^\infty(\mathbb{R}^n)} \le 1$ for all $R > 0$, by Lebesgue's Dominated Convergence Theorem we get that

$$\lim_{R\to+\infty} \int_{\mathbb{R}^n} |\eta_R(y)-\eta_R(x)|\,\frac{|\varphi(y)-\varphi(x)|}{|y-x|^{n+\alpha}}\,dy = 0 \tag{3.8}$$

for all $x \in \mathbb{R}^n$. Moreover, for a.e. $x \in \mathbb{R}^n$ we have

$$\int_{\mathbb{R}^n} |\eta_R(y)-\eta_R(x)|\,\frac{|\varphi(y)-\varphi(x)|}{|y-x|^{n+\alpha}}\,dy \le 2\,\mathcal{D}^\alpha\varphi(x). \tag{3.9}$$

Therefore, combining (3.8) and (3.9), again by Lebesgue's Dominated Convergence Theorem we get (3.7). Thus (3.5) is proved whenever $\varphi \in W^{1,q}(\mathbb{R}^n;\mathbb{R}^n) \cap \mathrm{Lip}_b(\mathbb{R}^n;\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n;\mathbb{R}^n)$.

Step 2. Now assume $\varphi \in W^{1,q}(\mathbb{R}^n;\mathbb{R}^n) \cap C_b(\mathbb{R}^n;\mathbb{R}^n)$ (if $q = 1$, then we instead take $\varphi \in BV(\mathbb{R}^n;\mathbb{R}^n) \cap C_b(\mathbb{R}^n;\mathbb{R}^n)$) and let $(\varrho_\varepsilon)_{\varepsilon>0} \subset C_c^\infty(\mathbb{R}^n)$ be a family of standard mollifiers as in [7,Section 3.3]. Then $\varphi_\varepsilon = \varrho_\varepsilon * \varphi \in W^{1,q}(\mathbb{R}^n;\mathbb{R}^n) \cap \mathrm{Lip}_b(\mathbb{R}^n;\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ and so, by Step 1, we can write

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi_\varepsilon \, dx = -\int_{\mathbb{R}^n} \varphi_\varepsilon \cdot dD^\alpha f$$

for all $\varepsilon > 0$. On the one hand, it is not difficult to see that $\operatorname{div}^\alpha \varphi_\varepsilon = \varrho_\varepsilon * \operatorname{div}^\alpha \varphi$ for all $\varepsilon > 0$, so that

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi_\varepsilon \, dx = \int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \, dx.$$

On the other hand, since $\varphi \in C_b(\mathbb{R}^n;\mathbb{R}^n)$, we get

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} |\varphi_\varepsilon - \varphi| \, d|D^\alpha f| = 0$$

by Lebesgue's Dominated Convergence Theorem (with respect to the finite measure $|D^\alpha f|$). This concludes the proof of (3.5). □

3.6 Integrability of $\mathcal{D}^\alpha$ in Lorentz space for $BV^{1,p}$ functions

We now need to focus on the weak integrability properties of the operator $\mathcal{D}^\alpha$ defined in (3.3) when applied to functions belonging to $BV^{1,p}(\mathbb{R}^n)$. This analysis will be useful for studying the integrability properties of the fractional gradient of the characteristic function of a ball, see Corollary 1 below. This result, in turn, will be useful in the proof of the integration-by-parts formula of $BV^{\alpha,p}$ functions on balls in Theorem 9.

Theorem 8 (Weak integrability of $\mathcal{D}^\alpha$) *Let $\alpha \in (0,1)$, $p \in [1,+\infty]$ and define*

$$p_\alpha = \begin{cases} \frac{p}{1-\alpha+\alpha p} & \text{if } p \in [1,+\infty), \\ \frac{1}{\alpha} & \text{if } p = +\infty. \end{cases}$$

The operator $\mathcal{D}^\alpha \colon BV^{1,p}(\mathbb{R}^n) \to L^{p_\alpha,\infty}(\mathbb{R}^n)$ is well defined and satisfies

$$\|\mathcal{D}^\alpha f\|_{L^{p_\alpha,\infty}(\mathbb{R}^n)} \le c_{n,\alpha,p} \, \|f\|_{L^p(\mathbb{R}^n)}^{1-\alpha} |Df|(\mathbb{R}^n)^\alpha \tag{3.10}$$

for all $f \in BV^{1,p}(\mathbb{R}^n)$, where $c_{n,\alpha,p} > 0$ is a constant depending on n, α and p only.

Proof The case $p = 1$ is easy, since (3.10) holds in the following stronger form

$$\|\mathcal{D}^\alpha f\|_{L^1(\mathbb{R}^n)} \le c_{n,\alpha} \, \|f\|_{L^1(\mathbb{R}^n)}^{1-\alpha} |Df|(\mathbb{R}^n)^\alpha$$

for all $f \in BV(\mathbb{R}^n)$, whose simple proof is left to the reader (for instance, one can follow the strategy of the proof of [10,Proposition 2.2]). In the following, we thus assume that $p > 1$. We now divide the proof into three steps.

Step 1. Assume $f \in W^{1,1}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n) \cap \mathrm{Lip}_b(\mathbb{R}^n)$. By [32,Lemma 3.2], there exists a constant $C_{n,\alpha} > 0$ such that

$$\mathcal{D}^\alpha f(x) \le C_{n,\alpha} (Mf(x))^{1-\alpha} \, (M|\nabla f|(x))^\alpha$$

for all $x \in \mathbb{R}^n$. Since $f \in L^p(\mathbb{R}^n)$, by Hölder's inequality in Lorentz spaces (see [23,Theorem 3.4] and [33,Theorem 5.1]) and the well-known continuity properties of the maximal function, we get that

$$\begin{aligned}
\|\mathcal{D}^\alpha f\|_{L^{p_\alpha,\infty}(\mathbb{R}^n)} &\le C_{n,\alpha} \, \|(Mf)^{1-\alpha} \, (M|\nabla f|)^\alpha\|_{L^{p_\alpha,\infty}(\mathbb{R}^n)} \\
&\le \frac{C_{n,\alpha} \, p_\alpha}{p_\alpha - 1} \, \|(Mf)^{1-\alpha}\|_{L^{\frac{p}{1-\alpha}}(\mathbb{R}^n)} \, \|(M|\nabla f|)^\alpha\|_{L^{\frac{1}{\alpha},\infty}(\mathbb{R}^n)} \\
&\le c_{n,\alpha,p} \, \|f\|_{L^p(\mathbb{R}^n)}^{1-\alpha} \, |Df|(\mathbb{R}^n)^\alpha,
\end{aligned}$$

where $c_{n,\alpha,p} > 0$ is a constant depending only on n, α and p.

Step 2. Now assume $f \in BV(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ and define $f_\varepsilon = f * \varrho_\varepsilon$ for all $\varepsilon > 0$. Then $f_\varepsilon \in W^{1,1}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n) \cap \mathrm{Lip}_b(\mathbb{R}^n)$ with $\|f_\varepsilon\|_{L^p(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)}$ for all $\varepsilon > 0$ and $|Df_\varepsilon|(\mathbb{R}^n) \to |Df|(\mathbb{R}^n)$ as $\varepsilon \to 0^+$. Moreover, by the Fatou Lemma, we have

$$\mathcal{D}^\alpha f(x) \le \liminf_{\varepsilon \to 0^+} \mathcal{D}^\alpha f_\varepsilon(x)$$

for a.e. $x \in \mathbb{R}^n$. Hence, by [13,Exercise 1.1.1(b)], we get

$$\begin{aligned}
\|\mathcal{D}^\alpha f\|_{L^{p_\alpha,\infty}(\mathbb{R}^n)} &\le \liminf_{\varepsilon \to 0^+} \|\mathcal{D}^\alpha f_\varepsilon\|_{L^{p_\alpha,\infty}(\mathbb{R}^n)} \\
&\le c_{n,\alpha,p} \lim_{\varepsilon \to 0^+} \|f_\varepsilon\|_{L^p(\mathbb{R}^n)}^{1-\alpha} \, |Df_\varepsilon|(\mathbb{R}^n)^\alpha \\
&\le c_{n,\alpha,p} \, \|f\|_{L^p(\mathbb{R}^n)}^{1-\alpha} \, |Df|(\mathbb{R}^n)^\alpha
\end{aligned}$$

thanks to Step 1.

Step 3. Finally, assume $f \in BV^{1,p}(\mathbb{R}^n)$. Let $(\eta_R)_{R>0} \subset C_c^\infty(\mathbb{R}^n)$ be a family of cut-off functions as in [7,Section 3.3] and define $f_R = f\eta_R$ for all $R > 0$. Then $f_R \in BV(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ with $\|f_R\|_{L^p(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)}$ for all $R > 0$ and $|Df_R|(\mathbb{R}^n) \to |Df|(\mathbb{R}^n)$ as $R \to +\infty$. Moreover, by the Fatou Lemma, we have

$$\mathcal{D}^\alpha f(x) \le \liminf_{R \to +\infty} \mathcal{D}^\alpha f_R(x)$$

for a.e. $x \in \mathbb{R}^n$. Inequality (3.10) thus follows again by [13,Exercise 1.1.1(b)], thanks to Step 2. This concludes the proof. □

From Theorem 8, we immediately deduce the following integrability properties of the fractional gradient of the indicator function of a ball.

Corollary 1 **(Integrability of $\nabla^\alpha \chi_{B_r(x)}$)** *Let $\alpha \in (0,1)$ and let $p \in \left[1, \frac{1}{\alpha}\right)$. There exists a constant $c_{n,\alpha,p} > 0$, depending on n, α and p only, such that*

$$\|\nabla^\alpha \chi_{B_r(x)}\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)} = c_{n,\alpha,q}\, r^{\frac{n}{p}-\alpha} \tag{3.11}$$

for all $x \in \mathbb{R}^n$ and $r > 0$.

Proof Let $x \in \mathbb{R}^n$ and $r > 0$ be fixed. Since

$$\nabla^\alpha \chi_{B_r(x)}(y) = r^{-\alpha}(\nabla^\alpha \chi_{B_1(0)})\left(\frac{y-x}{r}\right)$$

by the rescaling property of ∇^α, we immediately get that

$$\|\nabla^\alpha \chi_{B_r(x)}\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)} = \|\nabla^\alpha \chi_{B_1}\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)}\, r^{\frac{n}{p}-\alpha}.$$

Since $\nabla^\alpha \chi_{B_1} \in L^1(\mathbb{R}^n;\mathbb{R}^n) \cap L^{\frac{1}{\alpha},\infty}(\mathbb{R}^n;\mathbb{R}^n)$ by [7,Proposition 4.8] and Theorem 8, the conclusion follows by observing that $L^1(\mathbb{R}^n) \cap L^{\frac{1}{\alpha},\infty}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ with continuous inclusion for all $p \in \left[1, \frac{1}{\alpha}\right)$. This interpolation result can be proved for instance by arguing as in the proof of [13,Proposition 1.1.14] with some minor modifications. We leave the simple details to the interested reader. □

Remark 1 *(The case $n = 1$ in Corollary 1)* The estimate in (3.11) in the case $n = 1$ can be obtained by a direct computation from the explicit formula

$$\nabla^\alpha \chi_{(x-r,\,x+r)}(y) = \frac{\mu_{1,\alpha}}{\alpha}\left(|y-x+r|^{-\alpha} - |y-x-r|^{-\alpha}\right), \quad y \in \mathbb{R},$$

given by [7,Example 4.11]. In particular, we deduce that

$$\nabla^\alpha \chi_{(x-r,\,x+r)} \in L^{\frac{1}{\alpha},\infty}(\mathbb{R};\mathbb{R}) \setminus L^{\frac{1}{\alpha},s}(\mathbb{R};\mathbb{R})$$

for all $s \in [1, +\infty)$.

From Proposition 2, we immediately deduce the following improvement of Theorem 8. We leave its simple proof to the reader.

Corollary 2 **(Improved weak integrability of $\mathcal{D}^\alpha$)** *Let $\alpha \in (0,1)$, $n \in \mathbb{N}$ and $p \in [1,+\infty)$ be such that $p \le \frac{n}{n-1}$ for $n \ge 2$. If $f \in BV^{1,p}(\mathbb{R}^n)$, then*

$$\mathcal{D}^\alpha f \in L^{\frac{p}{1-\alpha+\alpha p},\infty}(\mathbb{R}^n) \cap L^{\frac{n}{n+\alpha-1},\infty}(\mathbb{R}^n).$$

3.7 Integration by parts of $BV^{\alpha,p}$ functions on balls

We are now ready to state and prove the following integration-by-parts of $BV^{\alpha,p}$ functions on balls, which is a generalization of [7,Theorem 5.2] to $BV^{\alpha,p}$ functions for $p \in \left(\frac{1}{1-\alpha}, +\infty\right]$. This result will be the central ingredient of the proof of the decay estimates for $BV^{\alpha,p}$ functions in Theorem 10 below.

Theorem 9 (Integration by parts on balls) *Let $\alpha \in (0,1)$ and $p \in \left(\frac{1}{1-\alpha}, +\infty\right]$. If $f \in BV^{\alpha,p}(\mathbb{R}^n)$, $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, then*

$$-\int_{B_r(x)} \varphi \cdot dD^\alpha f = \int_{B_r(x)} f \operatorname{div}^\alpha \varphi \, dy + \int_{\mathbb{R}^n} f\, \varphi \cdot \nabla^\alpha \chi_{B_r(x)} \, dy + \int_{\mathbb{R}^n} f \operatorname{div}^\alpha_{\mathrm{NL}}(\chi_{B_r(x)}, \varphi)\, dy \tag{3.12}$$

for $\mathscr{L}^1$-a.e. $r > 0$.

Proof Let $x \in \mathbb{R}^n$ and $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$ be fixed. We divide the proof into two parts, dealing with the cases $p = +\infty$ and $p \in \left(\frac{1}{1-\alpha}, +\infty\right)$ separately.

Case 1: $p = +\infty$. Let $\varepsilon > 0$ and define the function $h_{\varepsilon,r,x} \in \mathrm{Lip}(\mathbb{R}^n)$ by setting

$$h_{\varepsilon,r,x}(y) = \begin{cases} 1 & \text{if } 0 \le |y-x| \le r, \\ \dfrac{r+\varepsilon-|y-x|}{\varepsilon} & \text{if } r < |y-x| < r+\varepsilon, \\ 0 & \text{if } |y-x| \ge r+\varepsilon, \end{cases}$$

for all $y \in \mathbb{R}^n$. By [7,Lemma 5.1], we know that $\nabla^\alpha h_{\varepsilon,r,x} \in L^1(\mathbb{R}^n;\mathbb{R}^n)$ with

$$\nabla^\alpha h_{\varepsilon,r,x}(y) = \frac{\mu_{n,\alpha}}{\varepsilon(n+\alpha-1)} \int_{B_{r+\varepsilon}(x)\setminus B_r(x)} \frac{x-z}{|x-z|}\, |z-y|^{1-n-\alpha}\, dz \tag{3.13}$$

for $\mathscr{L}^n$-a.e. $y \in \mathbb{R}^n$. On the one hand, since $h_{\varepsilon,r,x}\,\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$, by Proposition 3 we have

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha(h_{\varepsilon,r,x}\,\varphi)\, dy = -\int_{\mathbb{R}^n} h_{\varepsilon,r,x}\,\varphi \cdot dD^\alpha f. \tag{3.14}$$

Since $h_{\varepsilon,r,x}(y) \to \chi_{\overline{B_r(x)}}(y)$ as $\varepsilon \to 0^+$ for all $y \in \mathbb{R}^n$ and $|D^\alpha f|(\partial B_r(x)) = 0$ for $\mathscr{L}^1$-a.e. $r > 0$, we can compute

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} (h_{\varepsilon,r,x}\,\varphi) \cdot dD^\alpha f = \int_{B_r(x)} \varphi \cdot dD^\alpha f$$

for $\mathscr{L}^1$-a.e. $r > 0$. On the other hand, by [8,Lemma 2.5], we have

$$\operatorname{div}^\alpha(h_{\varepsilon,r,x}\,\varphi) = h_{\varepsilon,r,x} \operatorname{div}^\alpha \varphi + \varphi \cdot \nabla^\alpha h_{\varepsilon,r,x} + \operatorname{div}^\alpha_{\mathrm{NL}}(h_{\varepsilon,r,x}, \varphi). \tag{3.15}$$

We deal with each term of the right-hand side of (3.15) separately. For the first term, since $0 \le h_{\varepsilon,r,x} \le \chi_{B_{r+1}(x)}$ for all $\varepsilon \in (0,1)$ and $h_{\varepsilon,r,x} \to \chi_{B_r(x)}$ in $L^q(\mathbb{R}^n)$ as

$\varepsilon \to 0^+$ for any $q \in [1, +\infty)$, by [7,Corollary 2.3] and Lebesgue's Dominated Convergence Theorem we can compute

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} f\, h_{\varepsilon,r,x} \operatorname{div}^\alpha \varphi \, dy = \int_{B_r(x)} f \operatorname{div}^\alpha \varphi \, dy. \tag{3.16}$$

For the second term, by (3.13) we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} f(y)\, \varphi(y) \cdot \nabla^\alpha h_{\varepsilon,r,x}(y)\, dy \\
&\quad = \frac{\mu_{n,\alpha}}{\varepsilon(n+\alpha-1)} \int_{\mathbb{R}^n} f(y)\, \varphi(y) \cdot \int_{B_{r+\varepsilon}(x)\setminus B_r(x)} \frac{x-z}{|x-z|} |z-y|^{1-n-\alpha}\, dz\, dy.
\end{aligned}$$

By Fubini's Theorem, we can compute

$$\begin{aligned}
&\int_{\mathbb{R}^n} f(y)\, \varphi(y) \cdot \int_{B_{r+\varepsilon}(x)\setminus B_r(x)} \frac{x-z}{|x-z|} |z-y|^{1-n-\alpha}\, dz\, dy \\
&\quad = \int_{B_{r+\varepsilon}(x)\setminus B_r(x)} \frac{x-z}{|x-z|} \cdot \int_{\mathbb{R}^n} f(y)\, \varphi(y)\, |z-y|^{1-n-\alpha}\, dy\, dz \\
&\quad = \int_r^{r+\varepsilon} \int_{\partial B_\varrho(x)} \frac{x-z}{|x-z|} \cdot \int_{\mathbb{R}^n} f(y)\, \varphi(y)\, |z-y|^{1-n-\alpha}\, dy\, d\mathscr{H}^{n-1}(z)\, d\varrho.
\end{aligned}$$

By Lebesgue's Differentiation Theorem, we have

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\mathbb{R}^n} f(y)\, \varphi(y) \cdot \int_{B_{r+\varepsilon}(x)\setminus B_r(x)} \frac{x-z}{|x-z|}\, |z-y|^{1-n-\alpha}\, dz\, dy \\
&\quad = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_r^{r+\varepsilon} \int_{\partial B_\varrho(x)} \frac{x-z}{|x-z|} \cdot \int_{\mathbb{R}^n} f(y)\, \varphi(y)\, |z-y|^{1-n-\alpha}\, dy\, d\mathscr{H}^{n-1}(z)\, d\varrho \\
&\quad = \int_{\partial B_r(x)} \frac{x-z}{|x-z|} \cdot \int_{\mathbb{R}^n} f(y)\, \varphi(y)\, |z-y|^{1-n-\alpha}\, dy\, d\mathscr{H}^{n-1}(z) \\
&\quad = \int_{\mathbb{R}^n} f(y)\, \varphi(y) \cdot \int_{\partial B_r(x)} \frac{x-z}{|x-z|}\, |z-y|^{1-n-\alpha}\, d\mathscr{H}^{n-1}(z)\, dy \\
&\quad = \int_{\mathbb{R}^n} f(y)\, \varphi(y) \cdot \int_{\mathbb{R}^n} |z-y|^{1-n-\alpha}\, dD\chi_{B_r(x)}(z)\, dy
\end{aligned}$$

for $\mathscr{L}^1$-a.e. $r > 0$. Therefore, by [7,Theorem 3.18, equation (3.26)], we get that

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} f\, \varphi \cdot \nabla^\alpha h_{\varepsilon,r,x}\, dy \\
&\quad = \frac{\mu_{n,\alpha}}{n+\alpha-1} \int_{\mathbb{R}^n} f(y)\, \varphi(y) \cdot \int_{\mathbb{R}^n} |z-y|^{1-n-\alpha}\, dD\chi_{B_r(x)}(z)\, dy \\
&\quad = \int_{\mathbb{R}^n} f\, \varphi \cdot \nabla^\alpha \chi_{B_r(x)}\, dy
\end{aligned} \tag{3.17}$$

for $\mathscr{L}^1$-a.e. $r > 0$. Finally, for the third term, we note that

$$\left|\frac{(z-y)\cdot(\varphi(z)-\varphi(y))(h_{\varepsilon,r,x}(z)-h_{\varepsilon,r,x}(y))}{|z-y|^{n+\alpha+1}}\right| \le 2\,\frac{|\varphi(z)-\varphi(y)|}{|z-y|^{n+\alpha}} \in L^1_z(\mathbb{R}^n)$$

for all $y \in \mathbb{R}^n$, so that

$$\lim_{\varepsilon\to 0} \operatorname{div}^\alpha_{\mathrm{NL}}(h_{\varepsilon,r,x},\varphi)(y) = \operatorname{div}^\alpha_{\mathrm{NL}}(\chi_{B_r(x)},\varphi)(y)$$

for $\mathscr{L}^n$-a.e. $y \in \mathbb{R}^n$ by Lebesgue's Dominated Convergence Theorem. Since

$$\left|\operatorname{div}^\alpha_{\mathrm{NL}}(h_{\varepsilon,r,x},\varphi)(y)\right| \le 2\int_{\mathbb{R}^n} \frac{|\varphi(z)-\varphi(y)|}{|z-y|^{n+\alpha}}\,dz \in L^1_y(\mathbb{R}^n),$$

again by Lebesgue's Dominated Convergence Theorem we can compute

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} f\operatorname{div}^\alpha_{\mathrm{NL}}(h_{\varepsilon,r,x},\varphi)\,dy = \int_{\mathbb{R}^n} f\operatorname{div}^\alpha_{\mathrm{NL}}(\chi_{B_r(x)},\varphi)\,dy. \tag{3.18}$$

Combining (3.14), (3.15), (3.16), (3.17) and (3.18), we obtain (3.12).

Case 2: $p \in \left(\frac{1}{1-\alpha},+\infty\right)$. Let $(\varrho_\varepsilon)_{\varepsilon>0}$ be a family of standard mollifiers (see [7,Section 3.3]) and define $f_\varepsilon = f * \varrho_\varepsilon$ for all $\varepsilon > 0$. By Theorem 4, we have that

$$f_\varepsilon \in BV^{\alpha,p}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\cap C^\infty(\mathbb{R}^n) \quad \text{with} \quad D^\alpha f_\varepsilon = (\varrho_\varepsilon * D^\alpha f)\mathscr{L}^n \tag{3.19}$$

for all $\varepsilon > 0$. Hence, by Step 1, for each $\varepsilon > 0$ we have

$$\begin{aligned}-\int_{B_r(x)} \varphi\cdot dD^\alpha f_\varepsilon = \int_{B_r(x)} f_\varepsilon \operatorname{div}^\alpha\varphi\,dy + \int_{\mathbb{R}^n} f_\varepsilon\,\varphi\cdot\nabla^\alpha\chi_{B_r(x)}\,dy \\ + \int_{\mathbb{R}^n} f_\varepsilon \operatorname{div}^\alpha_{\mathrm{NL}}(\chi_{B_r(x)},\varphi)\,dy\end{aligned}$$

for $\mathscr{L}^1$-a.e. $r > 0$. We now need to study the convergence as $\varepsilon \to 0^+$ of each term of the above equality. By Theorem 4, we know that $D^\alpha f_\varepsilon \rightharpoonup D^\alpha f$ as $\varepsilon \to 0^+$, so that

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\int_{B_r(x)} \varphi\cdot dD^\alpha f_\varepsilon = \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n} \chi_{B_r(x)}\,\varphi\cdot dD^\alpha f_\varepsilon = \int_{\mathbb{R}^n}\chi_{B_r(x)}\,\varphi\cdot dD^\alpha f \\ = \int_{B_r(x)}\varphi\cdot dD^\alpha f\end{aligned}$$

for $\mathscr{L}^1$-a.e. $r > 0$ thanks to [3,Proposition 1.62]. Since $\operatorname{div}^\alpha\varphi \in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ by [7,Corollary 2.3] and $\operatorname{div}^\alpha_{\mathrm{NL}}(\chi_{B_r(x)},\varphi) \in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ by [7,Lemma 2.7 and

Remark 2.8], we have

$$\lim_{\varepsilon\to 0^+}\int_{B_r(x)} f_\varepsilon \operatorname{div}^\alpha\varphi\, dy = \int_{B_r(x)} f \operatorname{div}^\alpha\varphi\, dy$$

and

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n} f_\varepsilon \operatorname{div}^\alpha_{\mathrm{NL}}(\chi_{B_r(x)},\varphi)\, dy = \int_{\mathbb{R}^n} f \operatorname{div}^\alpha_{\mathrm{NL}}(\chi_{B_r(x)},\varphi)\, dy$$

thanks to the fact that $f_\varepsilon \to f$ in $L^p(\mathbb{R}^n)$ as $\varepsilon\to 0^+$. Finally, thanks to Corollary 1, we have $\nabla^\alpha\chi_{B_r(x)} \in L^{p'}(\mathbb{R}^n;\mathbb{R}^n)$ for any $p\in\left(\frac{1}{1-\alpha},+\infty\right)$ and so

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n} f_\varepsilon\,\varphi\cdot\nabla^\alpha\chi_{B_r(x)}\, dy \to \int_{\mathbb{R}^n} f\,\varphi\cdot\nabla^\alpha\chi_{B_r(x)}\, dy$$

again by the convergence $f_\varepsilon\to f$ in $L^p(\mathbb{R}^n)$ as $\varepsilon\to 0^+$. The conclusion thus follows. □

4 Absolute continuity of the fractional variation

In this section, we prove our first main result Theorem 1. We divide the proof into two parts, dealing with the *subcritical regime* (i) and the *supercritical regime* (ii) separately, see Proposition 4(i) and Corollary 3 respectively. At the end of this section, we provide two examples to show the sharpness of our result in the one-dimensional case $n = 1$.

4.1 The subcritical regime $p\in\left[1,\frac{n}{1-\alpha}\right)$

Thanks to [7,Lemma 3.28], if $f\in BV^\alpha(\mathbb{R}^n)$ then $u = I_{1-\alpha}f\in bv(\mathbb{R}^n)\cap L^{\frac{n}{n-1+\alpha},\infty}(\mathbb{R}^n)$, with $Du = D^\alpha f$ in $\mathscr{M}(\mathbb{R}^n;\mathbb{R}^n)$. As a consequence, we immediately deduce that $|D^\alpha f| \ll \mathscr{H}^{n-1}$ for all $f\in BV^\alpha(\mathbb{R}^n)$.

In the following result, which is a generalization of [7,Lemma 3.28] to the present setting, we show that this phenomenon is typical of the functions belonging to $BV^{\alpha,p}(\mathbb{R}^n)$ in the subcritical regime $p\in\left[1,\frac{n}{1-\alpha}\right)$.

Proposition 4 (Relation between $BV^{\alpha,p}(\mathbb{R}^n)$ and $BV^{1,p}(\mathbb{R}^n)$) *Let* $\alpha\in(0,1)$, $p\in\left(1,\frac{n}{1-\alpha}\right)$ *and* $q=\frac{np}{n-(1-\alpha)p}$.

(i) If $f\in BV^{\alpha,p}(\mathbb{R}^n)$, *then* $u = I_{1-\alpha}f\in BV^{1,q}(\mathbb{R}^n)$ *with*

$$\|u\|_{L^q(\mathbb{R}^n)} \le c_{n,\alpha,p}\,\|f\|_{L^p(\mathbb{R}^n)} \quad \textit{and} \quad Du = D^\alpha f \textit{ in } \mathscr{M}(\mathbb{R}^n;\mathbb{R}^n).$$

As a consequence, we have $|D^\alpha f|\ll\mathscr{H}^{n-1}$ *for all* $f\in BV^{\alpha,p}(\mathbb{R}^n)$ *and the operator* $I_{1-\alpha}\colon BV^{\alpha,p}(\mathbb{R}^n)\to BV^{1,q}(\mathbb{R}^n)$ *is continuous.*

(ii) If $p \in \left(1, \frac{n}{n-\alpha}\right)$ *and* $u \in BV^{1,p}(\mathbb{R}^n)$*, then* $f = (-\Delta)^{\frac{1-\alpha}{2}} u \in BV^{\alpha,p}(\mathbb{R}^n)$ *with*

$$\|f\|_{L^p(\mathbb{R}^n)} \le c_{n,\alpha,p} \|u\|_{BV^{\alpha,p}(\mathbb{R}^n)} \quad \text{and} \quad D^\alpha f = Du \text{ in } \mathscr{M}(\mathbb{R}^n; \mathbb{R}^n).$$

As a consequence, the operator $(-\Delta)^{\frac{1-\alpha}{2}} : BV^{1,p}(\mathbb{R}^n) \to BV^{\alpha,p}(\mathbb{R}^n)$ *is continuous.*

Proof Let $s = \frac{p}{p-1}$ and note that $r = \frac{ns}{n+(1-\alpha)s} \in \left(1, \frac{n}{1-\alpha}\right)$. We prove the two properties separately.

Proof of (i). Let $f \in BV^{\alpha,p}(\mathbb{R}^n)$. By the Hardy–Littlewood–Sobolev inequality, we immediately get that

$$u = I_{1-\alpha} f \in L^q(\mathbb{R}^n).$$

Given $\varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$, we clearly have $I_{1-\alpha}|\mathrm{div}\varphi| \in L^s(\mathbb{R}^n)$, because $|\mathrm{div}\varphi| \in L^r(\mathbb{R}^n)$. Hence, by Fubini Theorem, we have

$$\int_{\mathbb{R}^n} f \,\mathrm{div}^\alpha \varphi \, dx = \int_{\mathbb{R}^n} f \, I_{1-\alpha} \mathrm{div}\varphi \, dx = \int_{\mathbb{R}^n} u \,\mathrm{div}\varphi \, dx \tag{4.1}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$, proving that $D^\alpha f = Du$ in $\mathscr{M}(\mathbb{R}^n; \mathbb{R}^n)$. The remaining part of the statement in (i) follows easily.

Proof of (ii). Let $p \in \left(1, \frac{n}{n-\alpha}\right)$ and $u \in BV^{1,p}(\mathbb{R}^n)$. Since $p < \frac{n}{n-\alpha}$, we can apply Theorem 7 to get that $BV^{1,p}(\mathbb{R}^n) \subset S^{1-\alpha,p}(\mathbb{R}^n)$ with continuous inclusion, so that $f = (-\Delta)^{\frac{1-\alpha}{2}} u \in L^p(\mathbb{R}^n)$ (thanks to the identification $S^{1-\alpha,p}(\mathbb{R}^n) = L^{1-\alpha,p}(\mathbb{R}^n)$ following from [6,Corollary 2.1], also see the discussion in [6,Section 2.1]) and thus $I_{1-\alpha} f \in L^q(\mathbb{R}^n)$ by the Hardy–Littlewood–Sobolev inequality. Since $p < \frac{n}{n-\alpha}$, we also have that $p < q < \frac{n}{n-1}$ and thus $BV^{1,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ with continuous inclusion by Proposition 2. Hence $u \in L^q(\mathbb{R}^n)$ and we can now claim that $I_{1-\alpha} f = u$ in $L^q(\mathbb{R}^n)$. Indeed, this is easily verified if $u \in C_c^\infty(\mathbb{R}^n)$ by applying the Fourier transform (see [19,Lemma 2.3] for instance), so that the claim follows by a plain approximation argument. Therefore, by applying Fubini Theorem again, we can write (4.1) and prove that $D^\alpha f = Du$ in $\mathscr{M}(\mathbb{R}^n; \mathbb{R}^n)$, reaching the conclusion. □

Remark 2 *(About Proposition* 4(ii)*)* The validity of Proposition 4(ii) when $p = 1$ was already proved in [7,Lemma 3.28]. We also refer the reader to [17,Proposition 3.1], in which the authors prove that, if $u \in W^{1,p}(\mathbb{R}^n)$ for some $p \in [1, +\infty]$, then $f = (-\Delta)^{\frac{1-\alpha}{2}} u \in S^{\alpha,p}(\mathbb{R}^n)$ with $\nabla^\alpha f = \nabla u$ in $L^p(\mathbb{R}^n; \mathbb{R}^n)$.

4.2 The supercritical regime $p \in \left[\frac{n}{1-\alpha}, +\infty\right]$

We now focus on the absolute continuity property of the fractional variation with respect to the Hausdorff measure for functions belonging to $BV^{\alpha,p}(\mathbb{R}^n)$ in the supercritical regime $p \in \left[\frac{n}{1-\alpha}, +\infty\right]$. The crucial tool in this case is provided by the following important consequence of Theorem 9, which extends [7,Theorem 5.3] to the present setting.

Theorem 10 (Decay estimates for $BV^{\alpha,p}$ functions for $p > \frac{1}{1-\alpha}$) *Let $\alpha \in (0,1)$ and $p \in \left(\frac{1}{1-\alpha}, +\infty\right]$. There exist two constants $A_{n,\alpha,p}, B_{n,\alpha,p} > 0$, depending on n, α and p only, with the following property. If $f \in BV^{\alpha,p}(\mathbb{R}^n)$ then, for $|D^\alpha f|$-a.e. $x \in \mathbb{R}^n$, there exists $r_x > 0$ such that*

$$|D^\alpha f|(B_r(x)) \le A_{n,\alpha,p} \|f\|_{L^p(\mathbb{R}^n)}\, r^{\frac{n}{q}-\alpha} \tag{4.2}$$

and

$$|D^\alpha (f\chi_{B_r(x)})|(\mathbb{R}^n) \le B_{n,\alpha,p} \|f\|_{L^p(\mathbb{R}^n)}\, r^{\frac{n}{q}-\alpha} \tag{4.3}$$

for all $r \in (0, r_x)$, where $q \in [1, +\infty)$ is such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof Since $f \in BV^{\alpha,p}(\mathbb{R}^n)$, by the Polar Decomposition Theorem for Radon measures there exists a Borel vector valued function $\sigma_f^\alpha \colon \mathbb{R}^n \to \mathbb{R}^n$ such that

$$D^\alpha f = \sigma_f^\alpha\, |D^\alpha f| \quad \text{with} \quad |\sigma_f^\alpha(x)| = 1 \text{ for } |D^\alpha f|\text{-a.e. } x \in \mathbb{R}^n. \tag{4.4}$$

We divide the proof into two steps, dealing with the two estimates separately.

Step 1: Proof of (4.2). Let $\sigma_f^\alpha : \mathbb{R}^n \to \mathbb{R}^n$ be as in (4.4) and let $x \in \mathbb{R}^n$ be such that $|\sigma_f^\alpha(x)| = 1$. Given $r > 0$, we define the vector field $\varphi \colon \mathbb{R}^n \to \mathbb{R}^n$ by setting

$$\varphi_{x,r}(y) = \begin{cases} \sigma_f^\alpha(x) & \text{if } y \in B_r(x), \\ \sigma_f^\alpha(x)\left(2 - \frac{|y-x|}{r}\right) & \text{if } y \in B_{2r}(x) \setminus B_r(x), \\ 0 & \text{if } y \notin B_{2r}(x), \end{cases} \tag{4.5}$$

for all $y \in \mathbb{R}^n$. We clearly have that $\varphi_{x,r} \in \mathrm{Lip}_c(\mathbb{R}^n; \mathbb{R}^n)$ with $\|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1$. Thus, on the one hand, we can find $r_x \in (0,1)$ such that

$$\int_{B_r(x)} \varphi_{x,r}(y) \cdot dD^\alpha f(y) = \int_{B_r(x)} \sigma_f^\alpha(x) \cdot \sigma_f^\alpha(y)\, d|D^\alpha f|(y) \ge \frac{1}{2} |D^\alpha f|(B_r(x)) \tag{4.6}$$

for all $r \in (0, r_x)$. On the other hand, by (3.12) we can write

$$\begin{aligned} \int_{B_r(x)} \varphi_{x,r} \cdot dD^\alpha f \le \left| \int_{B_r(x)} f \operatorname{div}^\alpha \varphi_{x,r}\, dy \right| + \left| \int_{\mathbb{R}^n} f\, \varphi_{x,r} \cdot \nabla^\alpha \chi_{B_r(x)}\, dx \right| \\ + \left| \int_{\mathbb{R}^n} f \operatorname{div}^\alpha_{\mathrm{NL}}(\chi_{B_r(x)}, \varphi_{x,r})\, dy \right| \end{aligned} \tag{4.7}$$

for $\mathscr{L}^1$-a.e. $r \in (0, r_x)$. We now estimate the three terms in the right-hand side of (4.7) separately. For the first term, recalling the definition of $\varphi_{x,r}$ in (4.5), we have

$$\begin{aligned}
\left|\int_{B_r(x)} f \operatorname{div}^\alpha \varphi_{x,r}\, dy\right| &\le \mu_{n,\alpha} \int_{B_r(x)} |f(y)| \int_{\mathbb{R}^n \setminus B_r(x)} \frac{|\varphi_{x,r}(z) - \sigma_f^\alpha(x)|}{|z-y|^{n+\alpha}}\, dz dy \\
&\le \mu_{n,\alpha} \|f\|_{L^p(B_r(x))} \left(\int_{B_r(x)} \left(\int_{\mathbb{R}^n \setminus B_r(x)} \frac{|\varphi_{x,r}(z) - \sigma_f^\alpha(x)|}{|z-y|^{n+\alpha}}\, dz\right)^q dy\right)^{\frac{1}{q}} \\
&\le 2\mu_{n,\alpha} \|f\|_{L^p(B_r(x))}\, r^{\frac{n}{q}-\alpha} \left(\int_{B_1} \left(\int_{\mathbb{R}^n \setminus B_1} \frac{1}{|z-y|^{n+\alpha}}\, dz\right)^q dy\right)^{\frac{1}{q}}.
\end{aligned}$$

After some elementary computations, we get

$$\left(\int_{\mathbb{R}^n \setminus B_1(-y)} \frac{1}{|z|^{n+\alpha}}\, dz\right)^q dy \le C_{n,\alpha} \int_0^1 \frac{t^{n-1}}{(1-t)^{\alpha q}}\, dt \tag{4.8}$$

for some constant $C_{n,\alpha} > 0$ depending only on n and α. Note that the integral appearing in the right-hand side of (4.8) converges if and only if $\alpha q < 1$, that is, $p > \frac{1}{1-\alpha}$. We thus get

$$\left|\int_{B_r(x)} f \operatorname{div}^\alpha \varphi_{x,r}\, dy\right| \le C_{n,\alpha,q}\, \|f\|_{L^p(\mathbb{R}^n)}\, r^{\frac{n}{q}-\alpha} \tag{4.9}$$

for some constant $C_{n,\alpha,q} > 0$ depending only on n, α and q. For the second term in the right-hand side of (4.7), we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} f \varphi_{x,r} \cdot \nabla^\alpha \chi_{B_r(x)}\, dy\right| &\le \|f\|_{L^p(B_{2r}(x))} \|\nabla^\alpha \chi_{B_r(x)}\|_{L^q(\mathbb{R}^n;\mathbb{R}^n)} \\
&\le C_{n,\alpha,q} \|f\|_{L^p(B_{2r}(x))}\, r^{\frac{n}{q}-\alpha}
\end{aligned} \tag{4.10}$$

thanks to Corollary 1, for some constant $C_{n,\alpha,q} > 0$ depending only on n, α and q. Finally, observing that $\varphi_{x,r}(x + ry) = \varphi_{0,1}(y)$ for all $y \in \mathbb{R}^n$, a simple change of variables gives

$$\|\operatorname{div}^\alpha_{\mathrm{NL}}(\chi_{B_r(x)}, \varphi_{x,r})\|_{L^q(\mathbb{R}^n)} = r^{\frac{n}{q}-\alpha}\, \|\operatorname{div}^\alpha_{\mathrm{NL}}(\chi_{B_1}, \varphi)\|_{L^q(\mathbb{R}^n)}. \tag{4.11}$$

Thus, for the third and last term in the right-hand side of (4.7), we have

$$\left|\int_{\mathbb{R}^n} f \operatorname{div}^\alpha_{\mathrm{NL}}(\chi_{B_r(x)}, \varphi_{x,r})\, dy\right| \le C_{n,\alpha,q}\, \|f\|_{L^p(\mathbb{R}^n)} r^{\frac{n}{q}-\alpha}, \tag{4.12}$$

where $C_{n,\alpha,q} = \|\operatorname{div}^\alpha_{\mathrm{NL}}(\chi_{B_1}, \varphi)\|_{L^q(\mathbb{R}^n)}$ (which is finite thanks to [7,Lemma 2.7 and Remark 2.8]). Combining (4.6) with (4.7), (4.9), (4.10) and (4.12), we get (4.2) with a simple continuity argument.

Step 2: Proof of (4.3). Let $x \in \mathbb{R}^n$ be such that $|\sigma_u^\alpha(x)| = 1$. Given $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n; \mathbb{R}^n)$ such that $\|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1$, from (3.12) we deduce that

$$\left| \int_{B_r(x)} f \operatorname{div}^\alpha \varphi \, dy \right| \le |D^\alpha f|(B_r(x)) + \|f\|_{L^p(\mathbb{R}^n)} \|\nabla^\alpha \chi_{B_r(x)}\|_{L^q(\mathbb{R}^n;\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)} \|\operatorname{div}^\alpha_{\mathrm{NL}}(\chi_{B_r(x)}, \varphi)\|_{L^q(\mathbb{R}^n)}$$

for $\mathscr{L}^1$-a.e. $r \in (0, r_x)$. Exploiting (4.2), (3.11) and (4.11), we conclude that

$$|D^\alpha (f \chi_{B_r(x)})|(\mathbb{R}^n) \le C_{n,\alpha,q} \, r^{\frac{n}{q}-\alpha}$$

for $\mathscr{L}^1$-a.e. $r \in (0, r_x)$, where $C_{n,\alpha,q} > 0$ is a constant depending only on n, α and q. Inequality (4.3) thus follows for all $r \in (0, r_x)$ by a simple continuity argument. □

Thanks to Theorem 10 and extending [7,Corollary 5.4] to the present setting, we are now ready to state and prove the following absolute continuity property of the fractional variation for $BV^{\alpha,p}$ functions with $p \in \left(\frac{1}{1-\alpha}, +\infty\right]$. Note that the result below is truly interesting only for $p \in \left[\frac{n}{1-\alpha}, +\infty\right]$, due to Theorem 1(i) (see also Proposition 4) and the fact that

$$n - \alpha - \frac{n}{p} \ge n - 1 \iff p \ge \frac{n}{1-\alpha}.$$

Corollary 3 ($|D^\alpha f| \ll \mathscr{H}^{\frac{n}{q}-\alpha}$ **for** $p > \frac{1}{1-\alpha}$) *Let* $\alpha \in (0,1)$ *and* $p \in \left(\frac{1}{1-\alpha}, +\infty\right]$. *If* $f \in BV^{\alpha,p}(\mathbb{R}^n)$, *then there exists a* $|D^\alpha f|$*-negligible set* $Z_f^{\alpha,p} \subset \mathbb{R}^n$ *such that*

$$|D^\alpha f| \le 2^{\frac{n}{q}-\alpha} \frac{A_{n,\alpha,p}}{\omega_{\frac{n}{q}-\alpha}} \|f\|_{L^p(\mathbb{R}^n)} \, \mathscr{H}^{\frac{n}{q}-\alpha} \llcorner \mathbb{R}^n \setminus Z_f^{\alpha,p}, \tag{4.13}$$

where $A_{n,\alpha,p}$ *is as in* (4.2) *and* $q \in [1, +\infty)$ *is such that* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof By Theorem 10, there exists a set $Z_f^{\alpha,p} \subset \mathbb{R}^n$ such that $|D^\alpha f|(Z_f^{\alpha,p}) = 0$ and (4.2) holds for any $x \notin Z_f^{\alpha,p}$. Thanks to the Borel regularity of the Radon measure $|D^\alpha f|$, we can assume that $Z_f^{\alpha,p}$ is a Borel set without loss of generality. Hence, for all $x \in \mathbb{R}^n \setminus Z_f^{\alpha,p}$, we have

$$\Theta^*_{\frac{n}{q}-\alpha}(|D^\alpha f|, x) = \limsup_{r \to 0^+} \frac{|D^\alpha f|(B_r(x))}{\omega_{\frac{n}{q}-\alpha} r^{\frac{n}{q}-\alpha}} \le \frac{A_{n,\alpha,p}}{\omega_{\frac{n}{q}-\alpha}} \|f\|_{L^p(\mathbb{R}^n)}.$$

Inequality (4.13) thus follows from [3,Theorem 2.56]. □

Remark 3 *(The case $n=1$ and $p=\frac{1}{1-\alpha}$)* Note that Corollary 3 covers the supercritical regime $p\in\left[\frac{n}{1-\alpha},+\infty\right]$ for $n\geq 2$, while for $n=1$ the boundary case $p=\frac{1}{1-\alpha}$ is missing. However, if $n=1$ and $p=\frac{1}{1-\alpha}$, then $q=\frac{p}{p-1}=\frac{1}{\alpha}$ and so $\mathscr{H}^{\frac{n}{q}-\alpha}=\mathscr{H}^0$, so that $|D^\alpha f|\ll\mathscr{H}^0$ for all $f\in BV^{\alpha,\frac{1}{1-\alpha}}(\mathbb{R})$ trivially. We do not know if this result is sharp.

Remark 4 *(The limit as $\alpha\to 1^-$)* It is somewhat interesting to observe that Corollary 3 still holds true if we send $\alpha\to 1^-$. Indeed, such a limit case would apply only to functions $f\in BV^{1,\infty}(\mathbb{R}^n)$, for which it is well known (see [3,Theorem 3.77, Theorem 3.78 and equation (3.90)], for instance) that

$$|Df|\leq 2\|f\|_{L^\infty(\mathbb{R}^n)}\mathscr{H}^{n-1}\llcorner J_f,$$

where J_f is the jump set, so that $Z_f^{1,\infty}$ could be any $|Df|$-negligible subset of $\mathbb{R}^n\setminus J_f$.

4.3 Two examples in one dimension

We conclude this section by discussing the optimality of the absolute continuity properties of the fractional variation stated in Theorem 1 in the one-dimensional case $n=1$.

We begin with the following example, which is borrowed from [7,Theorem 3.26].

Example 1 *(Proposition 4(i) is sharp for $n=1$)* Let $\alpha\in(0,1)$ and consider

$$f_\alpha(x)=\mu_{1,-\alpha}\left(|x|^{\alpha-1}\operatorname{sgn}x-|x-1|^{\alpha-1}\operatorname{sgn}(x-1)\right),\quad x\in\mathbb{R}.$$

By [7,Theorem 3.26], we have $f_\alpha\in BV^\alpha(\mathbb{R})$ with $D^\alpha f_\alpha=\delta_0-\delta_1$. Moreover, by [8,Theorem 3.8 and Remark 3.9], we have $f_\alpha\in L^{\frac{1}{1-\alpha},\infty}(\mathbb{R})\setminus L^{\frac{1}{1-\alpha},q}(\mathbb{R})$ for all $q\geq 1$. In particular, since $f_\alpha\in L^1(\mathbb{R})\cap L^{\frac{1}{1-\alpha},\infty}(\mathbb{R})$, by interpolation we get that $f_\alpha\in L^p(\mathbb{R})$ for all $p\in\left[1,\frac{1}{1-\alpha}\right)$. Hence $f_\alpha\in BV^{\alpha,p}(\mathbb{R})$ for all $p\in\left[1,\frac{1}{1-\alpha}\right)$ with $|D^\alpha f_\alpha|\not\ll\mathscr{H}^\varepsilon$ for all $\varepsilon>0$. This proves that the absolute continuity property of the fractional variation stated in Theorem 1(i) is sharp for $n=1$.

We now prove the following result, which combines the properties of the function f_α introduced in Example 1 with the decay properties of a finite Radon measure.

Proposition 5 **(The function $u_\alpha=f_\alpha*\nu$)** *Let $\alpha\in(0,1)$, let f_α be as in Example 1, and let $\nu\in\mathscr{M}(\mathbb{R})$. Then we have*

$$u_\alpha=f_\alpha*\nu\in BV^{\alpha,p}(\mathbb{R})\quad\textit{for all } p\in\left[1,\frac{1}{1-\alpha}\right),$$

with

$$D^\alpha u_\alpha=\nu-(\tau_1)_\#\nu, \tag{4.14}$$

where $\tau_x(y) = y + x$ for all $x, y \in \mathbb{R}$. In addition, if there exist $C, \varepsilon > 0$ such that

$$\nu\big((x-r, x+r)\big) \le C r^{\varepsilon} \quad \text{for all } x \in \mathbb{R} \text{ and } r > 0, \tag{4.15}$$

then

$$u_\alpha \in BV^{\alpha,p}(\mathbb{R}) \quad \text{for all } p \in \begin{cases} \left[1, \frac{1-\varepsilon}{1-\alpha-\varepsilon}\right) & \text{if } \varepsilon \in (0, 1-\alpha), \\ [1, +\infty) & \text{if } \varepsilon = 1-\alpha, \\ [1, +\infty] & \text{if } \varepsilon \in (1-\alpha, 1]. \end{cases} \tag{4.16}$$

Proof We divide the proof into two steps.

Step 1. Let $\nu \in \mathscr{M}(\mathbb{R})$. We start by showing that $u_\alpha \in BV^{\alpha,p}(\mathbb{R})$ for all $p \in \left[1, \frac{1}{1-\alpha}\right)$ and that it satisfies (4.14). Indeed, by Young's inequality (for Radon measures) we can estimate

$$\|u_\alpha\|_{L^1(\mathbb{R})} \le \|f_\alpha\|_{L^1(\mathbb{R})} |\nu|(\mathbb{R}).$$

Moreover, thanks to the translation invariance of div^α and exploiting the explicit expression of f_α given in Example 1, we can write

$$\begin{aligned}
\int_{-\infty}^{\infty} u_\alpha(x)\,\mathrm{div}^\alpha \varphi(x)\,dx &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_\alpha(x-y)\,\mathrm{div}^\alpha\varphi(x)\,d\nu(y)\,dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_\alpha(x-y)\,\mathrm{div}^\alpha\varphi(x)\,dx\,d\nu(y) \\
&= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(x+y)\,d\left(\delta_0(x) - \delta_1(x)\right)\,d\nu(y) \\
&= -\int_{-\infty}^{\infty} \left(\varphi(y) - \varphi(y+1)\right)\,d\nu(y)
\end{aligned}$$

for all $\varphi \in C_c^\infty(\mathbb{R})$. Thus $u_\alpha \in BV^{\alpha,1}(\mathbb{R})$ with $D^\alpha u_\alpha = \nu - (\tau_1)_\# \nu$. In addition, by Jensen's inequality and Tonelli's Theorem we can estimate

$$\begin{aligned}
\int_{-\infty}^{\infty} |u_\alpha(x)|^p\,dx &\le \int_{-\infty}^{\infty} |\nu|(\mathbb{R})^{p-1}\int_{-\infty}^{\infty} |f_\alpha(x-y)|^p\,d|\nu|(y)\,dx \\
&= |\nu|(\mathbb{R})^p\,\|f_\alpha\|_{L^p(\mathbb{R})}^p < +\infty
\end{aligned}$$

for all $p \in \left[1, \frac{1}{1-\alpha}\right)$, thanks to the integrability properties of f_α given in Example 1.

Step 2. We prove that (4.15) implies (4.16). To this aim, let $\delta > 0$ and $q = \frac{p}{p-1}$. Since $|f_\alpha| = |f_\alpha|^{\frac{\delta}{q}} |f_\alpha|^{1-\frac{\delta}{q}}$, by Hölder's inequality we get

$$\begin{aligned} |u_\alpha(x)|^p &\leq \left(\int_{-\infty}^{\infty} |f_\alpha(x-y)|^{\frac{\delta}{q}} \, |f_\alpha(x-y)|^{1-\frac{\delta}{q}} \, d|\nu|(y) \right)^p \\ &\leq \left(\int_{-\infty}^{\infty} |f_\alpha(x-y)|^{\delta} \, d|\nu|(y) \right)^{\frac{p}{q}} \left(\int_{-\infty}^{\infty} |f_\alpha(x-y)|^{p\left(1-\frac{\delta}{q}\right)} \, d|\nu|(y) \right) \end{aligned}$$

for $x \in \mathbb{R}$. We now recall the explicit expression of f_α in Example 1 and write

$$\begin{aligned} \int_{-\infty}^{\infty} |f_\alpha(x-y)|^{\delta} \, d|\nu|(y) &= \int_{\left(-\infty, x-\frac{3}{2}\right) \cup \left(x+\frac{1}{2}, \infty\right)} |f_\alpha(x-y)|^{\delta} \, d|\nu|(y) \\ &\quad + \sum_{j=1}^{\infty} \int_{I_j\left(x, \frac{1}{2}\right) \cup I_j\left(x-1, \frac{1}{2}\right)} |f_\alpha(x-y)|^{\delta} \, d|\nu|(y), \end{aligned} \tag{4.17}$$

where we have set

$$I_j(x, r) = (x - r^j, x + r^j) \setminus (x - r^{j+1}, x + r^{j+1})$$

for all $x \in \mathbb{R}$, $r \in (0, 1)$ and $j \in \mathbb{N}$ for brevity. Now, on the one hand, if $y \in \left(-\infty, x - \frac{3}{2}\right) \cup \left(x + \frac{1}{2}, \infty\right)$, then $x - y \in \left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{3}{2}, \infty\right)$, so that

$$|f_\alpha(x-y)| \leq \mu_{1,-\alpha} \left(2^{1-\alpha} + 2^{1-\alpha}\right) = \mu_{1,-\alpha} 2^{2-\alpha}$$

for all $y \in \left(-\infty, x - \frac{3}{2}\right) \cup \left(x + \frac{1}{2}, \infty\right)$. Therefore, we can estimate

$$\int_{\left(-\infty, x-\frac{3}{2}\right) \cup \left(x+\frac{1}{2}, \infty\right)} |f_\alpha(x-y)|^{\delta} \, d|\nu|(y) \leq \left(\mu_{1,-\alpha} 2^{2-\alpha}\right)^{\delta} |\nu|(\mathbb{R}) \tag{4.18}$$

for all $x \in \mathbb{R}$. On the other hand, for all $x \in \mathbb{R}$ and $j \in \mathbb{N}$, we have

$$\int_{I_j\left(x, \frac{1}{2}\right)} |f_\alpha(x-y)|^{\delta} \, d|\nu|(y) \tag{4.19}$$

$$\begin{aligned} &\leq \mu_{1,-\alpha}^{\delta} \int_{I_j\left(x, \frac{1}{2}\right)} \left(|x-y|^{\alpha-1} + |x-y-1|^{\alpha-1} \right)^{\delta} d|\nu|(y) \\ &\leq \mu_{1,-\alpha}^{\delta} \left(2^{(j+1)(1-\alpha)} + \left(1 - 2^{-j}\right)^{\alpha-1} \right)^{\delta} |\nu| \left((x - 2^{-j}, x + 2^{j}) \right) \\ &\leq \mu_{1,-\alpha}^{\delta} \left(2^{(j+1)(1-\alpha)} + 2^{1-\alpha} \right)^{\delta} C \, 2^{-j\varepsilon}. \end{aligned} \tag{4.20}$$

Reasoning analogously, we obtain

$$\int_{I_j\left(x-1,\frac{1}{2}\right)} |f_\alpha(x-y)|^\delta \, d|\nu|(y) \le C\mu_{1,-\alpha}^\delta \left(2^{(j+1)(1-\alpha)} + 2^{1-\alpha}\right)^\delta 2^{-j\varepsilon} \tag{4.21}$$

for all $x \in \mathbb{R}$ and $j \in \mathbb{N}$. Therefore, inserting (4.18), (4.20) and (4.21) in (4.17), we conclude that

$$\int_{-\infty}^{\infty} |f_\alpha(x-y)|^\delta \, d|\nu|(y) \le C_{\alpha,\varepsilon,\delta} \tag{4.22}$$

for all $x \in \mathbb{R}$, where $C_{\alpha,\varepsilon,\delta} > 0$ is constant depending on α, ε, and δ which is finite provided that we choose $\delta < \frac{\varepsilon}{1-\alpha}$, as we are assuming from now on. We thus get

$$\begin{aligned}\int_{-\infty}^{\infty} |u_\alpha(x)|^p \, dx &\le C_{\alpha,\varepsilon,\delta}^{p-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f_\alpha(x-y)|^{p\left(1-\frac{\delta}{q}\right)} \, d|\nu|(y)\, dx \\ &= C_{\alpha,\varepsilon,\delta}^{p-1} |\nu|(\mathbb{R}) \int_{-\infty}^{\infty} |f_\alpha(x)|^{p\left(1-\frac{\delta}{q}\right)} \, dx.\end{aligned}$$

Now, recalling Example 1, we immediately see that

$$\int_{-\infty}^{\infty} |f_\alpha(x)|^{p\left(1-\frac{\delta}{q}\right)} \, dx < +\infty \iff \begin{cases} p < \frac{1}{(1-\alpha)(1-\delta)} - \frac{\delta}{1-\delta} = \frac{1-\delta+\alpha\delta}{(1-\alpha)(1-\delta)}, \\ p > \frac{1}{(2-\alpha)(1-\delta)} - \frac{\delta}{1-\delta} = \frac{1-2\delta+\alpha\delta}{(2-\alpha)(1-\delta)}. \end{cases} \tag{4.23}$$

Since one easily recognizes that

$$\frac{1-2\delta+\alpha\delta}{(2-\alpha)(1-\delta)} < 1 \quad \text{for all } \alpha \in (0,1) \text{ and } \delta > 0,$$

the second condition on p in (4.23) can be dropped. As for the first condition on p in (4.23), it is readily seen that

$$\varepsilon \in (0, 1-\alpha) \implies \delta < \frac{\varepsilon}{1-\alpha} < 1 \implies p \in \left[1, \frac{1-\varepsilon}{1-\alpha-\varepsilon}\right)$$

and, similarly,

$$\varepsilon \in [1-\alpha, 1] \implies \delta(1-\alpha) < \varepsilon \text{ for all } \delta \in (0,1) \implies p \in [1, +\infty).$$

Finally, in the case $\varepsilon \in (1-\alpha, 1]$, we exploit (4.22) for $\delta = 1$ in order to conclude that

$$|u_\alpha(x)| \le \int_{-\infty}^{\infty} |f_\alpha(x-y)| \, d|\nu|(y) = C_{\alpha,\varepsilon} < +\infty$$

for all $x \in \mathbb{R}$, which implies that $u_\alpha \in L^\infty(\mathbb{R})$. The conclusion thus follows. □

Thanks to Proposition 5, we can now give the following example.

Example 2 *(Corollary 3 is sharp for $n = 1$)* Let $\alpha \in (0, 1)$ and let ν and u_α be as in Proposition 5. By [12,Corollary 4.12], there exists a compact set $K \subset \mathbb{R}$ such that $\nu = \mathscr{H}^\varepsilon \llcorner K$, so that $D^\alpha u_\alpha \not\ll \mathscr{H}^s$ for all $s > \varepsilon$. Now we observe that, by (4.16), we have the following situations:

- if $\varepsilon \in (0, 1 - \alpha)$, then $p < \frac{1-\varepsilon}{1-\alpha-\varepsilon} < \frac{1}{1-\alpha-\varepsilon}$ and thus $\varepsilon > \frac{1}{q} - \alpha$;
- if $\varepsilon = 1 - \alpha$, then $p \in [1, +\infty)$ and thus $\varepsilon > \frac{1}{q} - \alpha$;
- if $\varepsilon \in (1 - \alpha, 1]$, then $p \in [1, +\infty]$ and so, for $p = +\infty$, if $s > 1 - \alpha$ then we can take $\varepsilon \in (1 - \alpha, s)$.

Therefore, the absolute continuity property of the fractional variation stated in Theorem 1(ii) is sharp for $n = 1$.

5 Fractional capacity and precise representative

In this last section, we study the fractional capacity and the existence of the precise representatives of $BV^{\alpha,p}$ functions.

5.1 The (α, p)-capacity

We begin with the definition of *fractional capacity*, see [1,Chapter 2]. For the classical integer case $\alpha = 1$, we also refer the reader to [11,Sections 4.7 and 5.6.3], [15,Chapter 2], [21,Section 2.1] and [22,Section 2.2]. Here and in the following, we repeatedly use the identification $S^{\alpha,p}(\mathbb{R}^n) = L^{\alpha,p}(\mathbb{R}^n)$ for $\alpha \in (0, 1)$ and $p \in (1, +\infty)$ proved in [6,Corollary 2.1].

Definition 1 (**The** (α, p)-**capacity**) Let $\alpha \in (0, 1)$ and $p \in [1, +\infty)$. We let

$$\mathrm{Cap}_{\alpha,p}(K) = \inf\left\{\|f\|^p_{S^{\alpha,p}(\mathbb{R}^n)} : f \in C^\infty_c(\mathbb{R}^n),\ f \geq \chi_K\right\}$$

be the (α, p)-*capacity* of the compact set $K \subset \mathbb{R}^n$.

The mapping $\mathrm{Cap}_{\alpha,p}$ can be extended to more general sets via the following standard routine. If $A \subset \mathbb{R}^n$ is an open set, then we set

$$\mathrm{Cap}_{\alpha,p}(A) = \sup\left\{\mathrm{Cap}_{\alpha,p}(K) : K \subset A,\ K \text{ compact}\right\}$$

and so, given any set $E \subset \mathbb{R}^n$, we let

$$\mathrm{Cap}_{\alpha,p}(E) = \sup\left\{\mathrm{Cap}_{\alpha,p}(A) : A \supset E,\ A \text{ open}\right\}.$$

We now recall the notion of (α, p)-*quasievery point*, see [1,Definition 2.2.5].

Definition 2 ((α, p)-**quasievery point**) Let $\alpha \in (0, 1)$ and $p \in [1, +\infty)$. We say that a property $\mathscr{P}(x)$ is true for (α, p)-*quasievery* $x \in \mathbb{R}^n$ if

$$\mathrm{Cap}_{\alpha,p}(\left\{x \in \mathbb{R}^n : \mathscr{P}(x) \text{ is false}\right\}) = 0.$$

Recall that, if $\alpha \in (0,1)$ and $p \in \left(\frac{n}{\alpha}, +\infty\right)$, then $S^{\alpha,p}(\mathbb{R}^n) \subset C_b(\mathbb{R}^n)$ continuously by the fractional Sobolev Embedding Theorem, see [1,Theorem 1.2.4(c)] for instance. For this reason, the notion of (α, p)-capacity becomes interesting only when $\alpha p \le n$ (see the discussion below [1,Proposition 2.6.1]). Precisely, if $\alpha \in (0,1)$ and $p \in \left(1, \frac{n}{\alpha}\right]$, then $\mathscr{H}^{n-\alpha p+\varepsilon} \ll \mathrm{Cap}_{\alpha,p}$ for all $\varepsilon > 0$, see [1,Theorem 5.1.13 and Corollary 5.1.14].

5.2 The precise representative

We now study the precise representatives of $BV^{\alpha,p}$ functions by combining the embedding proved in Theorem 7 with the results already known in the literature for the precise representatives of functions in fractional Bessel potential spaces.

We begin by recalling the definition of *quasicontinuity*. For the integer case $\alpha = 1$, we refer the reader to [1,Definition 6.1.1] and [11,Definition 4.11].

Definition 3 (**(α, p)-quasicontinuity**) We say that a function $f : \mathbb{R}^n \to [-\infty, +\infty]$ defined (α, p)-quasieverywhere is (α, p)-*quasicontinuous* if, for each $\varepsilon > 0$, there exists an open set $A_\varepsilon \subset \mathbb{R}^n$ such that $\mathrm{Cap}_{\alpha,p}(A_\varepsilon) < \varepsilon$ and $f|_{\mathbb{R}^n \setminus A_\varepsilon}$ is continuous.

Here and in the following, the *precise representative* of a function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ is defined as

$$u^\star(x) = \lim_{r \to 0^+} \fint_{B_r(x)} u(y)\, dy, \quad x \in \mathbb{R}^n,$$

if the limit exists, otherwise $u^\star(x) = 0$ by convention. The following result provides a precise description of the continuity properties of the precise representative of a function in $S^{\alpha,p}(\mathbb{R}^n)$ for $p \in \left(1, \frac{n}{\alpha}\right]$. We refer the reader to [1,Theorem 6.2.1] for the proof.

Theorem 11 (**Quasicontinuity of $S^{\alpha,p}$ functions**) *Let $\alpha \in (0,1)$ and $p \in \left(1, \frac{n}{\alpha}\right]$. If $f \in S^{\alpha,p}(\mathbb{R}^n)$, then $f^\star$ is an (α, p)-quasicontinuous representative of f and*

$$\lim_{r \to 0^+} \fint_{B_r(x)} |f(y) - f^\star(x)|^q\, dy = 0$$

for (α, p)-quasievery $x \in \mathbb{R}^n$, where

$$q \in \begin{cases} \left[1, \frac{np}{n-\alpha p}\right] & \text{if } \alpha p < n, \\ [1, +\infty) & \text{if } \alpha p = n. \end{cases}$$

Thanks to the embedding proved in Theorem 7, we immediately deduce the following result concerning the quasicontinuity of the functions in $BV^{\alpha,p}(\mathbb{R}^n)$.

Corollary 4 (**Quasicontinuity of $BV^{\alpha,p}$ functions for $p < \frac{n}{n-\alpha}$**) *Let $\alpha, \beta \in (0,1)$, with $\beta < \alpha$, and let $p, q \in [1, +\infty]$ be such that $p \le q < \frac{n}{n+\beta-\alpha}$, with $q > 1$. If $f \in BV^{\alpha,p}(\mathbb{R}^n)$, then $f^\star$ is a (β, q)-quasicontinuous representative of f and*

$$\lim_{r \to 0^+} \fint_{B_r(x)} |f(y) - f^\star(x)|^t\, dy = 0$$

for (β, q)*-quasievery* $x \in \mathbb{R}^n$ *and for all* $t \in \left[1, \frac{nq}{n-\beta q}\right]$.

In order to provide an extension of Corollary 4 also for all exponents $p \in [1, +\infty]$, we need the following localization result for $BV^{\alpha,p}$ functions.

Lemma 1 **(Localization for $BV^{\alpha,p}$ functions for $p \in [1, +\infty]$)** *Let* $\alpha \in (0, 1)$ *and let* $p \in [1, +\infty]$. *If* $f \in BV^{\alpha,p}(\mathbb{R}^n)$ *and* $\eta \in \mathrm{Lip}_c(\mathbb{R}^n)$, *then* $f\eta \in BV^{\alpha,q}(\mathbb{R}^n)$ *for all* $q \in [1, p]$, *with*

$$D^\alpha(f\eta) = \eta\, D^\alpha f + f\, \nabla^\alpha \eta\, \mathscr{L}^n + \nabla^\alpha_{\mathrm{NL}}(f, \eta)\, \mathscr{L}^n \quad in \ \mathscr{M}(\mathbb{R}^n; \mathbb{R}^n). \tag{5.1}$$

Proof By Hölder's inequality, we clearly have that $f\eta \in L^q(\mathbb{R}^n)$ for all $q \in [1, p]$, so that we only need to prove (5.1). First of all, note that the right-hand side of (5.1) is well posed because $\eta \in \mathrm{Lip}_c(\mathbb{R}^n)$. In particular, we have $\nabla^\alpha \eta \in L^1(\mathbb{R}^n; \mathbb{R}^n) \cap L^\infty(\mathbb{R}^n; \mathbb{R}^n)$ by [7,Corollary 2.3] and $\nabla^\alpha_{\mathrm{NL}}(f, \eta) \in L^1(\mathbb{R}^n)$, since by Minkowski's and Hölder's (generalized) inequalities we can estimate

$$\begin{aligned}
\|\nabla^\alpha_{\mathrm{NL}}(f, \eta)\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} &\le \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{\| |f(\cdot + h) - f|\, |\eta(\cdot + h) - \eta| \|_{L^1(\mathbb{R}^n)}}{|h|^{n+\alpha}}\, dh \\
&\le 2\mu_{n,\alpha}\, \|f\|_{L^p(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{\|\eta(\cdot + h) - \eta\|_{L^{p'}(\mathbb{R}^n)}}{|h|^{n+\alpha}}\, dh \\
&\le c_{n,\alpha,p}\, \|f\|_{L^p(\mathbb{R}^n)}\, \|\eta\|_{W^{1,p'}(\mathbb{R}^n)}
\end{aligned} \tag{5.2}$$

(for the validity of the last inequality, see [18,Theorem 17.33] for instance). Now let $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n; \mathbb{R}^n)$ be given. By [7,Lemma 2.7], we can write

$$\mathrm{div}^\alpha(\eta\varphi) = \eta\, \mathrm{div}^\alpha \varphi + \varphi \cdot \nabla^\alpha \eta + \mathrm{div}^\alpha_{\mathrm{NL}}(\eta, \varphi),$$

so that

$$\begin{aligned}
\int_{\mathbb{R}^n} f\eta\, \mathrm{div}^\alpha \varphi\, dx = \int_{\mathbb{R}^n} f\, \mathrm{div}^\alpha(\eta\varphi)\, dx - \int_{\mathbb{R}^n} f\varphi \cdot \nabla^\alpha \eta\, dx \\
- \int_{\mathbb{R}^n} f\, \mathrm{div}^\alpha_{\mathrm{NL}}(\eta, \varphi)\, dx.
\end{aligned}$$

In addition, since $\eta\varphi \in \mathrm{Lip}_c(\mathbb{R}^n; \mathbb{R}^n)$ and $f \in BV^{\alpha,p}(\mathbb{R}^n)$, we immediately see that

$$\int_{\mathbb{R}^n} f\, \mathrm{div}^\alpha(\eta\varphi)\, dx = -\int_{\mathbb{R}^n} \eta\varphi \cdot dD^\alpha f.$$

Finally, let $(f_\varepsilon)_{\varepsilon>0} \subset BV^{\alpha,p}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ be such that $f_\varepsilon = f * \varrho_\varepsilon$ for all $\varepsilon > 0$ as in Theorem 4. Note that $f_\varepsilon \in \mathrm{Lip}_b(\mathbb{R}^n; \mathbb{R}^n)$, so that

$$\int_{\mathbb{R}^n} f_\varepsilon\, \mathrm{div}^\alpha_{\mathrm{NL}}(\eta, \varphi)\, dx = \int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha_{\mathrm{NL}}(f_\varepsilon, \eta)\, dx.$$

for all $\varepsilon > 0$ by [8,Lemmas 2.4 and 2.5]. Now, arguing as in the proof of (5.2), we can infer that

$$\lim_{\varepsilon \to 0^+} \nabla^\alpha_{\mathrm{NL}}(f_\varepsilon, \eta) = \nabla^\alpha_{\mathrm{NL}}(f, \eta) \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n), \tag{5.3}$$

so that we can pass to the limit as $\varepsilon \to 0^+$ in (5.3) to get that

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha_{\mathrm{NL}}(g, \varphi)\, dx = \int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha_{\mathrm{NL}}(f, g)\, dx.$$

In conclusion, we have that

$$\int_{\mathbb{R}^n} f\eta \operatorname{div}^\alpha \varphi\, dx = -\int_{\mathbb{R}^n} \eta\varphi \cdot dD^\alpha f - \int_{\mathbb{R}^n} f\varphi \cdot \nabla^\alpha \eta\, dx - \int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha_{\mathrm{NL}}(f, \eta)\, dx$$

for any given $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n; \mathbb{R}^n)$ and the proof is complete. □

Corollary 5 **(Quasicontinuity of $BV^{\alpha,p}$ functions for $p \in [1, +\infty]$)** *Let $\alpha, \beta \in (0, 1)$ be such that $\beta < \alpha$ and let $p \in [1, +\infty]$ and $q \in \left[1, \frac{n}{n+\beta-\alpha}\right)$. If $f \in BV^{\alpha,p}(\mathbb{R}^n)$, then $f^\star$ is a (β, q)-quasicontinuous representative of f and*

$$\lim_{r \to 0^+} \fint_{B_r(x)} |f(y) - f^\star(x)|^t\, dy = 0 \tag{5.4}$$

for (β, q)-quasievery $x \in \mathbb{R}^n$ and for all $t \in \left[1, \frac{nq}{n-\beta q}\right]$. In particular, the precise representative of f is well defined $\mathscr{H}^{n-\alpha+\varepsilon}$-a.e. for all $\varepsilon > 0$.

Proof By Lemma 1, we know that $f\eta \in BV^{\alpha,1}(\mathbb{R}^n)$ for all $\eta \in \mathrm{Lip}_c(\mathbb{R}^n)$. Hence, Corollary 4 implies the existence of a (β, q)-quasicontinuous representative of $f\eta$ for all $\beta \in (0, \alpha)$ and $q \in \left(1, \frac{n}{n+\beta-\alpha}\right)$. In particular, if $\eta(x) = 1$ for all $x \in B_R$ for some given $R > 0$, then we get the existence of $(f\eta)^\star(x) = f^\star(x)$ for (β, q)-quasievery $x \in \mathbb{R}^n$, together with (5.4). Since $R > 0$ is arbitrary, $f^\star(x)$ must exist for (β, q)-quasievery $x \in \mathbb{R}^n$. Finally, since $q < \frac{n}{\beta}$, we have $\mathscr{H}^{n-\beta q+\delta} \ll \mathrm{Cap}_{\beta,q}$ for all $\delta > 0$. Thus, by optimizing in $\beta \in (0, \alpha)$ and in $q \in \left(1, \frac{n}{n+\beta-\alpha}\right)$, the existence of $f^\star(x)$ follows for $\mathscr{H}^{n-\alpha+\varepsilon}$-a.e. $x \in \mathbb{R}^n$ and the proof is complete. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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