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Research paper

Kinematic chains with preserved mobility for arbitrary arrangements of their links

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ABSTRACT

Kinematic chains made from links whose lengths and twist angles satisfy a certain proportionality rule are considered. A notion of exchange symmetry for the links of any such chain affords a new understanding of the mobility of well-known overconstrained mechanisms, exemplified by Bennett's four-bar linkage, and provides a unifying framework for explaining the mobility of a recently discovered class of nontrivial underconstrained mechanisms. These findings allow for the design of mechanisms consisting of links that can be arbitrarily reordered without altering their number of internal degrees of freedom and, thus, of modular mechanical systems that can be assembled in different ways to serve different purposes. A nine-hinged linkage that can be assembled in ninety-four distinct ways, each evincing a single internal degree of freedom, is presented as an example of such a modular system.

1. Introduction

Thirty years ago, Wohlhart [1] identified conditions under which two adjacent links in a mechanism can be exchanged to yield a mechanism of the same type but with distinct input–output relations. We report some previously overlooked, yet fundamental, consequences of Wohlhart's [1] discoveries. Link exchanges like those introduced by Wohlhart [1] are possible only in the presence of symmetries of the kind exhibited by the overconstrained four-bar linkage of Bennett [2] and the related five- and six-bar linkages of Myard [3] and Goldberg [4]. Chen and You [5] gave a comprehensive review of overconstrained single- and multi-loop mechanisms, showing that many known overconstrained mechanisms are based on the Bennett linkage. Mavroidis and Roth [6] provided a general classification of overconstrained mechanisms and a unified approach to their design, finding that the mobility of overconstrained mechanisms relies on the presence of at least one "Bennett joint". Such a joint consists of three revolute hinges and two links with Denavit–Hartenberg parameters constrained so that the ratios of their lengths to the sines of their twist angles are identical. Observing that Bennett's [2] four-bar linkage consists of two pairs of such links, Wohlhart [1] referred to the operative proportionality rule as the "Bennett condition" and two adjacent links that satisfy the Bennett condition as a "Bennett pair". Moreover, and most importantly, he found that the spatial configuration of a kinematic chain containing a Bennett pair is invariant if the links comprising that pair are exchanged.

We provide an explicit description of the operation upon which Wohlhart's [1] exchange operation is based. We consequently find that Bennett's [2] four-bar linkage can be constructed from a trivial linkage by an application of the exchange operation, and we thus establish the deep connection between the exchange symmetry and that linkage. For mechanisms entirely made from links obeying the Bennett condition, which we designate as "Bennett chains", the exchange symmetry has the immediate consequence that any two links in the chain can be exchanged via consecutive exchanges of adjacent links.

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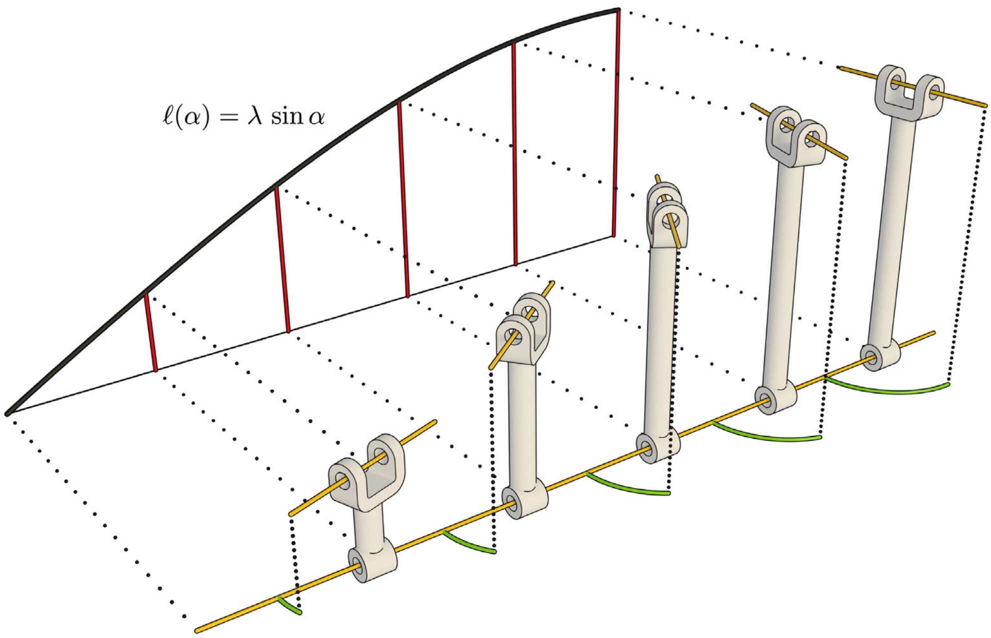

Fig. 1. Illustration of links of length ℓ (red) and twist angle α (green) that satisfy the Bennett condition (1) requiring that the ratio $\ell / \sin\alpha$ be fixed. The corresponding function ℓ of α is indicated as a black curve in the background with λ set to be of unit length. Links are shown for α given, from left to right, by 18°, 36°, 54°, 72°, and 90°.

Rather than being an exclusive property of overconstrained kinematic chains, the exchange symmetry is exhibited by any kinematic chain designed according to the Bennett condition, including, in particular, the family of underconstrained mechanisms found recently by Schönke and Fried [7], which evince a single internal degree of freedom. Using the exchange symmetry, we give a plausibility argument explaining why every underconstrained mechanism in that family should be mobile. Ultimately, to exemplify a newly discovered family of underconstrained linkages comprised by links that can be ordered arbitrarily and have a single internal degree of freedom, we present a nine-hinged linkage comprised by three distinct types of links that can be assembled in ninety-four different ways.

2. Bennett chains and the exchange symmetry

2.1. General considerations

We consider kinematic chains made from rigid bodies called links connected through hinges. We assume that the $N \geq 2$ links of a chain satisfy the Bennett condition, according to which the twist angle α_i and length ℓ_i of each link i are related by the proportionality rule

$$\frac{\ell_i}{\sin\alpha_i} = \lambda, \qquad i = 1, \ldots, N, \tag{1}$$

where $\lambda > 0$ is an assigned constant carrying the dimension of length. The Bennett condition (1) for links with various twist angles and corresponding lengths is provided in Fig. 1. As a consequence of (1), the links of the chain share a proportionality relation. Moreover, the length ℓ_i of link i cannot be chosen independent of the twist angle α_i of link i and is bounded above by λ. Without loss of generality, we may set the constant λ on the right-hand side of (1) to be of unit length. Doing so amounts to fixing the overall spatial scaling. Rules like (1) have appeared previously in the literature, most prominently in the work of Bennett [2] and later, for example, in the studies of Wohlhart [1], who referred to (1) as the "Bennett condition", and Mavroidis and Roth [6], who designated two adjacent links obeying (1) as a "Bennett set". It therefore seems fitting to call any kinematic chain made entirely from links obeying (1) a "Bennett chain".

Following Bates et al. [8, pp. 75–76], we describe a chain with N links by $N+1$ unit vectors $\boldsymbol{h}_i$, $i = 1, \ldots N+1$, that encode the hinge orientations. These vectors must satisfy the following system of equations:

$$\boldsymbol{h}_i \cdot \boldsymbol{h}_i = 1, \qquad \boldsymbol{h}_i \cdot \boldsymbol{h}_{i+1} = \cos\alpha_i, \qquad \sum_{1}^{N} \boldsymbol{h}_i \times \boldsymbol{h}_{i+1} = \boldsymbol{p}. \tag{2}$$

Whereas the "twist" relation $(2)_2$ guarantees that each link i has the specified twist angle α_i, the "path" relation $(2)_3$ ensures that the relative position of the terminal hinge $i = N+1$ with respect to the initial hinge $i = 1$ is equal to the prescribed vector $\boldsymbol{p}$ entering

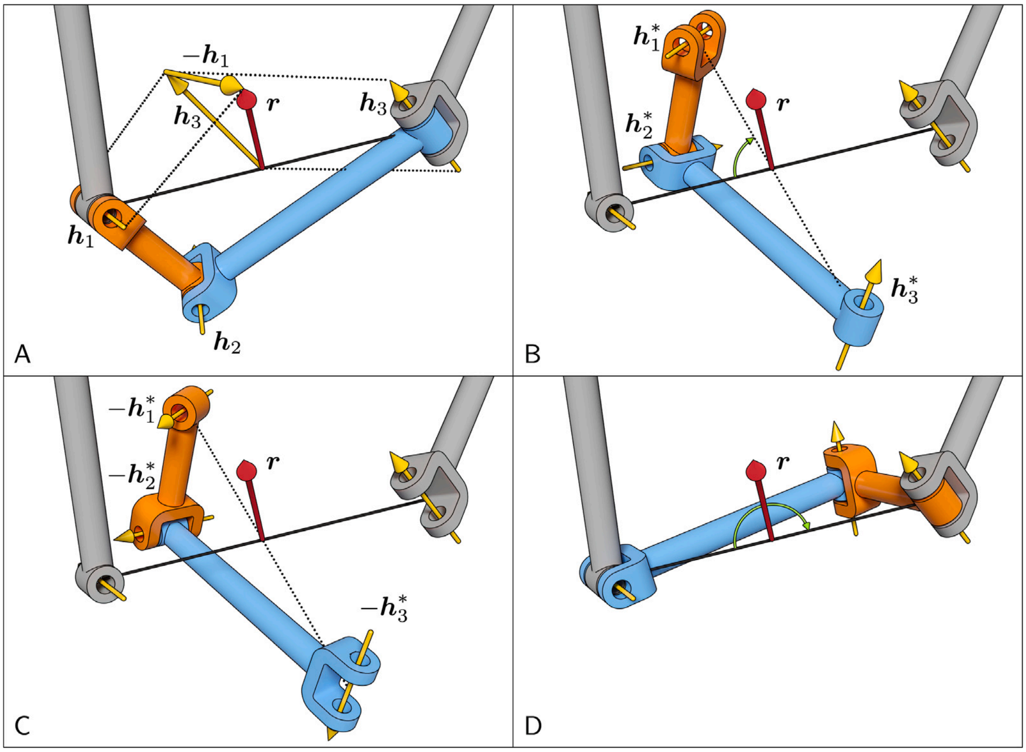

Fig. 2. The exchange of two links (orange and blue) obeying the Bennett condition (1) via a π-rotation. (A) The initial configuration of the links. The rotation axis $\boldsymbol{r}$ (red) is given by the difference vector $\boldsymbol{h}_3 - \boldsymbol{h}_1$ of the two hinge vectors (yellow) and goes through the middle of the line segment (solid black line) between hinges 1 and 3. (B) Partial rotation of both links about the axis $\boldsymbol{r}$ (red) indicated by the green arrow. (C) Individual π-rotations of the orange and blue links about the local axes $\boldsymbol{h}_2^* - \boldsymbol{h}_1^*$ and $\boldsymbol{h}_3^* - \boldsymbol{h}_2^*$, leading to a simple flip of each individual link and ensuring that the other links in the chain can be reconnected. (D) Full π-rotation of both links about the axis $\boldsymbol{r}$ (red) indicated by the green arrow. This completes the exchange operation. The orange and blue links in their initial (A) and final (D) position together also constitute a Bennett linkage. The exchange operation can therefore be used to construct Bennett's [2] four-bar linkage.

$(2)_3$. By allowing the twist angle to be chosen in the range from 0° to 180°, we cover the entire range of possible twist angles. Simple path relations of the form $(2)_3$ arise only for Bennett chains. For a kinematic chain consisting of links unconstrained by a condition such as (1), it would be necessary to specify the lengths of the links and those lengths would be included as factors of the summands in the counterpart of the left-hand side of $(2)_3$. This underlines the special, elementary nature of Bennett chains.

The particular form of $\boldsymbol{p}$ is dictated by the type of kinematic chain considered, as are any restrictions met by the initial and terminal hinge vectors $\boldsymbol{h}_1$ and $\boldsymbol{h}_{N+1}$ of the kinematic chain. We concentrate hereafter on closed kinematic chains — commonly called linkages — for which $\boldsymbol{p} = \boldsymbol{0}$ and the initial and terminal hinge vectors must be parallel or antiparallel. With this provision, the system (2) is augmented by the supplemental boundary condition

$$\boldsymbol{h}_{N+1} = \pm \boldsymbol{h}_1, \tag{3}$$

where the plus sign applies if the initial and terminal hinges are parallel and the minus sign applies otherwise. Our treatment can be extended to include open chains. However, the closure condition (3) affords a reduction that allows us to more easily expose and clarify the unique properties of Bennett chains.

We next show that two neighboring links of a closed kinematic chain consistent with the Bennett condition can be exchanged while maintaining the positions of all other links in the chain. A rendering of the exchange operation is provided in Fig. 2. Without loss of generality, we consider links $i = 1$ and $i = 2$. These links have terminal hinge vectors $\boldsymbol{h}_1$ and $\boldsymbol{h}_3$ and connecting hinge vector $\boldsymbol{h}_2$. By the Bennett condition (1), the relative center position of hinge $i = 3$ with respect to hinge $i = 1$ is given by the vector $\boldsymbol{v} = \boldsymbol{h}_1 \times \boldsymbol{h}_2 + \boldsymbol{h}_2 \times \boldsymbol{h}_3$. Our argument relies on the existence of a π-rotation $\boldsymbol{R}$ about the axis $\boldsymbol{r} = \boldsymbol{h}_3 - \boldsymbol{h}_1$ satisfying

$$\boldsymbol{R}\boldsymbol{h}_1 = -\boldsymbol{h}_3, \qquad \boldsymbol{R}\boldsymbol{h}_3 = -\boldsymbol{h}_1, \qquad \boldsymbol{R}\boldsymbol{v} = -\boldsymbol{v}. \tag{4}$$

To establish the existence of such a rotation, we first introduce the vector $\boldsymbol{s} = \boldsymbol{h}_1 + \boldsymbol{h}_3$. By $(2)_1$, $\boldsymbol{s}$ is orthogonal to $\boldsymbol{r}$. Any π-rotation $\boldsymbol{R}$ about $\boldsymbol{r}$ must therefore satisfy $\boldsymbol{R}\boldsymbol{r} = \boldsymbol{r}$ and $\boldsymbol{R}\boldsymbol{s} = -\boldsymbol{s}$. The relations $(4)_1$ and $(4)_2$ then follow on noting that $\boldsymbol{h}_1 = (\boldsymbol{s} - \boldsymbol{r})/2$ and that $\boldsymbol{h}_3 = (\boldsymbol{r} + \boldsymbol{s})/2$. Further, $(4)_3$ follows on noting that $\boldsymbol{v}$, which can be expressed as $\boldsymbol{v} = \boldsymbol{h}_2 \times \boldsymbol{r}$, is also orthogonal to $\boldsymbol{r}$. The relations in (4) ensure that, after the application of the rotation $\boldsymbol{R}$, hinge $i = 1$ is positioned and oriented in exactly the same way that hinge $i = 3$ was before the rotation and vice versa.

The rotation $\boldsymbol{R}$ leads to reversed hinge vector orientations and a reversed orientation of links $i = 1$ and $i = 2$ within the chain. This is restored by individual π-rotations of links $i = 1$ and $i = 2$ about the respective local axes $\boldsymbol{h}_2^* - \boldsymbol{h}_1^*$ and $\boldsymbol{h}_3^* - \boldsymbol{h}_2^*$. The axes pass through the midpoint of the respective link. Restoring the original orientation of the links within the chain is not only important to ensuring that the other links in the chain can be reconnected, but also necessary to preserve the orientations of the hinge vectors and, consequently, the choice of signs in closure condition (3).

In the two preceding paragraphs, we have shown that two neighboring links of a closed Bennett chain can be exchanged to yield another closed Bennett chain. An appropriate composition of such exchange operations can be performed to achieve any permutation of the original links of any given closed Bennett chain. We refer to this property of a closed Bennett chain as its 'exchange symmetry'.

The rotation $\boldsymbol{R}$ in (4) has the explicit representation

$$\boldsymbol{R} = \frac{1}{1 - \boldsymbol{h}_1 \cdot \boldsymbol{h}_3}(\boldsymbol{h}_3 - \boldsymbol{h}_1) \otimes (\boldsymbol{h}_3 - \boldsymbol{h}_1) - \boldsymbol{1}, \tag{5}$$

which is well defined except for the case $\boldsymbol{h}_1 = \boldsymbol{h}_3$, whereby $\boldsymbol{h}_1 \times \boldsymbol{h}_2 + \boldsymbol{h}_2 \times \boldsymbol{h}_3 = \boldsymbol{0}$. We infer that this degenerate case can only occur if the two links involved in the exchange operation are identical and have reversed paths. In this case the two links coincide exactly, and they can be removed from the linkage, resulting in a Bennett chain with $N - 2$ links.

2.2. Arbitrary reordering

The exchange symmetry has several far reaching consequences, the most immediate of which is that any permutation for the ordering of links in the linkage can be realized by consecutive exchanges of neighboring links. As such, regardless of the order in which the links are assembled, the chain will still close consistent with (3). This remarkable property will be illustrated in an example presented below.

Permutation groups can simplify the synthesis and analysis of kinematic chains. For example, Karger [9] used the dihedral group D_5 of spatial rotations and reflections to simplify the constraint equations on the class of 5R linkages and thereby obtained a complete classification of all overconstrained linkages in that class. For a Bennett linkage with N links, the exchange symmetry we describe gives rise to a much larger permutation group — the symmetric group S_N of $N!$ elements — and thus makes it possible to exchange any pair of links in such a linkage. It follows that any constraint on a closed Bennett chain that involves only parameters of the links (for example, the twist angle) must exhibit this symmetry, meaning that the constraint equation must be invariant under permutation of those parameters.

2.3. Preserved kinematic properties

Certain kinematic properties of a closed Bennett chain are preserved under the exchange operation described in Section 2.1. If, for example, such a linkage has one internal degree of freedom, then any reordering of the links results in a linkage that also has one internal degree of freedom. This property of closed Bennett chains can be understood by comparing the topology of the set of solutions to (2), which are considered to be differentiable manifolds in the configuration space of the linkage, before and after the reordering of links. With reference to (4), the exchange operation corresponds to mappings $\boldsymbol{h}_1 \mapsto -\boldsymbol{h}_3$ and $\boldsymbol{h}_3 \mapsto -\boldsymbol{h}_1$ which are evidently differentiable. Moreover, the mapping $\boldsymbol{h}_2 \mapsto \boldsymbol{h}_2^\star(\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3)$, which is simply a π-rotation, is also differentiable. The exchange operation underlying the reordering of the links is therefore differentiable and, since it is its own inverse, a diffeomorphism. The manifolds in the configuration space of the linkage before and after the reordering of links are consequently diffeomorphic. For example, if the original manifold were diffeomorphic to the unit circle S^1 (as is the case for a linkage with one internal degree of freedom) then the manifold after reordering would also be diffeomorphic to S^1.

2.4. Möbius Kaleidocycles and the mobility of underconstrained closed Bennett chains

Möbius Kaleidocycles are a class of underconstrained linkages made from $N \geq 7$ links of identical length and twist angle which possess a single internal degree of freedom first described by Schönke and Fried [7]. The decisive property of a Möbius Kaleidocycle is that its links have minimal twist angles. For each choice of N, there is a specific minimal twist angle below which the kinematic chain of links cannot be closed, meaning that the two terminal hinges cannot be aligned and brought to the same position. For the minimal twist angle the generally $N - 6$ internal degrees of freedom of the closed chain are reduced to a single one. Assigning a vector to each hinge, as described in Section 2.1, endows the kinematic chain with a sense of orientation. These linkages close nonorientably (obeying (3) with the minus sign) and have a topological twist of 3π, leading to the moniker "Möbius Kaleidocycles".

An underconstrained mechanism has a configuration space with real dimension less than its complex dimension. This can result in mechanisms with fewer internal degrees of freedom than predicted by a simple mobility analysis (as is the case for a Möbius Kaleidocycle), but there are also exceptional cases where there is no reduction in the number of internal degrees of freedom (this happens if the system is simultaneously underconstrained and overconstrained). For an introduction to the concept of underconstrained linkages, see Arponen et al. [10]. Since Möbius Kaleidocycles are made from identical links, they are closed Bennett chains. This allows us to use the notion of exchange symmetry to formulate a plausibility argument for the unexpected mobility of these linkages.

Our argument involves inserting an infinitesimal 'ϵ-link', obeying (1) with $\ell_0 = \sin\alpha_0 = \epsilon \ll 1$, between two links of a closed Bennett chain. Since a Möbius Kaleidocycle with $N \geq 7$ links is minimally twisted, an ϵ-link can always be inserted in any such linkage. This is possible because the ϵ-link increases the total twist of the chain, which can then still be closed. The resulting linkage

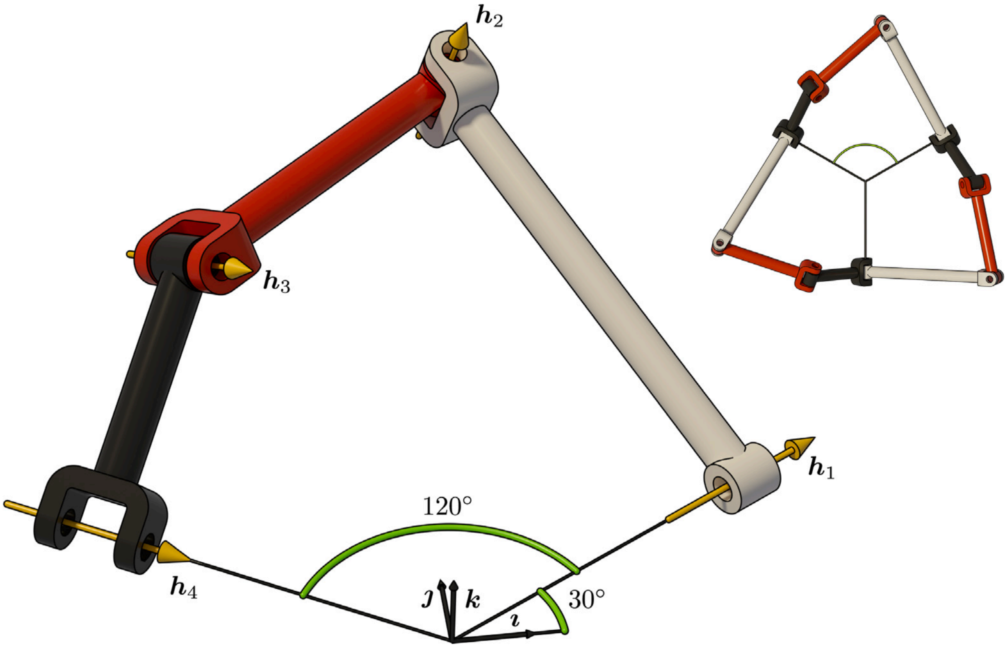

Fig. 3. Sketch of one third of a nine-hinged linkage with threefold rotational symmetry. The three distinct links obeying the Bennett condition (white, red, black) are shown together with the hinge vectors $\boldsymbol{h}_1$ to $\boldsymbol{h}_4$. The orientation of the hinge vectors $\boldsymbol{h}_1$ and $\boldsymbol{h}_4$ ensures that the linkage closes nonorientably. The inset in the top right shows the complete linkage.

is not necessarily critically twisted. However, the twist of its links can be progressively reduced until criticality is again met. This reduction in the twist angle should be on the order of ϵ. (The example of the 9R linkage presented below shows that a linkage can be minimally twisted even if its links are not identically twisted; the notion of minimal twist is a collective property of the linkage.) The final outcome of this procedure is an underconstrained linkage with an ϵ-link inserted, and that linkage is "infinitesimally close" to the original linkage without the ϵ-link. Next, consider an exchange cycle, meaning that the ϵ-link is exchanged consecutively with all other links until it returns to its initial position in the linkage. Such a cycle alters the configuration of the linkage infinitesimally, and a continuous, finite internal motion of the linkage is produced by repeating the cycle and passing to the limit $\epsilon \to 0$. The ϵ-link and the associated exchange cycle act as an infinitesimal generator of the motion.

3. Examples

3.1. Bennett four-bar linkage

The four-bar linkage of Bennett [2] is the only non-trivial spatial mobile four-bar linkage. It has been thoroughly studied (see, for example, Baker [11], Arponen et al. [12], or Dietmair [13]) and serves as the basic unit for constructing many other overconstrained linkages and deployable structures, as described, for example, by You and Chen [14].

A Bennett linkage consists of four links of two distinct types, say A and B, ordered according to $ABAB$. These links can be reordered according to $AABB$. If we apply the exchange operation to any two neighboring links of Bennett's linkage, we obtain a singular configuration in which the identical links occupy the same position with shared orientation. Moreover, the hinge involved in the exchange operation occupies the same position and has the same orientation as its formerly opposing hinge. This constitutes a degenerate linkage that is equivalent to an open two link chain that can undergo a rotating motion.

As discussed in Section 2.3, the kinematic properties of a Bennett chain are preserved under permutations of its links. This fact can be used to prove that a Bennett linkage is mobile. Since the singular configuration described in the previous paragraph has one internal degree of freedom in the form of a rotating motion, Bennett's four-bar linkage must also have one internal degree of freedom.

To give another perspective, it is instructive to consider the creation of a Bennett linkage as a generative procedure using the exchange operation. Consider an open chain of two arbitrary links obeying (1) in a given configuration. Now perform the exchange operation with the two links but in the sense of a "copy" process, adding the two links in the original position once more, resulting in a Bennett linkage. This approach to creating a Bennett mechanism was also formulated by Groeneveld [15, pp. 39–40].

3.2. Arbitrarily orderable nine-hinged linkages with a single degree of freedom

The linkage described next is reported here for the first time and is a generalization of the nine-hinged member of the family of Möbius Kaleidocycles described in Section 2.4. For the generalization, we consider a closed Bennett chain consisting of $N = 9$ links with as many as three distinct twist angles α_i, $i = 1, 2, 3$, and corresponding lengths constrained by the Bennett condition (1). We denote the corresponding cosines of the twist angles by a, b, c, and the three types of links by A, B, C, correspondingly. We assume that each link type appears three times in the linkage and that the (initial) ordering is $ABCABCABC$. For such an ordering, we assume that the links can be arranged in a configuration with threefold rotational symmetry in which three hinges (each three links apart) lie in a common plane and form pairwise angles of 120°, as ordained by the symmetry and illustrated in the upper right-hand corner of Fig. 3. It is then possible to explore the conditions that the links ABC in one 120° wedge of the linkage (constituting one third of the complete linkage) must satisfy in such a threefold symmetric configuration.

Referring to Fig. 3, we introduce a right-handed orthonormal basis $\{\boldsymbol{\imath}, \boldsymbol{\jmath}, \boldsymbol{k}\}$ with $\boldsymbol{k}$ chosen to be normal to the common plane in which $\boldsymbol{h}_1$ and $\boldsymbol{h}_4$ reside and $\boldsymbol{\jmath}$ "halving" the 120° angle in that plane. In the threefold symmetric configuration, the hinge vectors can then be expressed as

$$\boldsymbol{h}_1 = \tfrac{1}{2}(\sqrt{3}\,\boldsymbol{\imath} + \boldsymbol{\jmath}), \quad \boldsymbol{h}_2 = x\boldsymbol{\imath} + y\boldsymbol{\jmath} + z\boldsymbol{k}, \quad \boldsymbol{h}_3 = u\boldsymbol{\imath} + v\boldsymbol{\jmath} + w\boldsymbol{k}, \quad \boldsymbol{h}_4 = \tfrac{1}{2}(\sqrt{3}\,\boldsymbol{\imath} - \boldsymbol{\jmath}), \tag{6}$$

where the unknown coefficients x, y, z, u, v, and w must be consistent with (2). The normalization conditions $(2)_1$ for $\boldsymbol{h}_2$ and $\boldsymbol{h}_3$ then read

$$x^2 + y^2 + z^2 = 1, \qquad u^2 + v^2 + w^2 = 1, \tag{7}$$

and the twist relations $(2)_2$ amount to

$$\sqrt{3}\,x + y = 2a, \qquad xu + yv + zw = b, \qquad \sqrt{3}\,u - v = 2c, \tag{8}$$

where the cosines $a = \cos\alpha_1$, $b = \cos\alpha_2$, and $c = \cos\alpha_3$ of the twist angles remain to be determined. By virtue of the assumed threefold symmetry, the components of the path relation $(2)_3$ in the directions of $\boldsymbol{\jmath}$ and $\boldsymbol{k}$ must vanish. This leads to the conditions

$$\sqrt{3}\,y - x + 2xv - 2yu - u - \sqrt{3}\,v = 0, \qquad \sqrt{3}\,z - 2zu + 2xw - \sqrt{3}\,w = 0. \tag{9}$$

The component of $(2)_3$ in the $\boldsymbol{\imath}$ direction is restricted only by the requirement that it does not vanish.

Together, (7)–(9) constitute a second-order polynomial system of seven equations for the nine unknown scalar quantities x, y, z, u, v, w, a, b, and c. Despite the complexity of that system, it is feasible to compute a Gröbner basis, using, for example, Singular [16]. Choosing an appropriate ordering of the variables, the first element of the Gröbner basis yields a condition relating the cosines a, b, and c of the twist angles in form of a cubic polynomial, namely

$$8abc + a^2 + b^2 + c^2 - 6(ab + bc + ca) + 6(a + b + c) - 7 = 0. \tag{10}$$

This necessary condition (10) for the constructability of the nine-hinged linkage exhibits a symmetry as it is invariant under permutations of $\{a, b, c\}$. This is to be expected because links can be arbitrarily exchanged, as discussed before. This is why (10) must necessarily be a symmetric polynomial.

We first consider some special limits of (10). Starting with the homogeneous linkage $a = b = c$ corresponding to the situation in which all nine links are identical, we see that (10) simplifies to

$$8a^3 - 15a^2 + 18a - 7 = 0, \tag{11}$$

and, thus, leads to the nine-hinged Möbius Kaleidocycle presented by Schönke and Fried [7]. The corresponding twist angle of the links and a particular configuration of the linkage in terms of its hinge vectors are given explicitly in the Supplementary Information S3 of that reference.

As another limit of (10), we consider the removal of the links of type C by setting $c = 1$. This is tantamount to taking the links of type C to be untwisted and of vanishing length, resulting in a six-hinged linkage with the ordering $ABABAB$. The resulting reduction leads to the condition $a + b = 0$ which, interestingly, encompasses not only the solution for the classical six-hinged kaleidocycle with 90° twist angles (for which $a = b = 0$) but also the family of threefold-symmetric Bricard linkages with alternating twist orientations (for which $a = -b \neq 0$) as described, for example, by Chen, You and Tarnai [17].

For the general form (10) of the solvability condition, two of the three twist angles can be selected arbitrarily while leaving the remaining one to be determined consistent through (10). Although there are obviously infinitely many ways to choose angles in this manner, we illustrate the idea by singling out a simple rational solution of (10) as an example (with $a = 0$, $b = 5/7$, $c = 6/7$) while recognizing that there are also infinitely many rational solutions. The result is an underconstrained linkage with threefold rotational symmetry, as shown in Fig. 4 (left). A Maxwell–Calladine analysis identical to that applied to Möbius Kaleidocycles in the Supplementary Information S3 of Schönke and Fried [7] then provides compelling evidence that any linkage obeying (10) has a single internal degree of freedom and, thus, is underconstrained.

It can also be verified that the links of this family of linkages are minimally twisted, as discussed thoroughly by Schönke and Fried [7] (see, also, the Supplementary Information S1 of that reference). An issue that becomes relevant in chains made out of disparate links is that the minimal twist condition is a collective property of the linkage, as opposed to a property of the individual

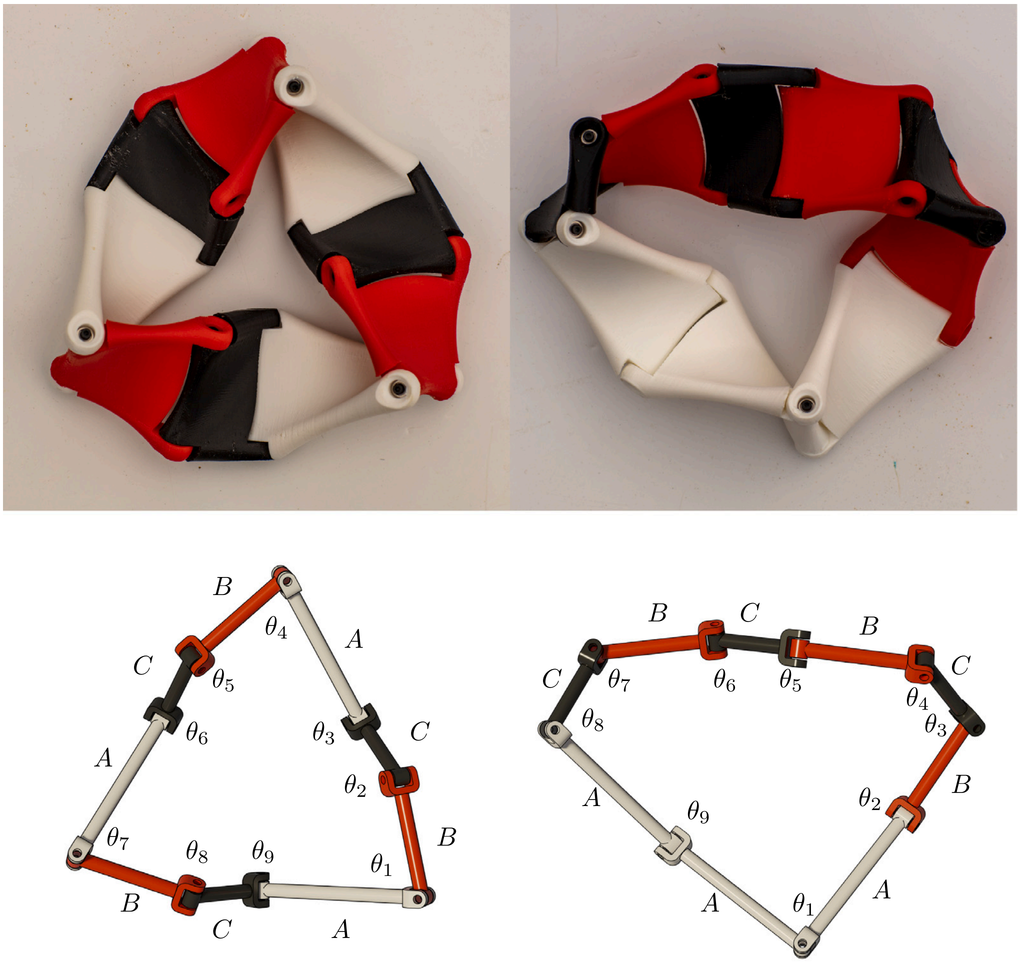

Fig. 4. (Top) Images of a 3D-printed nine-hinged linkage made from three distinct types of links obeying (1) with cosines of the twist angles $\cos\alpha_1 = 0$ (white) for link type A, $\cos\alpha_2 = 5/7$ (red) for link type B, and $\cos\alpha_3 = 6/7$ (black) for link type C, or equivalently, $\alpha_1 = 90°, \alpha_2 \approx 44.4°, \alpha_3 \approx 31.0°$. (Bottom) Technical sketches of the linkages shown above. The link types A, B, and C, as well as the hinge angles $\theta_{1,\ldots,9}$ are indicated. (Left) An ordering $ABCABCABC$ of the links with threefold rotational symmetry. (Right) An asymmetric ordering $AAABCBCBC$ of the same links. Both orderings lead to underconstrained (minimally twisted) linkages with a single internal degree of freedom. This kinematic property is conserved for any ordering of the links.

links. We define a minimal twist condition for an arbitrary linkage in the following way: a linkage is minimally twisted if its kinematic constraints do not allow for a decrease in the twist angle of any one of its links while keeping all other links fixed.

A question that cannot be answered conclusively at this juncture concerns why a condition such as (10) suffices to ensure that any member of the family of nine-hinged linkages introduced here is underconstrained and minimally twisted. A clue of potential importance is tied to the existence of the specific threefold symmetric configuration in which three planar hinges form a 120° angle (as illustrated in Fig. 3), upon which the preceding analysis is based. This configuration likely constitutes the "most severe constraint" during the everting motion of the linkage. There appears to be a deep connection between certain spatial symmetries of the linkage and a geometric locking phenomenon facilitated by the minimal twist angle.

Since the linkage consists only of links consistent with the Bennett condition (1), it is possible to create a large variety of distinct linkages by arbitrarily reordering those links. The combinatorial problem of determining how many distinguishable orderings of the links exist involves dealing with cyclic 9-letter words using each letter A, B, and C three times, the starting point and the reading direction being arbitrary. With a program to perform a brute force search, we found ninety-four distinct orderings.

Any ordering of the links apart from the threefold symmetric ordering $ABCABCABC$ breaks the threefold rotational symmetry of the linkage. However, since the linkage with $ABCABCABC$ ordering is minimally twisted, any reordered version of the linkage must also be minimally twisted. Moreover, since mobility is preserved under any reordering of the links and the linkage with $ABCABCABC$ ordering has a single internal degree of freedom, any reordered version of the linkage must also have only a single internal degree of freedom. An image of a linkage with $AAABCBCBC$ ordering is shown in Fig. 4 (right). Thus, although the solvability condition (10) was derived with reference to the threefold rotational symmetric ordering $ABCABCABC$, the foregoing

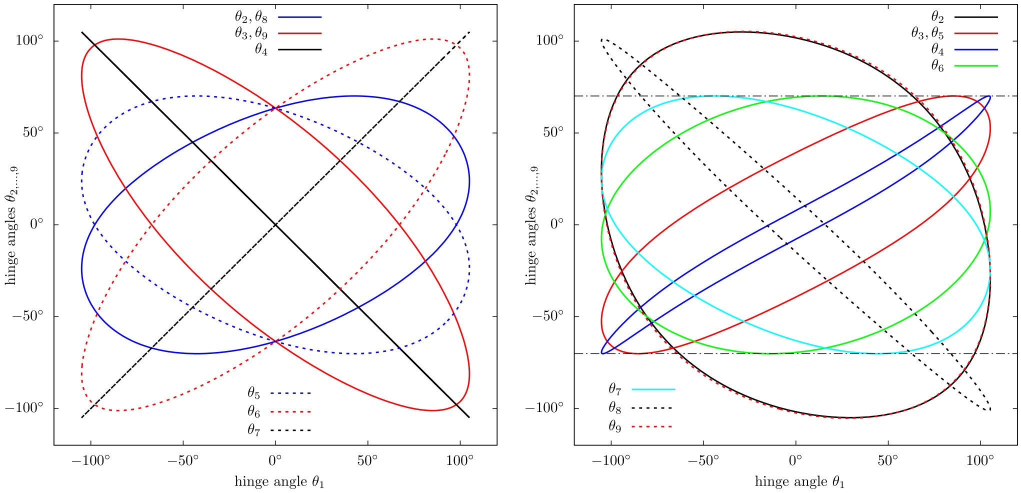

Fig. 5. Input–output relations of the two linkages shown in Fig. 4. These relations exemplify distinct motion behavior. Hinge angles θ_{2-9} are plotted against hinge angle θ_1, see Fig. 4 for the assignment of angles. (Left) Ordering $ABCABCABC$ of the links has threefold rotational symmetry (Fig. 4 (left)) and yields triples of identical hinge angles. However, it is clear from Fig. 3 that the sign of the second angle in each triple is reversed due to the nonorientability of the linkage. The triples are $\{\theta_1, \theta_4, \theta_7\}$, $\{\theta_2, \theta_5, \theta_8\}$, and $\{\theta_3, \theta_6, \theta_9\}$. Whenever one triple of hinge angles is zero (in this plot $\theta_1 = \theta_4 = \theta_7 = 0$), the remaining triples of hinge angles have identical absolute values. We conjecture that this is true for any choice of a, b, c consistent with (10) in the threefold symmetric ordering $ABCABCABC$. (Right) Asymmetric ordering $AAABCBCBC$ of the links (Fig. 4 (right)). While the motion is much more intricate in comparison to the threefold symmetric ordering, it appears from the plot that hinge angles of hinges positioned between the same type of link pair have the same minimum and maximum angles. In this case, it is apparent from Fig. 4 that the hinges corresponding to θ_3, θ_4, θ_5, θ_6, and θ_7 each lie between one link of type B and one link of type C. Their identical minimum and maximum values are highlighted by the two black horizontal dash-dotted lines. While the curves for θ_2 and θ_9 are very close, they are not identical and their minimum and maximum values differ by approximately 0.3°.

observations show that (10) is the general condition that ensures that any closed Bennett chain consisting of nine links of three different types, with each type appearing exactly three times, is underconstrained and minimally twisted.

The two linkages with different link orderings presented in Fig. 4 also exhibit different input–output relations. The system is too complex to allow us to obtain general analytical results for their input–output relations. However, we performed numerical computations and present the corresponding results in Fig. 5. While the linkage with $ABCABCABC$ ordering shows a regular behavior reflecting its threefold rotational symmetry, the linkage with $AAABCBCBC$ ordering exhibits a more intricate motion. One phenomenon apparent in the $AAABCBCBC$ ordering is that hinge angles of hinges positioned between the same type of link pair have the same minimum and maximum angles. This fact can be proven in the following way. Let θ_1 be the hinge angle between links 1 and 2 of a linkage of N links constructed consistent with the Bennett condition (1). Successive exchange operations can then be performed on links 3 through N of that linkage to reach any permutation of the original sequence of links that leaves links 1 and 2 unchanged. Because links 1 and 2 are not changed in this procedure, the hinge angle between them is also unchanged. An analogous procedure can be applied to an arbitrary configuration of the original linkage, and in particular for the configurations in which θ_1 achieves its extrema. Therefore, we infer that the range of possible values for any hinge angle may not depend on the ordering of links that are not in contact with the corresponding hinge.

4. Conclusions

We explore the profound consequences of an exchange operation and an associated symmetry introduced by Wohlhart [1]. In addition to showing that the exchange operation can be used to construct the Bennett [2] four-bar linkage, we show that the mobility of that linkage can be explained using the exchange symmetry. We also establish general results for any closed kinematic chain comprised entirely of links that satisfy the Bennett condition (1), namely a Bennett chain, demonstrating in particular that it is possible to arbitrarily reorder links of such a chain without altering the internal degrees of freedom. Moreover, we present a tentative explanation for the single internal degree of freedom in any underconstrained closed chain with links satisfying the Bennett condition, as exemplified by the recently discovered Möbius Kaleidocycles described by Schönke and Fried [7].

The opportunity to reorder links in a linkage arbitrarily while preserving its mobility makes it a candidate for use as a modular system in settings where different motion paths have to be realizable with a given fixed set of links. To illustrate this point, consider the closed Bennett chain made from $N = 9$ links of three distinct types. Since two of the three twist angles can be chosen freely (while the remaining twist angle is determined by solving the cubic (10)), the resulting parameter space is sufficiently large to accommodate a broad spectrum of application-tailored designs.

The theory of bonds introduced by Hegedüs et al. [18] might provide further insights and possibly a rigorous proof of the exceptional mobility of the 9R linkage presented here.

Finally, we mention that a similar analysis can be performed for a closed Bennett chain with $N = 12$ links of four distinct types. For such a chain, the solvability condition analogous to (10) is a formidable twelfth-order polynomial in four unknowns. Again, the analysis relies on the threefold rotational symmetry of the initial ordering as for the case of $N = 9$ links. An analytical treatment of other cases, such as $N = 7$ or $N = 8$, might be possible but will likely be more challenging because of the absence of threefold rotational symmetry. We are currently exploring the feasibility of obtaining other analytical results and are developing a general numerical approach to studying underconstrained closed Bennett chains.

CRediT authorship contribution statement

Johannes Schönke: Conception and design of study, Analysis and/or interpretation of data, Writing – original draft, Writing – review & editing. **Martín Forsberg Conde:** Conception and design of study, Analysis and/or interpretation of data, Writing – original draft, Writing – review & editing. **Eliot Fried:** Conception and design of study, Analysis and/or interpretation of data, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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