

# A Note on Shape Invariant Potentials for Discretized Hamiltonians

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## Abstract

Using the method of the "exact discretization" of the Schrödinger equation, we propose a particular discretized version of the  $N = 2$  Supersymmetric Quantum Mechanics. After defining the corresponding shape invariance condition, we show that the energy spectra and wavefunctions for discretized Quantum Mechanical systems can be found using the technique of  $N = 2$  Supersymmetric Quantum Mechanics exactly the same way as it is done for their continuous counterparts. As a demonstration of the present method we find the energy spectrum for a discretized Coulomb potential and its ground state wave function.

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# 1 Introduction

$N = 2$  Supersymmetric Quantum Mechanics (SQM) [1] – [3] being originally introduced as a tool for studying of the problem of supersymmetry breaking in more complicated supersymmetric field theories, has many interesting applications both in Physics and Mathematics (see for example [4] – [5] for reviews).

One such application is finding of the energy spectra of “usual” nonsupersymmetric Hamiltonians by using supersymmetry as an auxiliary tool. Recall, that  $N = 2$  supersymmetry algebra contains mutually hermitian conjugated supercharges  $Q$  and  $Q^\dagger$  and the Hamiltonian  $H$ , with only nonzero anticommutation relation

$$\{Q, Q^\dagger\} = H \quad (1.1)$$

This algebra can be realized in a matrix form as follows

$$Q = \begin{pmatrix} 0 & 0 \\ A_1 & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & A_1^\dagger \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} H^{(1)} - E_0^{(1)} & 0 \\ 0 & H^{(2)} - E_0^{(1)} \end{pmatrix} \quad (1.2)$$

In fact, the supersymmetric Hamiltonian  $H$  splits into two Hamiltonians  $H^{(1)}$  and  $H^{(2)}$ , each being factorized in terms of a ground state energy  $E_0^{(1)}$  for the first Hamiltonian and two ladder operators  $A_1^\dagger$  and  $A_1$

$$H^{(1)} = A_1^\dagger A_1 + E_0^{(1)}, \quad H^{(2)} = A_1 A_1^\dagger + E_0^{(1)} \quad (1.3)$$

For the case of one dimensional space, the operators  $A_1^\dagger$  and  $A_1$  have the form

$$A_1^\dagger = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W_1(x), \quad A_1 = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W_1(x) \quad (1.4)$$

where an arbitrary function  $W_1(x)$ , called the superpotential, is expressed via the ground state wavefunction of the first Hamiltonian

$$W_1(x) = \frac{\hbar}{\sqrt{2m}} \frac{d \ln \psi_0^{(1)}(x)}{dx} \quad (1.5)$$

As a consequence of relations (1.3), the Hamiltonians  $H^{(1)}$  and  $H^{(2)}$  have the form

$$H^{(1,2)} = -\frac{d^2}{dx^2} + V^{(1,2)}(x) \quad (1.6)$$

where

$$V^{(1)}(x) = W_1^2(x) - \frac{\hbar}{\sqrt{2m}} \frac{dW_1(x)}{dx} + E_0^{(1)}, \quad (1.7)$$

$$V^{(2)}(x) = W_1^2(x) + \frac{\hbar}{\sqrt{2m}} \frac{dW_1(x)}{dx} + E_0^{(1)}$$

As it can be concluded from the discussion above,  $N = 2$  SQM has the following basic properties

- From the algebra (1.1) it follows that the spectrum of the Hamiltonian  $H$  is positive semi-definite.
- The wave functions and the corresponding energy levels of the Hamiltonians (1.6) are related as

$$\psi_n^{(2)} = (E_{n+1}^{(1)} - E_0^{(1)})A_1\psi_{n+1}^{(1)}, \quad E_n^{(2)} = E_{n+1}^{(1)} \quad (1.8)$$

- If one of the Hamiltonians (say  $H^{(1)}$ ) has an eigenstate with zero energy, then the corresponding wave function can be found from the equation

$$A_1\psi_0^{(1)} = 0 \quad (1.9)$$

After using the explicit form of  $A_1$  given in (1.4), this equation leads to (1.5).

- The Hamiltonian  $H^{(2)}$  can be “refactorized”

$$H^{(2)} = A_2^\dagger A_2 + E_0^{(2)} = A_2^\dagger A_2 + E_1^{(1)} \quad (1.10)$$

in terms of new ladder operators

$$A_2^\dagger = \frac{\hbar}{\sqrt{2m}} \left( \frac{d}{dx} + \frac{d \ln \psi_0^{(2)}(x)}{dx} \right), \quad (1.11)$$

$$A_2 = \frac{\hbar}{\sqrt{2m}} \left( -\frac{d}{dx} + \frac{d \ln \psi_0^{(2)}(x)}{dx} \right)$$

and a new superpotential, which is expressed in terms of the lowest energy wavefunction of the Hamiltonian  $H^{(2)}$ . Then one can construct the third Hamiltonian  $H^{(3)}$

$$H^{(3)} = A_2 A_2^\dagger + E_0^{(2)} = A_2 A_2^\dagger + E_1^{(1)} \quad (1.12)$$

which has the same energy levels as  $H^{(2)}$  except of the lowest energy of of  $H^{(2)}$ . Continuing in this manner one can construct a hierarchy of Hamiltonians. The total number of the Hamiltonians in the hierarchy is equal to a number  $n$  of the bound states of the Hamiltonian  $H^{(1)}$ . The energy levels and the wavefunctions of these Hamiltonians being

$$E_n^{(m)} = E_{n+1}^{(m-1)} = \dots = E_{n+m-1}^{(1)}, \quad (1.13)$$

$$\psi_n^{(m)} = (E_{n+m-1}^{(1)} - E_{m-2}^{(1)})^{-1/2} \dots (E_{n+m-1}^{(1)} - E_0^{(1)})^{-1/2} A_{m-1} \dots A_1 \psi_{n+m-1}^{(1)}$$

- Suppose, the potentials in (1.7) satisfy the so called shape invariance condition [6]

$$V^{(2)}(x, a_1) = V^{(1)}(x, a_2) + R(a_1) \quad (1.14)$$

where the parameters  $a_1$  and  $a_2$  are related to each other via some function  $a_2 = \phi(a_1)$  and the “rest”  $R(a_1)$  does not depend on the coordinate  $x$ . Then the spectrum of the Hamiltonian  $H^{(1)}$  can be found to be

$$E_n^{(1)} = \sum_{k=1}^n R(a_k) \quad (1.15)$$

Here  $a_k$  means that the function  $\phi(a_1)$  is applied  $k$  times. To obtain this result one constructs a hierarchy of Hamiltonians

$$H^{(s)} = -\frac{d^2}{dx^2} + V^{(1)}(x, a_s) + \sum_{k=1}^{s-1} R(a_k) \quad (1.16)$$

by repeated use of the shape invariance condition. Then, assuming that the ground state energy of the first Hamiltonian is zero and using the fact that spectra of the Hamiltonians  $H^{(s)}$  and  $H^{(s+1)}$  are the same, except of the lowest energy of  $H^{(s)}$ , one gets (1.15).

To summarize, the technique of  $N = 2$  SQM allows one to find a spectrum of a “usual” nonsupersymmetric Hamiltonian  $H^{(1)}$  by purely algebraic methods, provided the corresponding potential satisfies the shape invariance condition (1.14). Since the discussion above was for the case when the space coordinate  $x$  is continuous, it is natural to consider a modification of this method for the case when SQM is defined on a lattice [7]– [8]. To this end we use the procedure of an “exact discretization” of the Schrödinger equation given in [9]– [12] and find the factorization of the “discrete” Hamiltonian in terms of the of two ladder operators<sup>1</sup>. This step actually means that we are considering a discrete version of  $N = 2$  SQM. Then we present a discretized version of the shape invariance condition (1.14) which allows one to find the wave functions and energy spectra of Hamiltonians, whose potentials obey this condition. Finally, we illustrate this technique on the example of the Coulomb potential. The last Section contains a brief discussion of some open problems and possible generalizations of our results.

## 2 Discretization

We follow the procedure of the “exact discretization”, proposed by Tarasov [9]– [11]. The advantage of this discretization scheme is that an immediate correspondence between continuous differential equations and discrete difference equations can be achieved. Namely, the main algebraic properties of differential operators, like the semigroup property, Leibniz rule and differentiation of polynomials carry over to the difference operators. Furthermore, discretized solutions of differential equations are

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<sup>1</sup>For alternative methods of factorization of second order differential equations, see [13] and references therein.

solutions of the corresponding difference equations. All these properties are obtained without a limiting process, in which the lattice constant of the discrete model goes to zero.

In the following we summarize the main steps of the “exact discretization” [9]–[11].

Given a wave function  $\tilde{\psi}(x)$  one starts by computing its Fourier transform

$$\mathcal{F}\{\tilde{\psi}(x)\} = \tilde{\psi}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \tilde{\psi}(x) e^{-ikx} \quad (2.1)$$

and defines a new function

$$\psi(k) = \tilde{\psi}(k) \left( \Theta\left(k + \frac{\pi}{a}\right) - \Theta\left(k - \frac{\pi}{a}\right) \right) \quad (2.2)$$

where  $a > 0$  is a lattice constant and  $\Theta(k)$  is the Heaviside step function,  $\Theta(k) = 1$  for  $k \geq 0$ , and  $\Theta(k) = 0$  for  $k < 0$ . The new function is equivalent to the former one  $\psi(k) = \tilde{\psi}(k)$  within the interval  $k \in [-\frac{\pi}{a}, +\frac{\pi}{a}]$ . The Fourier transform of the function (2.2) can be thought of as the coefficients of a discrete Fourier series

$$\psi[n] = \frac{a}{2\pi} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk \psi(k) e^{ikna} \equiv \mathcal{F}_a^{-1}\{\psi(k)\} \quad (2.3)$$

Using the relation  $\mathcal{F}\{i\frac{\partial}{\partial x}\} = k$  one can define a difference operator  $\Delta^m$  which is a discrete analog of the  $m_{\text{th}}$  derivative  $D^m$

$$\frac{\mathcal{F}^{-1}\{\mathcal{F}_a\{\Delta^m\}\}}{a^m} = D^m. \quad (2.4)$$

When acting on a “trial” function  $f[n]$  one gets

$$\frac{\mathcal{F}^{-1}\{\mathcal{F}_a\{\Delta^m f[n]\}\}}{a^m} = D^m f[n] = \frac{\partial^m}{\partial x^m} \mathcal{F}_a^{-1}\{\tilde{f}(k)\} \quad (2.5)$$

and it follows that

$$\mathcal{F}_a\{\Delta^m f[n]\} = (iak)^m \mathcal{F}_a\{f[n]\} \quad (2.6)$$

This suggests that  $\Delta^m f[n]$  has to be a discrete convolution

$$\Delta^m f[n] = \sum_{j=-\infty}^{\infty} K_m[j] f[n-j] = \sum_{j=-\infty}^{\infty} K_m[j+n] f[n] \quad (2.7)$$

with  $K_m$  being the kernel. Using (2.6) one finds that the Fourier transform of the kernel should yield  $\mathcal{F}_a\{K_m\} = (iak)^m$ . Finally, the solutions of the equation (2.6) for the first and for the second order difference operators are

$$\Delta^1 f[n] = \sum_{j=-\infty, j \neq 0}^{\infty} \frac{(-1)^j}{j} f[n-j] \quad (2.8)$$

$$\Delta^2 f[n] = - \sum_{j=-\infty, j \neq 0}^{\infty} \frac{2(-1)^j}{j^2} f[n-j] - \frac{\pi^2}{3} f[n] \quad (2.9)$$

One can show that these difference operators obey the semigroup property

$$\Delta^2 = \Delta^1 \Delta^1 \quad (2.10)$$

as well as the Leibniz rule

$$\Delta^1(fg) = \Delta^1(f)g + f\Delta^1(g) \quad (2.11)$$

In terms of the difference operator (2.9) and the Fourier transforms (2.3) one finds a discretized Schrödinger equation

$$i\hbar \frac{d\psi[n](t)}{dt} = \frac{\hbar^2}{ma^2} \sum_{j=-\infty, j \neq 0}^{\infty} \frac{2(-1)^j}{j^2} \psi[n-j](t) + V\psi[n](t) \equiv H\psi[n](t) \quad (2.12)$$

with

$$V[n] = \frac{2\pi^2}{3a^2} + \tilde{V}[n]. \quad (2.13)$$

and  $\tilde{V}[n]$  being a discretization of the potential which enters the continuous Schrödinger equation.

### 3 N=2 SQM on a lattice

In order to construct a discretized version of  $N = 2$  SQM, we use the difference operator (2.8) and define the ladder operators

$$A_1[n] = \frac{\hbar}{\sqrt{2m}} \Delta^1 + W[n], \quad A_1[n]^\dagger = -\frac{\hbar}{\sqrt{2m}} \Delta^1 + W[n] \quad (3.1)$$

From these ladder operators we construct a lattice Hamiltonian

$$H^{(1)}[n] = A_1^\dagger[n]A_1[n] = -\frac{\hbar^2}{2m} \Delta^2 + V^{(1)}[n] \quad (3.2)$$

with the potential

$$V^{(1)}[n] = W^2[n] - \frac{\hbar}{\sqrt{2m}} \Delta^1 W[n]. \quad (3.3)$$

Equation (3.2) holds because the difference operator fulfills the semigroup property (2.10) and the Leibniz rule (2.11).

The ground state wave function is given by a discrete analogue of the equation (1.9), which yields

$$W[n]\psi_0^{(1)}[n] = -\frac{\hbar}{\sqrt{2m}} \Delta^1 \psi_0^{(1)}[n] \quad (3.4)$$

As it was the case for the continuous  $N = 2$  SQM described in the Introduction, one can construct a hierarchy of Hamiltonians with the same properties of the energy levels and wavefunctions as in (1.13). In particular, the second Hamiltonian in the hierarchy is

$$H^{(2)}[n] = A_1[n]A_1^\dagger[n] = -\frac{\hbar^2}{2m}\Delta^2 + V^{(2)}[n] \quad (3.5)$$

with the potential

$$V^{(2)}[n] = W^2[n] + \frac{\hbar}{\sqrt{2m}}\Delta^1 W[n] \quad (3.6)$$

and so on. Finally, in order to find the spectrum of the original Hamiltonian  $H^{(1)}[n]$  we introduce a condition of the shape invariance,

$$W^2[n](a_1) + \Delta^1 W[n](a_1) = W^2[n](a_2) - \Delta^1 W[n](a_2) + R(a_1) \quad (3.7)$$

where  $\phi$  is a function of the parameters  $a_1$  and  $a_2$  which are present in the Hamiltonians  $H^{(1)}$  and  $H^{(2)}$ .

## 4 Solving the discrete Coulomb potential

As it follows from the discussion given in Sections 2 and 3, one can immediately apply the technique of discretized  $N = 2$  SQM for finding of energy spectra of the one dimensional Quantum Mechanical models, whose list (in case of continuous space coordinate  $x$ ) is given in [14]. Indeed, the crucial requirement for applying of the technique of  $N = 2$  SQM for finding of the energy spectra is that corresponding potential should satisfy the shape invariance condition (1.14). On the other hand, the procedure of the “exact discretization”, essentially maps this equation into its discrete counterpart (3.7). Therefore, the potentials given in [14] will satisfy the discretized shape invariance condition after Taylor expanding and replacing the continuous coordinate with the lattice constant times an integer  $n$ . Finally, the properties (2.10) and (2.11) of the difference operators provide a possibility of the corresponding factorization and for a construction of the hierarchy of Hamiltonians which leads to the expression for the energy spectrum given in (1.15).

As an illustration of the discussion above, let us consider as an example the discretization of the Coulomb potential in some detail. The Coulomb potential is an example of a Quantum Mechanical system which in the continuous case has been solved using the technique of  $N = 2$  SQM [15]. Here we shall present a similar discussion for the energy spectrum of the corresponding discretized model. Recall, that by exploiting spherical symmetry, the Coulomb problem becomes effectively one dimensional and the potential can be written in natural units as

$$V(r) = -\frac{1}{r} + \frac{\ell(\ell+1)}{2r^2}, \quad r > 0. \quad (4.1)$$

In the discrete case we replace the continuous radius by an integer  $r \rightarrow an$ . For simplicity we shall set the lattice constant to unity  $a = 1$ . This potential can be factorized by the superpotential

$$W[n] = \frac{1}{\ell + 1} - \frac{\ell + 1}{n}. \quad (4.2)$$

Using the power law property of the first difference operator<sup>2</sup>  $\Delta^1 n^k = kn^{k-1}$  and the fact that for a constant  $c$  one has  $\Delta^1 c = 0$ , we construct two partner potentials

$$V^{(1)}[n] = \frac{1}{(\ell + 1)^2} + \frac{(\ell + 2)(\ell + 1)}{n^2} - \frac{2}{n} \quad (4.3)$$

$$V^{(2)}[n] = \frac{1}{(\ell + 1)^2} + \frac{\ell(\ell + 1)}{n^2} - \frac{2}{n}. \quad (4.4)$$

These two potentials are indeed related by the shape invariance condition (3.7). The role of the parameter  $a_1$  is played by  $\ell$ , while the function  $\phi$  which relates the parameters  $a_2$  and  $a_1$  in the partner Hamiltonians is given by  $a_2 = \phi(\ell) = \ell + 1$ . We can further identify the ground state energy  $E_0^{(1)} = \frac{1}{(\ell+1)^2}$  and the "rest" function as

$$R(\ell) = \frac{2\ell + 3}{(\ell + 1)^2(\ell + 2)^2}. \quad (4.5)$$

According to (1.15) the full spectrum can then be found

$$E_n^{(1)} = E_0^{(1)} + \sum_{\ell=1}^n R(\ell) = \frac{1}{(\ell + 1)^2} + \sum_{m=1}^n \frac{2(\ell + m) + 1}{(\ell + m)^2(\ell + m + 1)^2}. \quad (4.6)$$

The ground state wave function can also be easily obtained in the discrete case by using the equation (3.4). Making an ansatz for the ground state wave function

$$\psi_0^{\ell=0}[n] = \sum_{j=0}^{\infty} c_j n^j \quad (4.7)$$

one finds the condition for the unknown coefficients

$$\frac{c_0}{n} + \sum_{j=0}^{\infty} (j c_{j+1} + c_j) n^j = 0. \quad (4.8)$$

This equation means that  $c_0 = 0$ , while the other coefficients obey recurrence relations

$$j c_{j+1} + c_j = 0, \quad (4.9)$$

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<sup>2</sup>The proof for differences of a power law function,  $\Delta^1 n^k = kn^{k-1}$ , given in [10] holds only for integer  $k \in \mathbb{N}$ . However, one can easily extend it for  $k \in \mathbb{R}$  using generalized binomial coefficients. The reason that it works stems from the fact that only a single term in the summation over the binomial coefficients remains finite.



which can be solved by

$$c_j = \frac{(-1)^{j-1}}{j!} N \quad (4.10)$$

with  $N$  some undetermined constant. We finally find the ground state wave function

$$\psi_0^{\ell=0}[n] = N n e^{-n} \quad (4.11)$$

which represents a discrete analog of the well known result of the continuum case.

## 5 Conclusions

In the present paper we considered the problem of finding of the energy spectrum for lattice Hamiltonians in terms a discretized version of  $N = 2$  Supersymmetric Quantum Mechanics, by applying of the corresponding shape invariance condition. We restricted ourselves with the consideration of the simplest case of the one dimensional Quantum Mechanical models. It would be very interesting to generalize this technique to the case of two and three dimensional systems, appropriately generalizing the procedure of “exact discretization” of the Schrödinger equation and considering a discretization of the shape invariance condition for higher dimensional  $N = 2$  SQM [16]. This generalization could allow an application of the suggested technique for finding of energy spectra of some realistic physical examples.

Another interesting generalization of our results could be a construction and further study of discrete versions of systems with higher number of supersymmetries. An example of such systems are one dimensional [17] and multidimensional [18]  $N = 4$  Supersymmetric Quantum Mechanics. Let us note, that in these systems the supersymmetry algebra can also contain nonzero central charges, which gives a interesting structure for vacuum ground states (so – called partial supersymmetry breaking)<sup>3</sup>. An application of our approach to models with nonlinear supersymmetry and hidden suppersymmatry (considered in the continuous case in [21]– [22]) is another interesting problem to address.

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<sup>3</sup>A discussion of isospectral Hamiltonians in the framework  $N = 4$  SQM can be found in [19]. An application to particular  $N = 4$  SQM models is given in [18]– [20].

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