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The third-order structure function in two dimensions: The Rashomon effect

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We study the third-order longitudinal structure function, $S_3(r)$, in two-dimensional turbulence. In three dimensions, there is considerable theoretical, experimental, and numerical consensus regarding the validity of Kolmogorov's arch-famous "$\frac{4}{5}$th law" for $S_3(r)$. By contrast, in two dimensions, two disparate cascades, changed dissipation anomalies, a large-scale drag, and other factors conspire to create several versions of the $S_3(r)$ "law." This single quantity can vary considerably when viewed from different perspectives, reminiscent of the "Rashomon effect" in anthropology. After reviewing the history and usage of $S_3(r)$ in two-dimensional turbulence, we show that $S_3(r)$ generically embodies a mixture of energy and enstrophy fluxes. Building on this result, we derive $S_3(r)$ laws for freely decaying and forced two-dimensional turbulent flows, where we also account for the effects of a large-scale drag, an inextricable feature of quasi two-dimensional turbulence in experimental and atmospheric flows. We draw attention to the caution needed in interpreting $S_3(r)$ in two-dimensional turbulence.
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I. INTRODUCTION

The beginning of the modern theory of turbulence may arguably be Kolmogorov's landmark paper[1] from 1941 in which he furnished a mathematical framework for the cascade of energy from large to small scales. Building on Richardson's intuition[2,3] and focusing on homogeneous and isotropic flows, Kolmogorov characterized the statistics of turbulent fluctuations at different spatial scales by introducing the longitudinal velocity differences, $\delta u \equiv (\vec{u}(\vec{x}+\vec{r}) - \vec{u}(\vec{x})) \cdot \vec{r}/r$, where $\vec{u}$ is the velocity measured at positions $\vec{x}+\vec{r}$ and $\vec{x}$, and $r = |\vec{r}|$.

Dimensionally, one may naively expect that the third moment of these velocity differences—kinetic energy per unit mass, δu^2, times velocity, δu—should be related to the flux of energy into or out of a volume $\propto r^3$. Starting from the Karman–Howarth equation[4] and focusing on the inertial range, Kolmogorov, in another remarkable paper[5] from 1941, derived an exact relation for the third moment $\langle \delta u^3 \rangle \equiv S_3(r) = -\frac{4}{5}\epsilon r$, where ϵ is the average energy dissipation rate per unit mass and the angular brackets denote the spatial average. This so-called "4/5th law," a cornerstone of turbulence theory, encapsulates the main features of three-dimensional turbulence. The minus sign indicates the flux of energy from large to small scales, which is now called a direct cascade. The power-law dependence on r is a signature of the self-similarity of the velocity field. The independence from any large forcing scale L embodies the universality of this result. Finally, the independence from viscosity hints at the energy dissipation anomaly,[1,6] where $\epsilon \neq 0$ in the inviscid limit. Commenting on the importance of the 4/5th law, Frisch[7] notes the following: "[this law] constitutes a kind of 'boundary condition' on theories of turbulence: such theories, to be acceptable, must either satisfy the four-fifths law, or explicitly violate the assumptions made in deriving it."

Given the foundational role $S_3(r)$ has played in three-dimensional (3D) turbulence, it may come as a surprise to find that the first mention (to our knowledge) of $S_3(r)$ for two-dimensional (2D) turbulence is in the appendix of a 1992 paper,[8] a quarter century after Kraichnan's seminal contribution in 2D turbulence,[9] and nearly 40 years after Fjørtoft[10] and Batchelor[11] first proposed that energy goes from small to large scales in 2D turbulence. An examination of the early years of research in 2D turbulence provides some hints for the reasons behind this neglect.

Kraichnan's work in 2D turbulence appears to have been exclusively in spectral space,[12–16] and other early researchers in the field followed suit.[17–19] In his pioneering 1967 paper[9] on the inertial ranges in 2D turbulence, Kraichnan analyzed the transfer of energy in spectral space using $T(k, p, q)$, where T measures the energy transfer rate out of wavenumber k, and $\vec{k}$, $\vec{p}$, and $\vec{q}$ are three wavenumbers in a triad ($\vec{k} = \vec{p} + \vec{q}$; $k = |\vec{k}|$, $p = |\vec{p}|$, and $q = |\vec{q}|$). Kraichnan laid the theoretical basis for two distinct inertial ranges, one with a flux of enstrophy (the norm of vorticity) to small scales and another with a flux of energy to large scales.[9] Much of the work that came after him built on this framework and tested its predictions,[13,14,17,19,20] and so we hazard to speculate that Kraichnan's influence helped to keep 2D turbulence out of real space.

Another reason for this preference for spectral space might have been that for many years, studies of turbulence in flatland existed only *in silico*.[21,22,59] Simulations with periodic

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boundary conditions naturally lend themselves to discrete Fourier analysis. Indeed, most simulations were carried out using a pseudo-spectral code,[23] where the governing Navier–Stokes equations are first transformed and then evolved in spectral space. A few authors gave a curt nod to real space in early work,[19] but with the advent of laboratory quasi-2D flows,[24,25] Babiano *et al.*[26] advocated a shift to real space and structure functions (statistical moments of δu). Experiments appear to have been the main reason for a renewed interest in real space, as structure functions are easier to calculate directly from experimental data than, say, $T(p, q, k)$.

Over the last two decades, $S_3(r)$ has been extensively used for another purpose. As Kraichnan pointed out,[9] "A principal reason for exploring two-dimensional turbulence has been the possible application to intermediate-scale meteorological flows." For such flows, the large aspect ratios involved suggest, but do not prove, two-dimensionality. The atmospheric energy spectrum,[27] $E(k)$, appears to show scaling consistent with the phenomenology of 2D turbulence, but with a puzzling twist. At small k, $E(k) \propto k^{-3}$, whereas at larger k, $E(k) \propto k^{-5/3}$. If the spectral exponents are interpreted in the framework of 2D turbulence, this is the inversion of the theoretical picture of the dual cascade (the cascade of energy to large scales and enstrophy to small scales[9,17,18]). There is considerable debate over what kind of cascade, if any, corresponds to these spectral exponents. The $k^{-5/3}$ portion, for example, is consistent with both the 3D direct energy cascade and the 2D inverse energy cascade. A second-order quantity like $E(k)$ cannot distinguish between these two. Lindborg[28] suggested that a third-order quantity proportional to energy or enstrophy flux might be useful to settle the issue and submitted $S_3(r)$ as a candidate to determine the flux directions. The ability of $S_3(r)$ to indicate the direction of energy or enstrophy flux brought it to the forefront of the debate, where it still remains.[29–31]

Because of its essential function in interpreting atmospheric turbulence data, as well as for its inherent fundamental interest, we study $S_3(r)$ laws in the inertial ranges of 2D turbulence, with particular focus on the energy and enstrophy fluxes. We restrict attention to the idealized theory of isotropic and homogeneous turbulence in an unbounded domain and in the limit of infinite Reynolds number (the inviscid limit), but we also consider the effects of a large-scale drag, an inextricable factor in experimental and atmospheric flows.

II. HISTORY OF $S_3(r)$ IN 2D TURBULENCE

It is common to cite a turbulence textbook, such as by Frisch,[7] by Monin and Yaglom,[32] and by Landau and Lifshitz,[33] and state that applying Kolmogorov's derivation[5] to 2D yields $S_3(r) > 0$. However, as pointed out by Paret and Tabeling,[34] if the same assumptions and conditions are used as in Kolmogorov's derivation, we will again be led to $S_3(r) < 0$, just as in 3D.

The first derivation (to our knowledge) of $S_3(r)$ serves as a useful example of Paret and Tabeling's point. The derivation is by Grossmann and Mertens,[8] who, like Babiano *et al.*,[26] were interested in shifting to real space. Although their work was devoted almost entirely to the second-order structure functions for velocity and vorticity, in an appendix, they derived a scaling law for $S_3(r)$ in arbitrary dimensions. Exactly paralleling the case in 3D (including considering the energy dissipation anomaly), their result for 2D reads as $S_3(r) = -\frac{3}{2}\epsilon r$, where the new factor of $\frac{3}{2}$ is a consequence of the different dimensionality, but the minus sign is the same as in 3D. The cascade is apparently a direct energy cascade. If an inverse energy cascade is to be inferred from $S_3(r)$, it is clear that assumptions appropriate to 2D will have to be explicitly invoked.

After Grossmann and Mertens,[8] there was a short hiatus until an *annus mirabilis* for the $S_3(r)$ laws in 2D occurred in 1999, which saw four independent studies of the laws for different flow conditions.[28,35–37] We turn next to the salient features of these four studies.

Belmonte *et al.*[35] closely followed the traditional Kolmogorov derivation. However, based on their experiments—perhaps the first experiments to measure $S_3(r)$ in 2D—they questioned one of the oft-used simplifying assumptions. In freely decaying 3D turbulence, on account of local equilibrium,[11] the contribution of the unsteady term for the small scales is assumed to be negligible. Belmonte *et al.* found experimentally that the unsteady term, $\frac{\partial}{\partial t}S_2(r)$ (where t is the time), is not negligible in 2D freely decaying turbulence, consistent with phenomenological considerations.[38] Although they did not obtain an $S_3(r)$ law, by computing the contribution of the unsteady term from the experimental data, they determined the shape of $S_3(r)$ and drew attention to the fact that the sign of $S_3(r)$ can be affected by the magnitude of the unsteady term.

Lindborg[28] derived $S_3(r)$ laws in detail, intending to apprehend the directions of energy and enstrophy fluxes in atmospheric flows. For turbulence sustained by a single forcing scale, he derived the $S_3(r)$ law for the inverse energy cascade, $S_3(r) = \frac{3}{2}Pr$ (where P is the average energy injection rate per unit mass), and the $S_3(r)$ law for the direct enstrophy cascade, $S_3(r) = \frac{1}{8}\beta r^3$ (where β is the average enstrophy dissipation rate). [Davidson[51] later adapted Lindborg's derivation to the freely decaying case and found the same $S_3(r)$ law for the direct enstrophy cascade.] For the k^{-3} region in the atmospheric energy spectrum, he found $S_3(r) \propto r^3$, corresponding to the direct enstrophy cascade. For the $k^{-5/3}$ region, his results for $S_3(r)$ did not provide clear answers. Later, Lindborg and Cho[39,40] reanalyzed the atmospheric data and found $S_3(r) \propto -r$ for the $k^{-5/3}$ region, leading them to interpret this region as corresponding to the direct energy cascade, thereby suggesting 3D turbulence.

Bernard[36] considered the 2D Navier–Stokes equations subject to a smooth-in-space and white-in-time forcing. Without a large-scale drag, he noted that the total kinetic energy of the flow grew linearly in time as Pt. Assuming local equilibrium for the structure functions and accounting for the lack of an energy dissipation anomaly and the presence of an enstrophy dissipation anomaly, he derived $S_3(r) = \frac{1}{8}\beta r^3$ for the region of the direct enstrophy cascade and $S_3(r) = \frac{3}{2}Pr$ for the region of the inverse energy cascade. (We discuss the anomalies in Sec. IV.) Including a large-scale drag, Bernard found that the $S_3(r)$ law for the direct enstrophy cascade remained unaffected (with the caveat that the law is expressed with the enstrophy dissipation rate, β, which is not equal to the enstrophy production rate; we return to this issue in Sec. IV D). For the inverse energy cascade, he found that the $S_3(r)$ law remained linear in r, but he did not determine the prefactor.

Yakhot[37] considered essentially the same conditions (except including a large-scale drag) as Bernard[36] (their papers were even published in the same issue of Physical Review E). He developed generating functions of the velocity differences and used them to derive the 2D "Kolmogorov relation" $\frac{1}{r^3}\frac{\partial}{\partial r}r^3 S_3(r) - 6P = 0$, wherein he accounted for the lack of an energy dissipation anomaly. Although he did not compute an $S_3(r)$ law (because he was interested in this relation only as a step to find general scaling relations for the structure functions), the Kolmogorov relation can easily be integrated to find the $S_3(r)$ law for the inverse energy cascade, $S_3(r) = \frac{3}{2}Pr$.

Hereafter, for 2D forced turbulence, we refer to the $S_3(r)$ law for the inverse energy cascade, $S_3(r) = \frac{3}{2}Pr$, and the $S_3(r)$ law for the direct enstrophy cascade, $S_3(r) = \frac{1}{8}\beta r^3$, as canonical $S_3(r)$ laws. Notable in most of the derivations discussed above is the absence of a large-scale drag, which is necessary to establish a steady-state inverse energy cascade.[41] Boffetta *et al.*[42] included such a drag term in their high-resolution simulations of 2D forced turbulence. They numerically verified the canonical $S_3(r)$ law for the inverse energy cascade. This seems to be the first clear confirmation of this law. Later, Boffetta and Musacchio[41] numerically verified the canonical $S_3(r)$ law for the direct enstrophy cascade. Note that these results suggest that the presence of a large-scale drag does not influence the canonical $S_3(r)$ laws, which were derived without any reference to such a term.

The influence of a large-scale drag as well as of lateral boundaries was carefully studied experimentally in a series of papers by Shats and co-workers.[29,43–45] They performed experiments in an electromagnetically driven double salt layer, where the flow in the top layer, which is immiscible with the bottom lubricating layer, is quasi-2D. Under these circumstances, the presence of lateral boundaries can lead to the development of a large-scale coherent flow, as was predicted by Kraichnan.[9] This coherent flow can suppress[43] or interact with the "background" turbulence.[29,44,45] In some cases, $S_3(r)$ calculated from the experimental data was reduced in magnitude;[43] more notably, in some cases, $S_3(r)$ changed sign to become negative.[44,45] Interestingly, if one subtracts out the coherent flow, the underlying inverse energy cascade can be revealed via $S_3(r) \propto r$. Finally, using $S_3(r)$ as an experimental probe, they showed how a large-scale coherent flow can make an ostensibly 3D flow into a 2D flow and *produce* inverse transfer of energy.[45] An important message from these studies is that even when an inverse cascade of energy is present, the coherent flow may change the sign of $S_3(r)$, thereby rendering $S_3(r)$ unsuitable for revealing the cascade. This finding has brought about a renewed interest[29–31] in the proper interpretation of atmospheric data and $S_3(r)$.

This brief review is not intended to be exhaustive, but we have attempted to illustrate the complexity of $S_3(r)$ in 2D turbulence. (For additional discussion, see Refs. 46–48.) In 2D turbulence, various factors, such as the presence of two disparate cascades, the unclear justification of local equilibrium in decaying turbulence, the influence of a large-scale drag, and the formation of a large-scale coherent flow, can all work to make this deceptively simple quantity much more complex, each factor adding its own imprint on $S_3(r)$. This multitude of storylines is reminiscent of Kurosawa's[49] *Rashomon*,[50] where varied perspectives combine to give a richer picture of reality. Each story is interesting in its own right, but at the same time, it is only one aspect of 2D turbulence as a whole.

III. WHY IS $S_3(r)$ RELATED TO ENERGY FLUX OR ENSTROPHY FLUX?

Here we seek to establish a mathematical relationship between $S_3(r)$ and the fluxes of energy and enstrophy. Following Kraichnan's lead, we begin our considerations in spectral space. In contrast to real space,[51] spectral space affords a spatially localized measure of energy and enstrophy, thereby permitting one to directly compute the attendant fluxes.

For 2D turbulence, the dynamical equation of the energy spectrum, $E(k)$, reads as[51]

$$\frac{\partial E(k,t)}{\partial t} = T(k,t) - 2\nu\Omega(k,t), \tag{1}$$

where $\Omega(k,t) = k^2 E(k,t)$ is the enstrophy spectrum. Equation (1) is the spectral equivalent of the 2D Karman–Howarth equation. It shows that the change in $E(k, t)$ with t is the sum of two terms. The first,[52] $T(k, t)$, is the transfer function, which stems from the non-linear terms in the Navier–Stokes equations, and the other is the viscous dissipation, where ν is the kinematic viscosity. [Hereafter, for the sake of brevity, we drop the explicit dependence on t from $E(k, t)$ and $T(k, t)$.] Note that $\int_0^\infty T(k)dk = 0$, which means the nonlinear terms serve only to move energy from wavenumber to wavenumber.[21,51] This establishes a concrete connection between $T(k)$ and the transfer of energy, and we define $T(k)$ as the rate of energy transfer out of k to larger wavenumbers (smaller scales). Also note that[21,51] $\int_0^\infty k^2 T(k)dk = 0$, where $k^2 T(k, t)$ is the rate of enstrophy transfer out of k. Often one works instead with the energy flux function, $\Pi(k) = -\int_0^k T(k')dk'$, which is the net energy transferred downscale through k, and the enstrophy flux function, $Z(k) = -\int_0^k k'^2 T(k')dk'$, which is the net enstrophy transferred downscale through k. We regard the flux functions in spectral space as accurate representations of the energy and the enstrophy fluxes.

To relate the flux functions with $S_3(r)$, we manipulate the 2D Karman–Howarth equation and obtain the following relation between $S_3(r)$ and $T(k)$ (see Appendix A):

$$S_3(r) = \frac{3}{2}r\int_0^{a/r} T(k)dk - \frac{1}{8}r^3\int_0^{a/r} k^2 T(k)dk + \cdots, \tag{2}$$

where a is an $O(1)$ constant. Written in terms of the flux functions, we arrive at a key result,

$$S_3(r) = -\frac{3}{2}\Pi\left(\frac{a}{r}\right)r + \frac{1}{8}Z\left(\frac{a}{r}\right)r^3 + \cdots. \tag{3}$$

This equation mathematically demonstrates how $S_3(r)$ embodies a mixture of the energy and enstrophy fluxes. The sign of $S_3(r)$ is affected by values of both fluxes—$S_3(r) > 0$ need not imply an inverse transfer of energy. To our knowledge, Eq. (3) is a new result. It will play a central role in our derivations of the $S_3(r)$ laws.

If the flux functions are known, using Eq. (3), we can easily calculate the canonical $S_3(r)$ laws in 2D turbulence. For the inertial range of an inverse energy cascade,[9] if we impose $\Pi(k \sim \frac{a}{r}) = -P$ and $Z(k \sim \frac{a}{r}) = 0$, we find $S_3(r) = \frac{3}{2}Pr$. For the inertial range of a direct enstrophy cascade,[9,18] if we impose $\Pi(k \sim \frac{a}{r}) = 0$ and $Z(k \sim \frac{a}{r}) = \beta$, we find $S_3(r) = \frac{1}{8}\beta r^3$. [Note that the exact value of a plays no role in the $S_3(r)$ laws because the fluxes in the inertial range are independent of k.] While the values of these fluxes are consistent with our notions of inertial ranges in the inverse energy cascade and the direct enstrophy cascade, here we have not provided any physical justification for the values. We simply posited them and obtained the $S_3(r)$ laws. In Sec. IV, we will compute the fluxes using physical arguments concerning the dynamics of energy and enstrophy.

A general remark on the sign of $S_3(r)$ may be in order. One is tempted, for example, when analyzing experimental or atmospheric data, to look only at the sign of $S_3(r)$ and interpret it as an indicator of the energy flux direction. However, in the inertial range of the direct enstrophy cascade, $S_3(r)$ is positive even though there is no energy flux. Knowing this, others have made careful comparisons[28,39,40,45] of not only the sign but also how $S_3(r)$ scales with r. Note, however, that even the scaling may be affected by non-ideal influences (see, e.g., Sec. IV D).

Finally, we note that Eq. (3) points to a severe difficulty with $S_3(r)$ in 2D flows. If the inertial ranges of the inverse energy and direct enstrophy cascades are not widely separated in r (or k), then $S_3(r)$ will represent a mixture of energy and enstrophy fluxes. For turbulence sustained by a single forcing scale, Kraichnan[9] and Leith[17] argued that spatially concurrent cascades of energy and enstrophy cannot occur. When there are two forcing scales, however, Lindborg showed that such a mixture is possible.[28] Further, as we will see in Sec. IV D, forced turbulence with a large-scale drag also allows for such a mixture.

IV. $S_3(r)$ LAWS IN 2D TURBULENCE

We now turn to our derivations of the $S_3(r)$ laws in 2D turbulence. Starting in spectral space, we first compute the flux functions $\Pi(k)$ and $Z(k)$ in the inertial range and then transform to real space by invoking Eq. (3). This yields $S_3(r)$.

To compute $\Pi(k)$, we analyze the dynamical equation for $E(k)$, Eq. (1), which reads as

$$\frac{\partial E(k)}{\partial t} = T(k) - 2\nu\Omega(k).$$

In both 2D and 3D, the viscous dissipation term is $2\nu\Omega(k)$ and $\epsilon = 2\nu \int_0^\infty \Omega(k)dk$. Because viscosity is the ultimate source of the energy dissipation, it may seem that in the limit $\nu \to 0$, we get $\epsilon \to 0$. Note, however, that this limit highlights a key difference between 2D and 3D. In 3D flows, the "vortex stretching term" in the vorticity equation furnishes a mechanism by which vorticity can be amplified without bound, akin to the increased rotational speed of an ice skater who pulls in their appendages. Consequently, $\Omega(k)$ has no upper bound, thereby permitting, as was first noted by Taylor,[6] the energy dissipation anomaly, $\lim_{\nu\to 0} \epsilon = \lim_{\nu\to 0} 2\nu \int_0^\infty \Omega(k)dk \neq 0$. In 2D flows, by contrast, the vortex stretching term is absent—the mechanism for vorticity amplification is gone and $\Omega(k)$ is bounded from above. Thus, the energy dissipation anomaly is gone, $\lim_{\nu\to 0} 2\nu \int_0^\infty \Omega(k)dk = 0$. We will invoke the lack of an energy dissipation anomaly in our derivations.

To compute $Z(k)$, we analyze the dynamical equation for $\Omega(k)$, which reads as

$$\frac{\partial \Omega(k)}{\partial t} = k^2 T(k) - 2\nu k^2 \Omega(k). \tag{4}$$

Note that the palinstrophy term, $k^2\Omega(k)$, which dissipates the enstrophy, is not bounded from above, and so $2\nu \int_0^\infty k^2\Omega(k)dk$ may remain finite as $\nu \to 0$, just as $2\nu \int_0^\infty \Omega(k)dk$ does in 3D. As noted by Batchelor:[18]

> "It may plausibly be supposed that the large value of $[\int_0^\infty k^2\Omega(k)dk]$ produced in this way, at times not close to the initial generation of turbulence, is such that, as $\nu \to 0$,
>
> $$[2\nu \int_0^\infty k^2\Omega(k) \to \beta \neq 0].$$
>
> We have no actual proof that material lines are extended (on average) in two-dimensional turbulence, just as we lack a proof that material lines are extended in three-dimensional turbulence. However, the heuristic arguments leading to this conclusion are of the same nature in the two cases, and empirically there is no doubt that material lines are extended in three-dimensional turbulence. If material lines were *not* extended in two-dimensional turbulence, the time required for the concentration of some conserved quantity such as salt dissolved in water to become uniform would be inversely proportional to the diffusivity, which does not seem credible even though we have so little experience with two-dimensional stirring. I propose to adopt the hypotheses that material lines are extended in two-dimensional turbulence, that there is a cascade process of transfer of mean-square vorticity to higher wavenumbers, and that the limiting value of the rate of dissipation of mean-square vorticity as $\nu \to 0$ is nonzero."

Thus, in 2D, we have the enstrophy dissipation anomaly, $\lim_{\nu\to 0} \beta = \lim_{\nu\to 0} 2\nu \int_0^\infty k^2\Omega(k) \neq 0$. We will also invoke the enstrophy dissipation anomaly in our derivations. As an aside, note that in the quote above Batchelor uses the language of "material lines" to refer to iso-vorticity lines that are stretched such that the vorticity remains constant but its gradient increases. This is a prevalent picture of the mechanism of the 2D direct enstrophy cascade.[51]

Armed with Eq. (1) (coupled with no energy dissipation anomaly) and Eq. (4) (coupled with enstrophy dissipation anomaly), we proceed to deriving $S_3(r)$ laws for four cases.

A. Freely decaying turbulence

Kolmogorov's derivation[5] of the 4/5th law concerned 3D freely decaying turbulence. Here we consider its 2D analog.

To compute $\Pi(k)$, we integrate the dynamical equation for $E(k)$, Eq. (1), and find the following:

$$\frac{\partial}{\partial t}\int_k^\infty E(k')dk' = \int_k^\infty T(k')dk' - 2\nu\int_k^\infty \Omega(k')dk', \quad (5)$$

where, to focus attention on the small scales, we consider $k \gg 1/L(t)$ [$L(t)$ is a characteristic large scale, e.g., the integral length scale; the region $k \gg 1/L(t)$ corresponds to the inertial range, which extends to arbitrarily large k in the limit $\nu \to 0$]. Analyzing the rhs, we note $\int_k^\infty T(k')dk' = \Pi(k)$ by definition and $2\nu\int_k^\infty \Omega(k')dk' \to 0$ for $\nu \to 0$ due to the lack of an energy dissipation anomaly. For the lhs, *if* we invoke local equilibrium for $k \gg 1/L(t)$, then the term becomes negligible. Thus, for $k \gg 1/L(t)$, $\nu \to 0$, and under local equilibrium, Eq. (5) yields

$$\Pi(k) = 0. \quad (6)$$

That is, in the limit of infinite Reynolds number ($\nu \to 0$), 2D freely decaying turbulence has no energy cascade, inverse or otherwise, a result that has already been noted by Kraichnan[9] and Batchelor.[18]

To compute $Z(k)$, we integrate the dynamical equation for $\Omega(k)$, Eq. (4), and find

$$\frac{\partial}{\partial t}\int_k^\infty \Omega(k')dk' = \int_k^\infty k'^2T(k')dk' - 2\nu\int_k^\infty k'^2\Omega(k')dk'. \quad (7)$$

Analyzing the lhs, we again invoke local equilibrium for $k \gg 1/L(t)$ and render its magnitude negligible. For the rhs, we note the first term is $\int_k^\infty k'^2T(k')dk' = Z(k)$ and write the second term as

$$2\nu\int_k^\infty k'^2\Omega(k')dk' = 2\nu\int_0^\infty k'^2\Omega(k')dk' - 2\nu\int_0^k k'^2\Omega(k')dk'. \quad (8)$$

In the limit $\nu \to 0$, the first term on the rhs $\to \beta$ (because of the enstrophy dissipation anomaly) and the second term $\to 0$ (because the integral is bounded for any finite value of k). Thus, for $k \gg 1/L(t)$, $\nu \to 0$, and under local equilibrium, Eq. (7) yields

$$Z(k) = \beta. \quad (9)$$

Substituting $\Pi(k) = 0$ and $Z(k) = \beta$ in Eq. (3), we obtain the $S_3(r)$ law for the direct enstrophy cascade in 2D freely decaying turbulence,

$$S_3(r) = \frac{1}{8}\beta r^3. \quad (10)$$

The law, it turns out, is the same as the canonical $S_3(r)$ law for the direct enstrophy cascade in forced turbulence (cf. Sec. II). This is analogous to the case in 3D turbulence where the 4/5th law is independent of whether the flow is freely decaying[5] or is forced.[7]

Note that our derivation of the $S_3(r)$ law is predicated on local equilibrium, which, as discussed in Sec. II, may be a questionable assumption, at least on phenomenological grounds.[38] We are not aware of any experiments or simulations that have tested this law for 2D freely decaying turbulence.

B. Freely decaying turbulence with a large-scale drag

We now consider a non-ideal effect. In experimental or atmospheric flows that are approximately 2D, 3D effects are invariably present. In a salt-layer experiment, for example, there is drag with the bottom boundary.[44,53] In atmospheric flows, too, the bottom boundary engenders drag.[54] In a soap film, there is drag between the film and the surrounding air.[55] The drag, in all these cases, typically acts at the large scales.

To compute the effect of a spatially localized, large-scale drag, we include it in the dynamical equations for $E(k)$ and $\Omega(k)$. The equation for $E(k)$ now reads as

$$\frac{\partial E(k)}{\partial t} = T(k) - 2\nu\Omega(k) - D(k), \quad (11)$$

where $D(k)$ is the drag term localized at $k \sim 1/l$ [l is the large scale where the drag acts; for simplicity, we restrict attention to $l \geq L(t)$]. Integrating, we find

$$\frac{\partial}{\partial t}\int_k^\infty E(k')dk' = \int_k^\infty T(k')dk' - 2\nu\int_k^\infty \Omega(k')dk' - \int_k^\infty D(k')dk'. \quad (12)$$

Invoking the same considerations as we used in obtaining Eq. (6) from Eq. (5), the unsteady term $\to 0$ for $k \gg 1/L(t)$ (local equilibrium), $\int_k^\infty T(k')dk' = \Pi(k)$ (definition), and the viscous term $\to 0$ for $\nu \to 0$ (no energy dissipation anomaly). Because the drag term is localized at $k \sim 1/l$, for $k \gg 1/L(t)$ (which ensures $k \gg 1/l$), $\int_k^\infty D(k')dk' \to 0$. Thus, just as in the aforementioned case without a large-scale drag, $\Pi(k) = 0$.

Now consider $\Omega(k)$. Including the large-scale drag, the dynamical equation for $\Omega(k)$ reads as

$$\frac{\partial \Omega(k)}{\partial t} = k^2T(k) - 2\nu k^2\Omega(k) - k^2D(k). \quad (13)$$

Integrating, we find

$$\frac{\partial}{\partial t}\int_k^\infty \Omega(k')dk' = \int_k^\infty k'^2T(k')dk' - 2\nu\int_k^\infty k'^2\Omega(k')dk' - \int_k^\infty k'^2D(k')dk'. \quad (14)$$

Again, except for the drag term, all considerations remain the same as we used in obtaining Eq. (9) from Eq. (7). For the drag term, we note that for $k \gg 1/L(t)$, $\int_k^\infty k'^2D(k')dk' \to 0$. [Here, we have assumed that like $D(k)$, $k^2D(k)$ is also localized at $k \sim 1/l$.] Thus, just as in the case without a large-scale drag, $Z(k) = \beta$.

Because $\Pi(k) = 0$ and $Z(k) = \beta$, the $S_3(r)$ law remains unchanged in the presence of the large-scale drag. That is, $S_3(r) = \frac{1}{8}\beta r^3$. To our knowledge, this is the first derivation of an $S_3(r)$ law in 2D freely decaying turbulence with a large-scale drag. As in the case without drag, we are not aware of any experiments or simulations that have tested this law.

Although the drag does not affect the fluxes partaking in the cascade, it does affect the form of $E(k)$ and $\Omega(k)$ at low k, which host the majority of energy and enstrophy, respectively. This can be seen by integrating Eqs. (11) and (13) from $\int_0^\infty dk$; in the limit $\nu \to 0$, we find

$$\frac{d}{dt}\int_0^\infty E(k)dk = -\int_0^\infty D(k)dk, \quad (15)$$

$$\frac{d}{dt}\int_0^\infty \Omega(k)dk = -\beta - \beta_l, \quad (16)$$

where $\beta_l \equiv \int_0^\infty k^2D(k)dk$ is the enstrophy dissipation rate due to the large-scale drag. This suggests that one must be careful when trying to determine the fluxes $\Pi(k)$ or $Z(k)$ from the time derivatives[41] $-\frac{\partial}{\partial t}\int_0^k E(k')dk'$ or $-\frac{\partial}{\partial t}\int_0^k \Omega(k')dk'$, respectively—these time derivatives include contributions from the large-scale drag.

C. Forced turbulence

Next we turn to forced turbulence. We will assume that the forcing term $\vec{f}$ in the Navier–Stokes equations acts locally at a scale l_f. Note that $\langle \vec{f}\cdot\vec{u}\rangle = P$, the energy injection rate per unit mass.

The dynamical equation for $E(k)$ reads as

$$\frac{\partial E(k)}{\partial t} = T(k) - 2\nu\Omega(k) + F(k), \tag{17}$$

where $F(k)$, the Fourier transform of $\langle \vec{f}\cdot\vec{u}\rangle$, is localized at $k \sim 1/l_f$. Integrating, we note that in the limit $\nu \to 0$, the total energy continuously grows at a rate

$$\frac{\partial}{\partial t}\int_0^\infty E(k)dk = \int_0^\infty F(k)dk = P, \tag{18}$$

where we have invoked the lack of an energy dissipation anomaly. Correspondingly, the large scale $L(t)$ also continuously grows with time; we limit attention to $L(t) \gg l_f$.

The dynamical equation for $\Omega(k)$ reads as

$$\frac{\partial \Omega(k)}{\partial t} = k^2T(k) - 2\nu k^2\Omega(k) + k^2F(k). \tag{19}$$

Integrating, we note that in the limit $\nu \to 0$, the rate of enstrophy is determined by

$$\frac{\partial}{\partial t}\int_0^\infty \Omega(k)dk = -\beta + Q, \tag{20}$$

where we have invoked the enstrophy dissipation anomaly to get β and $Q \equiv \int_0^\infty k^2F(k)dk$ is the enstrophy injection rate. To proceed, we note that although the flow is always unsteady at the large scales, at scales $\ll L(t)$, the flow at long times reaches a steady state.[9] Here, the enstrophy is carried by the scales $\sim l_f \ll L(t)$. Thus, we neglect the unsteady term in Eq. (20) and find $Q = \beta$. That is, the enstrophy injected by the forcing is balanced by the viscous dissipation.[36]

In deriving the $S_3(r)$ laws for forced turbulence, we consider the scales larger than l_f and smaller than l_f separately. First, consider $1/L(t) \ll k \ll 1/l_f$. To compute $\Pi(k)$, we integrate Eq. (17) and find

$$0 = \int_k^\infty T(k')dk' - 2\nu\int_k^\infty \Omega(k')dk' + \int_k^\infty F(k')dk', \tag{21}$$

where we have dropped the unsteady term because $k \gg 1/L(t)$. (Although we integrate until $k \to \infty$, we will shortly bring in the restriction $k \ll 1/l_f$.) Analyzing the rhs, the first term equals $\Pi(k)$ (definition), the second term $\to 0$ for $\nu \to 0$ (no energy dissipation anomaly), and the last term $\int_k^\infty F(k')dk' \to P$ for $k \ll 1/l_f$. Thus, we find $\Pi(k) = -P$.

To compute $Z(k)$, we integrate Eq. (19) and find

$$0 = \int_k^\infty k'^2T(k')dk' - 2\nu\int_k^\infty k'^2\Omega(k')dk' + \int_k^\infty k'^2F(k')dk', \tag{22}$$

where, again, we have dropped the unsteady term because $k \gg 1/L(t)$. Analyzing the rhs, the first term equals $Z(k)$ (definition), the second term $\to \beta$ for $\nu \to 0$ (enstrophy dissipation anomaly), and the last term $\int_k^\infty k'^2F(k')dk' \to Q$ for $k \ll 1/l_f$ [where we have assumed that like $F(k)$, $k^2F(k)$ is also localized at $k \sim 1/l_f$]. Thus, we find $Z(k) = \beta - Q$, which is equal to 0 because $\beta = Q$.

Substituting $\Pi(k) = -P$ and $Z(k) = 0$ in Eq. (3), we obtain the canonical $S_3(r)$ law for the inverse energy cascade in forced turbulence (cf. Sec. II),

$$S_3(r) = \frac{3}{2}Pr. \tag{23}$$

Now consider $k \gg 1/l_f$. To compute $\Pi(k)$, like before, we consider Eq. (21), but for the region $k \gg 1/l_f$. Again, the first term on the rhs equals $\Pi(k)$ and the second term $\to 0$ for $\nu \to 0$. But the last term now is different—$\int_k^\infty F(k')dk' \to 0$ for $k \gg 1/l_f$. Thus, we find $\Pi(k) = 0$. Next, to compute $Z(k)$, we consider Eq. (22), but for the region $k \gg 1/l_f$. Again, the first term on the rhs equals $Z(k)$ and the second term $\to \beta$ for $\nu \to 0$. The last term, again, is different—$\int_k^\infty k'^2F(k')dk' \to 0$ for $k \gg 1/l_f$. Thus, we find $Z(k) = \beta$. Substituting $\Pi(k) = 0$ and $Z(k) = \beta$ in Eq. (3), we obtain the canonical $S_3(r)$ law for the direct enstrophy cascade in forced turbulence (cf. Sec. II), which is the same law as in the freely decaying case [Eq. (10)].

Finally, we comment on the energy flux and the enstrophy flux for 2D forced turbulence. Our analysis yielded $\Pi(k) = -P$ and $Z(k) = 0$ [for large scales, $1/L(t) \ll k \ll 1/l_f$] and $\Pi(k) = 0$ and $Z(k) = \beta$ (for small scales, $k \gg 1/l_f$). These results are consistent with Kraichnan's[9] analysis and Leith's[17] analysis which showed that, for turbulence forced at a single scale, spatially concurrent cascades of energy and enstrophy cannot occur.

D. Forced turbulence with a large-scale drag

Unlike the previous case, 2D forced turbulence with a large-scale drag reaches a steady state[41] for all scales, large as well as small. Including a large-scale drag that acts locally at a scale $l \gg l_f$ (cf. Sec. IV B), the steady-state equation for $E(k)$ reads as

$$0 = T(k) - 2\nu\Omega(k) + F(k) - D(k). \tag{24}$$

Integrating $\int_0^\infty dk$, in the limit $\nu \to 0$, we find the equation of energy balance,

$$P = \int_0^\infty D(k)dk. \tag{25}$$

The energy injected by the forcing at the scale l_f is dissipated by the drag at the scale l. Integrating Eq. (24) $\int_0^k dk'$, we find

$$0 = \int_k^\infty T(k')dk' - 2\nu\int_k^\infty \Omega(k')dk' + \int_k^\infty F(k')dk' - \int_k^\infty D(k')dk', \tag{26}$$

which we will analyze shortly to compute $\Pi(k)$.

Including the large-scale drag, the steady-state equation for $\Omega(k)$ reads as

$$0 = k^2T(k) - 2\nu k^2\Omega(k) + k^2F(k) - k^2D(k). \tag{27}$$

Integrating $\int_0^\infty dk$, in the limit $\nu \to 0$, we find the equation of enstrophy balance,

$$Q = \beta + \beta_l. \tag{28}$$

The enstrophy injected by the forcing is dissipated by the large-scale drag as well as by viscosity (which acts at the small scales). Integrating Eq. (27) $\int_0^k dk'$, we find

$$0 = \int_k^\infty k'^2 T(k')dk' - 2\nu \int_k^\infty k'^2 \Omega(k')dk' + \int_k^\infty k'^2 F(k')dk' - \int_k^\infty k'^2 D(k')dk', \tag{29}$$

which we will analyze shortly to compute $Z(k)$.

First, consider $1/l \ll k \ll 1/l_f$. To compute $\Pi(k)$, we invoke the same considerations for the first three terms on the rhs of Eq. (26) as we did for Eq. (21) in the region $1/L(t) \ll k \ll 1/l_f$. For the last term, we note that $\int_k^\infty D(k')dk' \to 0$ for $k \gg 1/l$. Thus, we again find $\Pi(k) = -P$. Similarly, to compute $Z(k)$, we invoke the same considerations for the first three terms on the rhs of Eq. (29) as we did for Eq. (22) in the region $1/L(t) \ll k \ll 1/l_f$. For the last term, we note that $\int_k^\infty k'^2 D(k')dk' \to 0$ for $k \gg 1/l$. Thus, we again find $Z(k) = \beta - Q$, but now $\beta \neq Q$, so $Z(k) \neq 0$; instead, from Eq. (28), we find $Z(k) = -\beta_l$.

Substituting $\Pi(k) = -P$ and $Z(k) = -\beta_l$ in Eq. (3), we obtain the $S_3(r)$ law for the region $1/l \ll k \ll 1/l_f$ of forced turbulence with a large-scale drag,

$$S_3(r) = \frac{3}{2}Pr - \frac{1}{8}\beta_l r^3, \tag{30}$$

which represents a mixture of energy and enstrophy fluxes. To our knowledge, this law is a new result. Note that unlike the case of freely decaying turbulence with a large-scale drag, here the $S_3(r)$ law is modified by the drag. More importantly, unlike any other $S_3(r)$ law in 2D turbulence, Eq. (30) allows for $S_3(r) < 0$. This suggests that one must be cautious in interpreting the sign of $S_3(r)$ in flows, e.g., atmospheric flows, where a large-scale drag is present.

Now consider $k \gg 1/l_f$. Following our usual procedure, to compute $\Pi(k)$, we invoke the same considerations for the first three terms on the rhs of Eq. (26) as we did for Eq. (21) in the region $k \gg 1/l_f$. For the last term, we note that $\int_k^\infty D(k')dk' \to 0$ for $k \gg 1/l_f$. Thus, we again find $\Pi(k) = 0$. Similarly, to compute $Z(k)$, we invoke the same considerations for the first three terms on the rhs of Eq. (29) as we did for Eq. (22) in the region $k \gg 1/l_f$. For the last term, we note that $\int_k^\infty k'^2 D(k')dk' \to 0$ for $k \gg 1/l_f$. Thus, we again find $Z(k) = \beta$. Substituting $\Pi(k) = 0$ and $Z(k) = \beta$ in Eq. (3), we find that the $S_3(r)$ law for the region $k \gg 1/l_f$ of forced turbulence with a large-scale drag is unaffected by the drag, as was seen by Bernard.[36] This $S_3(r)$ law is the same as the canonical $S_3(r)$ law for direct enstrophy cascade in forced turbulence. The difference, however, is that in the case without the drag, Q and β are equal to each other, and thus they can be used interchangeably in the $S_3(r)$ law. In the case with drag, however, Q and β are not equal—the injected Q is dissipated by viscosity (β) as well as drag (β_l), as can be seen from the enstrophy balance [Eq. (28)] and from simulations.[41]

V. CONCLUDING REMARKS

We have discussed $S_3(r)$ laws in 2D turbulence. Approaching $S_3(r)$ from spectral space, we established that it generally embodies a mixture of energy and enstrophy fluxes. We determined the fluxes by analyzing the dynamical equations of energy spectrum and enstrophy spectrum, where we invoked the lack of an energy dissipation anomaly and the presence of an enstrophy dissipation anomaly. This yielded $S_3(r)$ laws for freely decaying and forced turbulence, with and without a large-scale drag. For the canonical cases—$S_3(r)$ laws for the direct enstrophy cascade and the inverse energy cascade in forced turbulence—we have attempted to provide simpler derivations than those found elsewhere. For the freely decaying case without a large-scale drag, our result is in accord with a previous study,[51] and for the freely decaying case with a large-scale drag, our result is new. Also, for the case of forced turbulence with a large-scale drag, our result for the large scales is new and our result for the small scales is in accord with a previous study.[36]

We have found that the canonical $S_3(r)$ law for the direct enstrophy cascade is a robust result. The law remains the same for forced turbulence and for freely decaying turbulence, analogous to the 4/5th law in 3D turbulence. Further, the law remains unaffected by the presence of a large-scale drag.

On the other hand, the canonical $S_3(r)$ law for the inverse energy cascade, which is only seen for the forced cases, is affected by the presence of a large-scale drag. In fact, in contrast to all the other cases in 2D where $S_3(r)$ is positive, for this case, the drag can engender $S_3(r) < 0$. Further, the scaling of $S_3(r)$ contains both r and r^3 terms, signifying a mixture of energy and enstrophy fluxes.

A general remark on our derivation may be in order. In deriving the $S_3(r)$ laws, we transformed from spectral space to real space via Eq. (3), which expresses the result as a power-series expansion in r. For the laws corresponding to the direct cascades, the limit $\nu \to 0$ extends the inertial range to $r \to 0$, which, with the proviso that the higher-order terms are not singular, makes Eq. (3) an exact result. But, the inertial range of an inverse cascade starts at $r \gtrsim l_f$ and extends to larger values of r. Thus, unless the higher-order terms are all negligible, we urge caution in interpreting $S_3(r)$ data for a large r using an $S_3(r)$ law.

In closing we note that, unlike 3D turbulence, which has only one $S_3(r)$ law, 2D turbulence has many laws, which correspond to different flow conditions. Using the framework of the idealized theory (albeit expanding its scope by including a large-scale drag), we have derived some of these laws. When interpreting experimental, numerical, or atmospheric data in light of the $S_3(r)$ laws, it is worth recalling a remark from Kraichnan and Montgomery:[21] "In some cases the ideali[z]ed theory may be more valid in providing a language for discussion rather than a true explanation." Beyond the idealized theory, the non-ideal factor we analyzed, a large-scale drag, is but a simplified model of the drag present in real flows. More realistic models of the drag may affect $S_3(r)$ differently. Further, other non-ideal factors—e.g., mean shear (see Appendix B), finite Reynolds number[23]—may drastically affect $S_3(r)$.

The complexities may be daunting, but the possibilities are rich. So it is with the Rashomon effect.

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APPENDIX A: SPECTRAL PERSPECTIVE ON $S_3(r)$ IN 2D

Here we derive Eq. (3). Our derivation is similar to how Davidson[51] relates $S_2(r)$ with $E(k)$. We find that $S_3(r)$ in 2D embodies contributions from both the energy flux and the enstrophy flux.

Manipulating the Karman–Howarth equation in 2D yields the following relation[51] between $T(k)$ and $S_3(r)$:

$$T(k) = \frac{k^3}{6}\int_0^\infty \frac{\partial}{\partial r}\left(r^3 S_3(r)\right)\frac{J_1(kr)}{2kr}dr, \tag{A1}$$

where $J_1(kr)$ is the 1st Bessel function of the first kind.[56] (Note that the above equation remains unchanged when additional terms, e.g., a large-scale drag, are added to the Karman–Howarth equation.) Integrating by parts, we get

$$T(k) = \frac{k^4}{6}\int_0^\infty r^3 S_3(r)\frac{J_2(kr)}{2kr}dr, \tag{A2}$$

where $J_2(kr)$ is the 2nd Bessel function of the first kind.[56] Taking advantage of the orthogonality condition for Bessel functions,

$$\int_0^\infty J_m(ax)J_m(bx)x dx = \frac{\delta(a-b)}{a}, \tag{A3}$$

where $\delta(a-b)$ is the delta function, we multiply Eq. (A2) by $\frac{J_2(kr)}{k^2}$ and integrate over k to get

$$S_3(r) = \frac{3}{2}r\int_0^\infty T(k)\frac{8J_2(kr)}{(kr)^2}dk. \tag{A4}$$

Noting that $\frac{8J_2(kr)}{(kr)^2}$ decays to zero as kr increases from zero to an $O(1)$ value, we approximate

$$\frac{8J_2(kr)}{(kr)^2} = \begin{cases} 1 - \frac{(kr)^2}{12} + \cdots, & kr < a, \\ 0, & kr > a, \end{cases} \tag{A5}$$

where these are the first two terms in the Taylor-series expansion[56] and a is an $O(1)$ constant. Thus, we find

$$S_3(r) = \frac{3}{2}r\int_0^{a/r} T(k)dk - \frac{1}{8}r^3\int_0^{a/r} k^2 T(k)dk + \cdots, \tag{A6}$$

which leads to Eq. (3). We note that for any homogeneous and isotropic 2D turbulent flow, $S_3(r)$ equals $\frac{3}{2}r$ times the net energy transfer through r, minus $\frac{1}{8}r^3$ times the net enstrophy transfer through r. For the case of 2D turbulence forced at two scales, Lindborg[28] also found that $S_3(r)$ contains a mixture of energy and enstrophy fluxes.

APPENDIX B: A NOTE ON THE EFFECT OF MEAN SHEAR

Stepping beyond the realm of the idealized theory, we have considered the effect of a large-scale drag. Another non-ideal effect present in real 2D flows is the shear in the mean flow. For example, in atmospheric flows and in soap-film flows, the turbulent fluctuations are often embedded in a strongly sheared mean flow. Although we have not included any derivations on how the mean shear affects $S_3(r)$, we note that simulations[57,58] of 2D turbulence in the presence of mean shear reveal that $\Pi(k)$ and $Z(k)$ are affected by the mean shear, which, via Eq. (3), affects $S_3(r)$. The overall effect will depend on the strength of the mean shear, but this reveals further complications for interpreting experimental and atmospheric data.[59]

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