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Conformal bootstrap analysis for the Yang–Lee edge singularity

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The Yang–Lee edge singularity is investigated by the determinant method of the conformal field theory. 3×3 minors are used for the evaluation of the scale dimension. The agreement with the Padé of ϵ expansion in the region $3 < D < 6$ is improved. The critical dimension D_c, where the scale dimension of scalar Δ_ϕ is vanishing, is used for the improvement of the Padé. For the understanding of the intersection point of zero loci of 3×3 minors, 2×2 minors are investigated in detail; these are connected through Plüker relations.

Subject Index I10

1. Introduction

The conformal field theory was developed a long time ago [1], and the modern numerical approach was initiated in Ref. [2]. Recent studies using this conformal bootstrap method have obtained some remarkable results for the 3D Ising model [3,4], Yang–Lee edge singularities [5,6], $O(N)$ models [7–10], and self-avoiding walks [11].

A brief summary of the determinant method for the conformal bootstrap theory is as follows. The conformal bootstrap theory is based on the conformal group $O(D, 2)$, and the conformal block $G_{\Delta,L}$ is the eigenfunction of the Casimir differential operator $\tilde{D}_2$. The eigenvalue of this Casimir operator is C_2:

$$\tilde{D}_2 G_{\Delta,L} = C_2 G_{\Delta,L},$$
$$C_2 = \frac{1}{2}[\Delta(\Delta - D) + L(L + D - 2)]. \tag{1}$$

The solutions of the Casimir equation have been studied in Refs. [18–20]. The conformal block $G_{\Delta,L}(u, v)$ has two variables u and v, which denote the cross ratios, $u = (x_{12}x_{34}/x_{13}x_{24})^2$, $v = (x_{14}x_{23}/x_{13}x_{24})^2$, where $x_{ij} = x_i - x_j$ (x_i is a 2D coordinate). They are expressed as $u = z\bar{z}$ and $v = (1 - z)(1 - \bar{z})$. For the particular point $z = \bar{z}$, the conformal block $G_{\Delta,L}(u, v)$ for the spin-zero ($L = 0$) case has a simple expression:

$$G_{\Delta,0}(u, v)|_{z=\bar{z}} = \left(\frac{z^2}{1-z}\right)^{\Delta/2} {}_3F_2\left[\frac{\Delta}{2}, \frac{\Delta}{2}, \frac{\Delta}{2} - \frac{D}{2} + 1; \frac{D+1}{2}, \Delta - \frac{D}{2} + 1; \frac{z^2}{4(z-1)}\right]. \tag{2}$$

The conformal bootstrap determines Δ by the condition of the crossing symmetry $x_1 \leftrightarrow x_3$. For practical calculations, the point $z = \bar{z} = 1/2$ is chosen and, by the recursion relation derived from the Casimir equation of Eq. (1), $G_{\Delta,L}(u, v)$ at $z = \bar{z} = 1/2$ can be obtained [3].

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The conformal bootstrap analysis with a small matrix size has been investigated for the Yang–Lee edge singularity with accurate results for the scale dimensions by Gliozzi and Rago [5,6]. For other models, this bootstrap method for the determinant has been considered [12,15,21,22]. In this paper, we emphasize the importance of the structure of minors along the Plüker relations, as shown in the appendix, and we apply it to the Yang–Lee model for a scale dimension of Δ_ϕ. The intersection point of the zero loci of 3×3 minors was evaluated and it gave a remarkable value of the critical exponent in Ref. [5]. In these 3×3 minors, if the value of the space dimension D is given, the value of $\Delta_\epsilon = \Delta_\phi$ is determined from the intersection point. However, if one goes to larger minors, for instance 4×4 or 5×5 minors, one need the values of additional scale dimensions of the operator product expansion (OPE). In the study of such large minors, however, the values of Δ_ϕ deviate from the ϵ expansion [6]. This discrepancy should be improved by the determinant method or by analysis of the ϵ expansion, since the conformal bootstrap method should be consistent with the ϵ expansion [24,25]. Recently, consistent results were obtained in the $O(N)$ vector model through the Mellin amplitude [13].

We consider this discrepancy by repeating Gliozzi's evaluation of the 3×3 minors [5] and find that 3×3 minors give accurate values that agree with the improved ϵ expansion (see Fig. 8). The nature of the determinant method is still not known, and the convergence to the true value seems to be slow. As we mentioned before, we have to assume the values of the scale dimensions of the OPE, such as spin 4, spin 6, etc., for the large minors. Unfortunately we do not know precisely these higher spin values at present. Therefore, we concentrate on 3×3 minors without any assumption of the other scale dimensions of the OPE.

The four-point correlation function is given by

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{g(u,v)}{|x_{12}|^{2\Delta_\phi}|x_{34}|^{2\Delta_\phi}} \tag{3}$$

and the amplitude $g(u,v)$ is expanded as the sum of conformal blocks:

$$g(u,v) = 1 + \sum_{\Delta,L} p_{\Delta,L} G_{\Delta,L}(u,v). \tag{4}$$

The crossing symmetry of $x_1 \leftrightarrow x_3$ implies

$$\sum_{\Delta,L} p_{\Delta,L} \frac{v^{\Delta_\phi} G_{\Delta,L}(u,v) - u^{\Delta_\phi} G_{\Delta,L}(v,u)}{u^{\Delta_\phi} - v^{\Delta_\phi}} = 1. \tag{5}$$

The minor method consists of the derivatives at the symmetric point $z = \bar{z} = 1/2$ of Eq. (5). By a change of variables $z = (a + \sqrt{b})/2$, $\bar{z} = (a - \sqrt{b})/2$, derivatives are taken about a and b. Since the number of equations becomes larger than the number of truncated variables Δ, we need to consider the minors for the determination of the values of Δ. The matrix elements of minors are expressed by

$$f_{\Delta,L}^{(m,n)} = \left(\partial_a^m \partial_b^n \frac{v^{\Delta_\phi} G_{\Delta,L}(u,v) - u^{\Delta_\phi} G_{\Delta,L}(v,u)}{u^{\Delta_\phi} - v^{\Delta_\phi}} \right) |_{a=1,b=0} \tag{6}$$

and the minors of 2×2, 3×3, e.g., d_{ij}, d_{ijk}, are determinants such as

$$d_{ij} = \det\left(f_{\Delta,L}^{(m,n)}\right), \quad d_{ijk} = \det\left(f_{\Delta,L}^{(m,n)}\right), \tag{7}$$

where i, j, k are numbers chosen differently from $(1, \ldots, 6)$, following the concise correspondence to (m, n) as $1 \to (2, 0)$, $2 \to (4, 0)$, $3 \to (0, 1)$, $4 \to (0, 2)$, $5 \to (2, 1)$, and $6 \to (6, 0)$.

In the Yang–Lee case, a fusion rule is simply written as [5]

$$[\Delta_\phi] \times [\Delta_\phi] = 1 + [\Delta_\phi] + [D, 2] + [\Delta_4, 4] + \cdots . \tag{8}$$

The scalar ($L = 0$) term in the bootstrap equation of Eq. (6) is denoted as

$$vs0 = \frac{v^{\Delta_\phi} G_{\Delta,0}(u, v) - u^{\Delta_\phi} G_{\Delta,0}(v, u)}{u^{\Delta_\phi} - v^{\Delta_\phi}}. \tag{9}$$

We use the notation (vsk) (k is a number, related to the degree of the differential, v is a vector, and s is a scalar) for the derivatives of the ($vs0$) rule in Eq. (6). For $L = 2$ we use (vtk), and for $L = 4$ we use (vqk). (vxk) is for $L = 6$.

The scalar ($L = 0$) case is expressed by the hypergeometric function ($\Delta = \Delta_\phi$ for the Yang–Lee edge singularity)

$$vs0 = -2^{-\Delta/2} ys + \frac{3 \times 2^{-1-\Delta/2}}{2} ys + \frac{2^{-\Delta/2}}{2\Delta} ys', \tag{10}$$

where

$$ys = {}_3F_2\left(\frac{\Delta}{2}, \frac{\Delta}{2}, \frac{\Delta}{2} - \frac{D}{2} + 1; \frac{\Delta}{2} + \frac{1}{2}, \Delta - \frac{D}{2} + 1 : \frac{a^2}{8(-2 + a)}\right) \tag{11}$$

and ys' is derivative of ys. We consider the derivative at a point $a = 1$.

For instance, we have 2×2 minors:

$$d_{13} = \det \begin{pmatrix} vs1 & vs3 \\ vt1 & vt3 \end{pmatrix}, \quad d_{23} = \det \begin{pmatrix} vs2 & vs3 \\ vt2 & vt3 \end{pmatrix}. \tag{12}$$

In this Yang–Lee edge singularity, we find that the 2×2 minor, d_{13}, becomes zero at $D = 6$, and it provides exact values of $\Delta_\phi = 2$.

2. 2×2 minors

The intersection points of zero loci of 3×3 minors are decomposed to 2×2 minors at the critical dimensions $D = 6$ for the free theory. In the appendix (relation 1), this decomposition is shown. Among several 2×2 minors, $d_{12}, d_{13}, d_{23}, \ldots$, the most fundamental minors may be d_{13} and d_{23}, which are made of lower derivatives of a and b. Note that, in the Yang–Lee model, the constraint $\Delta_\epsilon = \Delta_\phi$ is taken, and the 2×2 minor analysis corresponds to the 3×3 minor analysis of another model such as the Ising model, in which $\Delta_\epsilon \neq \Delta_\phi$. From the point of the truncation error in the OPE, as discussed in Ref. [15], it is interesting to start from 2×2 minors. The Yang–Lee singularity is connected to supersymmetry through the dimensional reduction of a branched polymer, where we have an exact relation of $\Delta_\epsilon = \Delta_\phi + 1$ [14,16,17]. We will discuss the dimensional reduction $D + 2 \to D$ behavior in the Yang–Lee model in 2×2 minors.

The 2×2 minors consist of two parameters, dimension D and the scaling dimension Δ_ϕ, since we have $\Delta_{\phi^2} = \Delta_\phi$ ($\Delta_\epsilon = \Delta_{\phi^2}$) for the Yang–Lee case, due to the ϕ^3 theory.

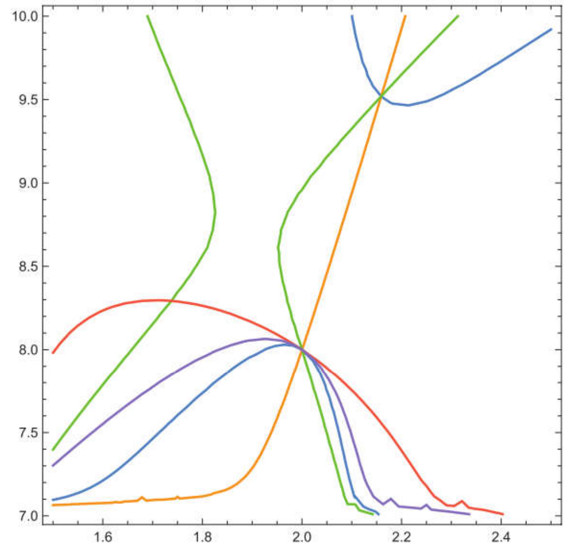

Fig. 1. The intersection of zero loci of 3 × 3 minors (d_{123}, d_{135}, d_{134}, d_{234}, d_{235}) for dimension $D = 6$. The axes are $(x, y) = (\Delta_\phi, Q)$. The intersection point of the five lines is $D = 6, \Delta_\phi = 2$ with $Q = 8$, which is decomposed to the 2 × 2 minor d_{13}. The upper fixed point is related to the minor d_{23}.

(i) Free field theory at $D = 6$

The zero loci of d_{13} gives exactly the scale dimension $\Delta_\phi = 2.0$ at $D = 6$. The minors for vq, vx, and vs' (with a scalar scale dimension Δ') can be zero:

$$\det\begin{pmatrix} vs1 & vq1 \\ vs3 & vq3 \end{pmatrix} = 0, \quad \det\begin{pmatrix} vs1 & vx1 \\ vs3 & vx3 \end{pmatrix} = 0, \quad \det\begin{pmatrix} vs1 & vs1' \\ vs3 & vs3' \end{pmatrix} = 0. \qquad (13)$$

For the value $\Delta_\phi = 2$, they yield $Q(= \Delta_4) = 8$, $\Delta_6 = 10$, and $\Delta' = 4$, respectively. These Δ_ϕ, Δ_4, Δ_6, Δ' are determined exactly by 2 × 2 minors of d_{13}. This is due to the free theory, i.e., no interactions between the scale dimensions. It is known that the six dimensions have algebraic identities, which are valid for a free theory in any dimension D [23].

Other minors yield different values; for instance, d_{23} gives $\Delta_\phi = 2.158$ at $D = 6$. These two different values of Δ_ϕ are understood when we evaluate 3 × 3 minors, as shown in Figs. 1 and 2. There are two fixed points of zero loci of 3 × 3 minors in the six dimensions. The 3 × 3 minors are denoted by d_{ijk} in Eq. (7); for instance, d_{123} means

$$d_{123} = \det\begin{pmatrix} vs1 & vt1 & vq1 \\ vs2 & vt2 & vq2 \\ vs3 & vt3 & vq3 \end{pmatrix}. \qquad (14)$$

(ii) $D = 3$

The zero loci of 2 × 2 minor do not provide good results except for $D = 6$ dimensions. This is due to the situation where the free theory breaks down in general dimensions. We have to consider the loci of 2 × 2 minors of a finite small value due to the interactions, instead of a zero value, for the non-trivial fixed point. The small non-vanishing value of the minor might have an important meaning for the bound of Δ_ϕ. The loci of two important 2 × 2 minors d_{13} and d_{23} are shown in Fig. 2, in which the value of d_{13} is changed by small values from −0.007 to 0.004, and the zero loci of d_{23} are also shown in Fig. 3 as a guide. The correct value of Δ_ϕ is realized in some

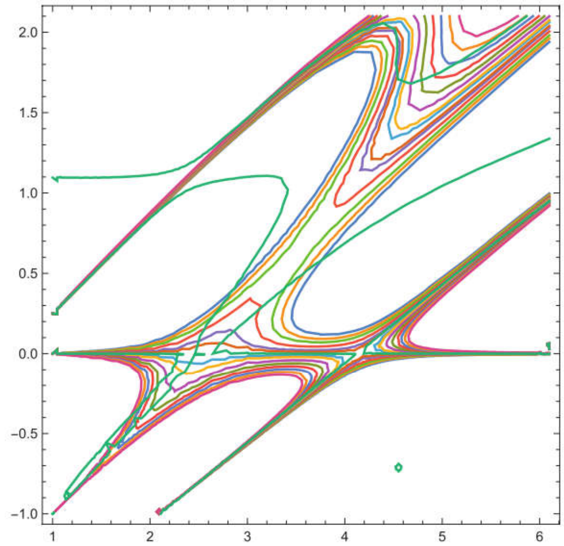

Fig. 2. The loci of 2×2 minors d_{13} with values from -0.007 to 0.004 are shown; $d_{23} = 0$ is also shown by a green line. The axes are $(x, y) = (D, \Delta_\phi)$.

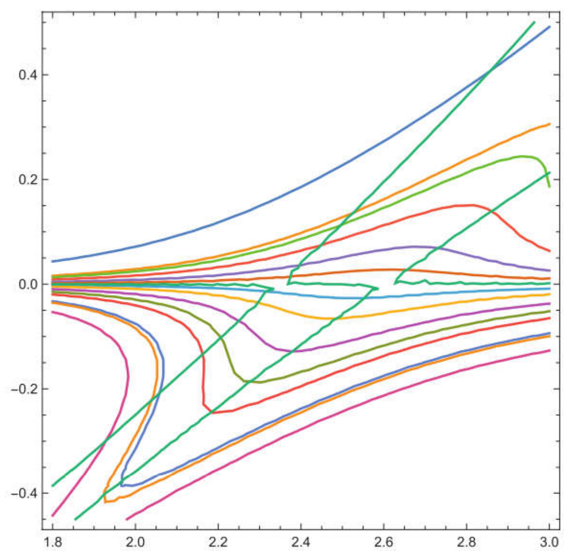

Fig. 3. A close-up of the contour map in Fig. 2 near the $\Delta_\phi = 0$ loci of minor d_{13}. The green line shows the zero loci of d_{23} as a guide. The axes are $(x, y) = (D, \Delta_\phi)$.

finite-value contours of d_{13} for $D < 6$. Near $D = 6$, deviation from the zero loci of d_{13} is expected since Δ_ϕ deviates from 2 as the $\epsilon = 6 - D$ expansion shows: $\Delta_\phi \simeq 2 - 0.5555\epsilon$.

In Fig. 2, the finite-value curves of d_{13} converge to a curve starting from $\Delta_\phi = 0.25$ (at $D = 1$) to 2.0 (at $D = 4$). This curve resembles the correct Δ_ϕ contour if the space dimension is shifted from D to $D+2$. (The curve is translated to the right in Fig. 2 by two space dimensions by keeping the same value of Δ_ϕ.) This dimensional reduction will be discussed in Sect. 4.

The finite-value contour map of d_{13} shows a valley shape, and one line is located at $D = 2$ near the exact value $\Delta_\phi = -0.4$, as shown in Fig. 3. In Fig. 3, the value of minor d_{13} is 0.0034, which gives $\Delta_\phi = -0.4$. The cusp point moves when the value of minor d_{13} is changed and its trace may give the bound of Δ_ϕ.

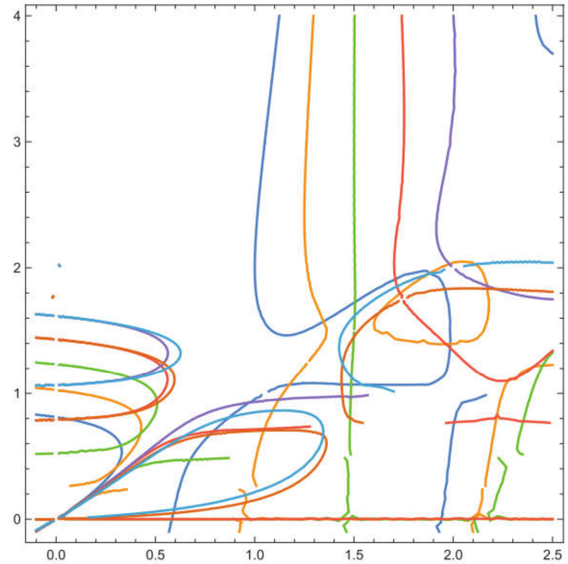

Fig. 4. The zero loci of $d[\Delta_\phi, \Delta_\epsilon]_{13}$ for different dimensions: $D = 6$ (dark blue), $D = 5.5$ (red), $D = 5$ (green), $D = 4.5$ (yellow), and $D = 4$ (blue). The axes are $(x, y) = (\Delta_\phi, \Delta_\epsilon)$.

In Figs. 2 and 3, contour maps of d_{13} with small non-vanishing values are shown.

3. Critical dimension for $\Delta_\phi = 0$

In the Yang–Lee edge singularity, the scale dimension Δ_ϕ of ϕ becomes zero at some dimension between 2 and 3, since it is -0.4 in two dimensions and approximately 0.2 in three dimensions. We call this dimension the critical dimension D_c, on which the scale dimension Δ_ϕ vanishes. In the determinant method, it is known that even small minors give accurate values of the scale dimensions [5,6]. We discuss here small determinants of 2×2, 3×3, and 4×4 matrices for this critical dimension D_c.

Figure 2 shows Z-shaped curves for zero or very small values of the loci of d_{13}. This figure can be understood when we plot Δ_ϵ versus Δ_ϕ for various values of fixed dimensions. The minor of d_{13} has so far been considered as a function of D and Δ_ϕ with the condition of $\Delta_\epsilon = \Delta_\phi$ due to the Yang–Lee model. However, in general we are able to consider Δ_ϵ and Δ_ϕ as two free parameters for d_{13}, and, for the Yang–Lee case, we put $\Delta_\epsilon = \Delta_\phi$.

In Fig. 4, we plot the zero loci of $d[\Delta_\phi, \Delta_\epsilon]_{13}$ for different values of D. This minor then becomes the usual one including the Ising case. A complicated contour for the zero loci of d_{13} is obtained especially around $D = 4.4$, where it has three solutions for $\Delta_\epsilon = \Delta_\phi$. This solution corresponds to the Z shape around $(D, \Delta) = (4, 1.5)$ in Fig. 2.

In addition, we find that the line of $\Delta_\epsilon = \Delta_\phi$ of the Yang–Lee condition goes through very near the zero loci of d_{13}, but it does not intersect. This corresponds to Fig. 2 with the finite-value loci of d_{13} around $D = 3$.

With some value of D, the contours in Fig. 4 (bottom left) and Fig. 5 intersect at the point $(\Delta_\epsilon, \Delta_\phi) = (0, 0)$. We find that this critical dimension from Fig. 5 is

$$D_c = 2.6199. \tag{15}$$

As seen in Fig. 3, for a small finite value of d_{13}, the contour of d_{13} approaches the line $\Delta_\phi = 0$ from both sides, above $\Delta_\phi > 0$ and $\Delta_\phi < 0$. The peak of the contours approaches the critical

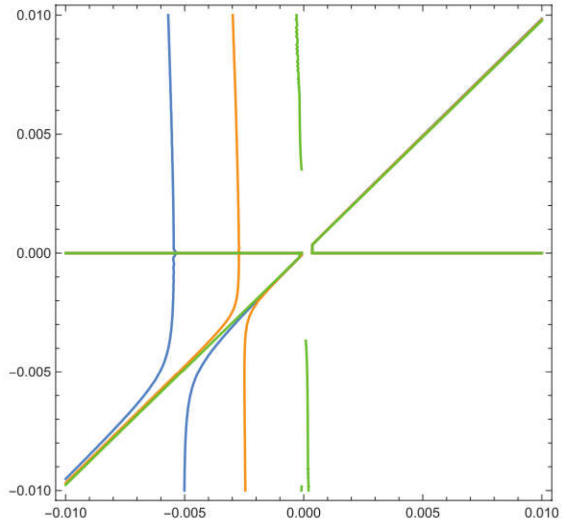

Fig. 5. A contour plot of the zero loci of $d[\Delta_\phi, \Delta_\epsilon]_{13}$ for different dimensions: $D = 2.6$ (blue), $D = 2.61$ (yellow), $D = 2.6199$ (green). The axes are $(x, y) = (\Delta_\phi, \Delta_\epsilon)$.

dimension D_c. We consider the derivative of d_{13} by the dimension D, and we estimate D_c as the point at which this derivative becomes zero. For $\Delta_\phi = 0.000\,001$, we obtain $D_c = 2.605\,74$. The derivative of d_{13} by the space dimension D gives the approximate solution of D_c in the limit $\Delta_\phi \to 0$. The minor d_{12} is proportional to Δ_ϕ in the limit $\Delta_\phi \to 0$, since $vs1, vs3 \sim \Delta_\phi^2$, and $vt1, vt3 \sim \Delta_\phi^{-1}$.

We consider how the minors become zero in the limit $\Delta_\phi \to 0$. For instance, at $D = 2.61$, we have, for small Δ_ϕ,

$$vs1 = 0.188\Delta_\phi^2, \;\; vs2 = 0.139\Delta_\phi^2, \;\; vs3 = 0.225\Delta_\phi^2$$
$$vt1 = \frac{1.055}{\Delta_\phi}, \;\; vt2 = \frac{0.667}{\Delta_\phi}, \;\; vt3 = \frac{1.078}{\Delta_\phi}. \tag{16}$$

The coefficients p_ϕ, p_t are obtained from

$$\begin{pmatrix} vs0 & vt0 \\ vs1 & vt1 \end{pmatrix} \begin{pmatrix} p_\phi \\ p_t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{17}$$

Since $vs0 = -\frac{1}{4}, vt0 = 0.7185/\Delta_\phi$ in the limit $\Delta_\phi \to 0$, we obtain a solution of $p_\phi = -4, p_t = a\Delta_\phi^2 \log\Delta_\phi$ (a is a constant), which gives the value of the central charge $C = 0$. With this choice of $p_t = p_{(D,2)}$, the central charge C becomes $C = \Delta_\phi^2/p_t = 0$ for this critical dimension D_c. This means that the energy–momentum tensor operator $T = \Delta_{(D,2)}$ can be neglected since the OPE coefficient of this operator p_t becomes zero. This is the well known c catastrophe, which leads to logarithmic conformal field theory [26]. When neglecting the $\Delta_{(D,2)}$ term, and only considering $vs0$ in Eq. (10), we obtain the critical dimension $D_c = 2.748\,632 \cdots$. However, this value is too large compared to the expected value and might not be correct. We need further operators to get the correct value. We hope to get the correct value of the critical dimension D_c by another sophisticated method by taking higher operators.

4×4 *minors* The numbers of zero loci of minors are ${}_6C_4 = 6!/4!2! = 15$. We investigated the critical dimensions D_c by changing D between $D = 2.58$ and $D = 2.59$. Remarkably, at

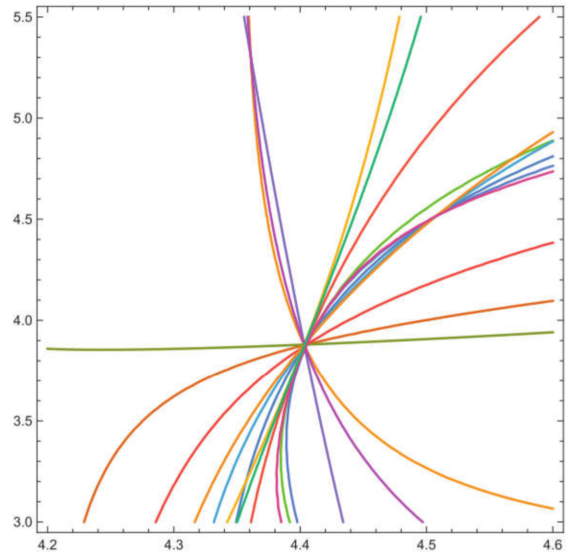

Fig. 6. The intersection of zero loci of minors at $D = 2.589\,53$, $\Delta_\phi = 0$. The axes are $(x, y) = (Q, \Delta')$.

$D = 2.589\,53$, all 15 lines intersect at a single point in the contour map of (Q, Δ') under the condition $\Delta_\phi = 0$. If we take this as the value of the critical dimension D_c, we find the following set of results:

$$D_c = 2.589\,53, \quad \Delta_\phi = 0, \quad Q = 4.403\,97, \quad \Delta' = 3.871\,27. \tag{18}$$

This critical dimension D_c obtained with 4×4 minors is close to $D_c = 2.6199$ from the 2×2 minor d_{13} with the intersection point of $\Delta_\epsilon = \Delta_\phi = 0$. In Fig. 6, all loci coincide but, if we include more relevant operators, this value may change.

For the $\Delta_\phi = 0$ case, in any dimension D, d_{ij} can become zero. When $\Delta_\phi = 0$, by the definition of the scale dimension Δ_ϕ, the two-point correlation function $G(r) = 1/r^{2\Delta_\phi}$ becomes a constant in the long-range limit $r \to \infty$.

In the Yang–Lee edge singularity, the exponent of the density σ is related to Δ_ϕ as

$$\sigma = \frac{\Delta_\phi}{D - \Delta_\phi}. \tag{19}$$

From this relation, σ is vanishing for $\Delta_\phi = 0$. This means that the density is constant at the transition point. Several interesting systems are known in which the density is constant but a phase transition occurs. One example is a localization problem under the random potential.

4. Dimensional reduction

The zero loci of the 2×2 minor d_{13} shows an interesting characteristic linear behavior for $D < 4$, which is approximated as $\Delta_\phi = (3D - 2)/5$. In Fig. 7, the shift of the zero loci of d_{13} for $D \to D + 2$ is shown (translation of two dimensions to the right in Fig. 7). The blue line of Fig. 7 for the zero loci of d_{13} almost coincides with the red line of the Yang–Lee edge singularity analyzed by the Padé approximation. The red line represents the result of 3×3 minors and it can be

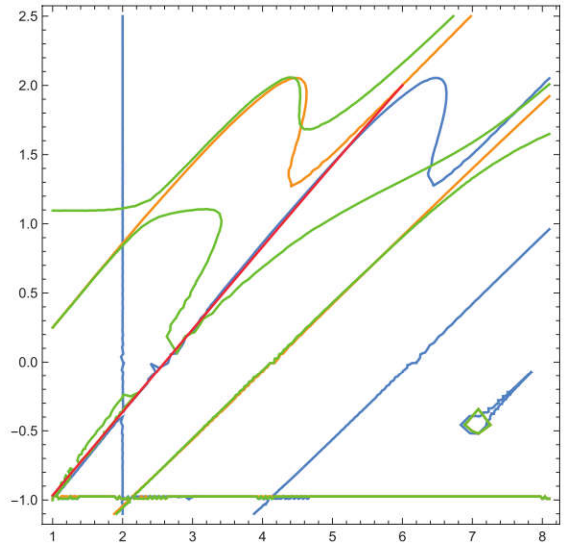

Fig. 7. The $D \to D + 2$ shifted line of the zero loci of d_{13} (yellow line) is shown by the blue line and is consistent with the estimated Yang–Lee line (red) by Padé analysis. The blue and red lines are almost the same for $1 < D < 5.5$.

approximated as

$$\Delta_\phi = \frac{3D - 8}{5}. \tag{20}$$

This expression satisfies the exact values of $D = 1$ ($\Delta_\phi = -1$), $D = 2$ ($\Delta_\phi = -0.4$), and $D = 6$ ($\Delta_\phi = 2$).

This dimensional shift seems accidental and only appears in a lower truncation, but there is a dimensional reduction that has been proven rigorously.

It is known that a branched polymer in $D + 2$ dimensions is equivalent to a D-dimensional Yang–Lee edge singularity for the critical phenomena. This dimensional reduction has been explained by supersymmetry [28]. A rigorous mathematical proof of this dimensional reduction was shown in Ref. [29]. We will study this dimensional reduction in a separate paper using the conformal bootstrap in a determinant method [14]. The Yang–Lee edge singularity requires $\Delta_\epsilon (= \Delta_{\phi^2}) = \Delta_\phi$. For a branched polymer due to dimensional reduction there is a relation $\Delta_\epsilon = \Delta_\phi + 1$. This relation is a manifestation of the supersymmetry of the system [16].

Since the blue line is very close to the red line in Fig. 7, one can make an estimation of the critical dimension D_c, in which $\Delta_\phi = 0$. The intersection of d_{13} with the $\Delta_\phi = 0$ line can be evaluated very precisely as $D = 0.599\,547\,1444$. Adding 2 to this value for the dimensional reduction, it gives the estimation of the critical dimension D_c as $D_c = 2.599\,547\,1444$, which is very close to other estimations of this paper. We discussed this value in the previous section as $D_c = 2.6199$ as in Eq. (18).

5. 3×3 minors and Padé analysis

Up to now, we have mainly discussed 2×2 minors. If we take a spin-4 operator and its scale dimension $Q = \Delta_4$, we need to do an analysis of the intersection of the zero loci of the 3×3 minors d_{ijk}.

Table 1. Estimated scale dimensions by zero loci of three dimensional minors. Comparison with Pade analysis is shown. The value of parenthesis is estimation from other zero loci nearby.

	Δ_ϕ	Q	Δ_ϕ (Padé)
$D = 3.0$	0.174 343 (0.187 825)	4.341 06 (3.771 24)	0.229 95
$D = 3.5$	0.499 401 (0.500 969)	5.041 95 (4.375 56)	0.531 53
$D = 4.0$	0.823 283 (0.815 623)	5.711 52 (4.922 83)	0.831 75
$D = 4.5$	1.137 55 (1.123 71)	6.333 95 (5.446 45)	1.1300
$D = 5.0$	1.438 07 (1.419 87)	6.917 16 (5.959 72)	1.4255
$D = 5.5$	1.724 69 (1.746 82)	7.469 85 (6.466 72)	1.7165
$D = 6.0$	2.0	8.0	2.0

From the formula of the minors in a 3×6 matrix in the appendix, we have a Plüker formula such as (Eq. (A.12))

$$[126][345] - [123][456] + [124][356] - [125][346] = 0. \tag{21}$$

For instance, at $D = 3$, we find that the zero loci of $d_{123} = [123]$, $d_{126} = [126]$, $d_{124} = [124]$, and $d_{346} = [346]$ intersect at a point, and the above Plüker formula is satisfied.

In Table 1, we present the intersection of three zero loci of minors d_{123}, d_{134}, and d_{124}. The value of Δ_ϕ is obtained from the intersection point of the zero loci of the minors. However, the intersection point is not the only one, and we find at least three different intersection points for each fixed dimension D. In Table 1, we present in parentheses such nearby different intersection points of the three zero loci of the minors.

In the $\epsilon = 6 - D$ expansion, Δ_ϕ is known up to the four-loop level [30]:

$$\Delta_\phi = 2 - 0.555\,55x - 0.029\,4925x^2 + 0.021\,845x^3 - 0.039\,4773x^4, \tag{22}$$

where $x = \epsilon = 6 - D$. Including the critical dimension D_c, for which Δ_ϕ is vanishing, the above expansion becomes

$$\Delta_\phi = (6 - x - D_c)\left[\frac{2}{6 - D_c} + \left(\frac{2}{(6 - D_c)^2} - \frac{0.55555}{6 - D_c}\right)x + O(x^2)\right]. \tag{23}$$

Inserting the value of $D_c = 2.6199$ from Sect. 4, it becomes

$$\Delta_\phi = (3.3801 - x)[0.591\,70 + 0.010\,695x - 0.005\,561\,25x^2 + 0.004\,817\,55x^3 - 0.010\,2541x^4 + \cdots]. \tag{24}$$

This expansion is approximated by the [2,2] Padé method,

$$\Delta_\phi = (3.3801 - x)\left[\frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x + b_2 x^2}\right], \tag{25}$$

with $a_0 = 0.591\,70$, $a_1 = 2.3916$, $a_2 = 1.0090$, $b_1 = 4.023\,85$, $b_2 = 1.6419$. The curve of this Padé is incorporated in Fig. 8.

For the values of Δ_ϕ of the Yang–Lee singularity in D dimensions, the previous results obtained by Gliozzi [5] are very close to the values of Table 1; for instance, at $D = 4$, $\Delta_\phi = 0.823$. In another evaluation with more primary operators [6], the values $\Delta_\phi = 0.8466$ in $D = 4$ and $\Delta_\phi = 1.455$ in

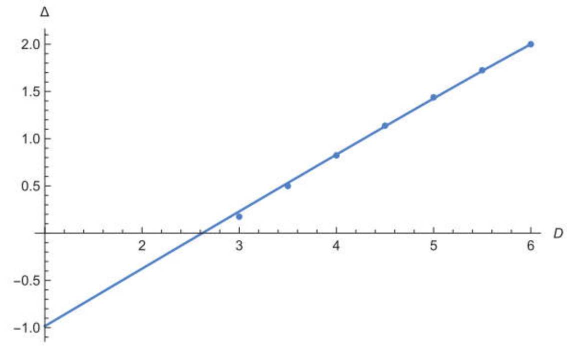

Fig. 8. Δ_ϕ is estimated by a [2,2] Padé with $D_c = 2.6199$. The dots are the values from Table 1 for a 3×3 minor analysis of Δ_ϕ.

$D = 5$ are obtained, which are larger than the values in Table 1, and a comparison with ϵ expansion shows the apparent deviations, as shown in Fig. 5 of Ref. [6]. Our new analysis of Padé with a fixed value at the critical dimension $\Delta_\phi = 0$ seems more precise, and the ϵ expansion and bootstrap determinant method agree well with each other, as shown in Fig. 8. One of the aims of this paper was to clarify this point by the evaluation of the critical dimension D_c. Unfortunately, what we have done is a numerical estimation of D_c by the bootstrap determinant method and it may be an approximation. It is important to obtain an exact value of D_c, and the comparison with ϵ expansion will be improved with an exact value of D_c.

6. Summary

We have considered the Yang–Lee edge singularity in this article by the bootstrap determinant method, initiated by Gliozzi. Although our results for the scale dimension Δ_ϕ are same as the 3×3 determinant result of Gliozzi [5], we have improved the consistency with the result of the ϵ expansion. We find that the basic 2×2 minor d_{13} is important for the determination of the above scale dimensions through the analysis of the intersection points of dimensions one to six, although we used the practical value of Δ_ϕ obtained by 3×3 minors. We have shown that the value obtained by the intersection of the zero loci of 3×3 minors agrees with the result of the ϵ expansion. The discrepancy between the ϵ expansion and the determinant method is reduced with the new Padé analysis with the introduction of the critical dimension D_c.

We obtained the critical dimension D_c from the minor d_{13} for the $\Delta_\phi = 0$ and $\Delta_\epsilon \to 0$ limits. The intersection of the zero loci of $d[D, \Delta_\phi, \Delta_\epsilon]_{13}$ at the point $\Delta_\epsilon = \Delta_\phi = 0$ gives the critical dimension $D_c = 2.6199$. We have estimated the critical dimension approximately by other methods, by the peak approaching the finite d_{13} ($D_c = 2.6050$), the dimensional reduction value ($D_c = 2.5995$), and 4×4 minor analysis ($D_c = 2.589$).

We emphasize that the estimation of the critical dimension D_c is useful in practical terms for the precise analysis of the Yang–Lee edge singularity between two and six dimensions by the Padé analysis of the ϵ expansion [30].

We have discussed the dimensional reduction property in the Yang–Lee model, using the fact that a branched polymer in $D + 2$ dimensions is equivalent to a D-dimensional Yang–Lee edge singularity.

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Appendix. Plüker formula

Relation 1 (3×3 *minors)* For the determinant d_{123}, which is defined as

$$d_{123} = \det \begin{pmatrix} vs1 & vs2 & vs3 \\ vt1 & vt2 & vt3 \\ vq1 & vq2 & vq3 \end{pmatrix}$$

$$= (vq3)\det \begin{pmatrix} vs1 & vs2 \\ vt1 & vt2 \end{pmatrix} - (vq2)\det \begin{pmatrix} vs1 & vs3 \\ vt1 & vt3 \end{pmatrix} + (vq1)\det \begin{pmatrix} vs2 & vs3 \\ vt2 & vt3 \end{pmatrix}, \quad \text{(A.1)}$$

if the first determinant (d_{12}) and second minor (d_{13}) on the right-hand side are zero, then the third minor (d_{23}) should be vanishing when $d_{123} = 0$.

Relation 2 (2×4 *matrix)* The 2×4 matrix is denoted as

$$M = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{pmatrix}. \quad \text{(A.2)}$$

The Plüker relation is

$$[12][34] - [13][24] + [14][23] = 0, \quad \text{(A.3)}$$

where the minor is denoted as $[ij] = (x_{1i})(x_{2j}) - (x_{1j})(x_{2i})$. The application of this formula to minors of our case can be taken as

$$M = \begin{pmatrix} vs1 & vs2 & vs3 & vs4 \\ vt1 & vt2 & vt3 & vt4 \end{pmatrix}. \quad \text{(A.4)}$$

Thus, in our notation for 2×2 minors,

$$d_{12} = [12], \ d_{13} = [13], \ d_{14} = [14], \ d_{23} = [23], \ d_{24} = [24], \ d_{34} = [34] \quad \text{(A.5)}$$

and we have an identity

$$d_{12}d_{34} - d_{13}d_{24} + d_{14}d_{23} = 0, \quad \text{(A.6)}$$

which is represented by the tableaux

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline \end{array} = 0. \quad \text{(A.7)}$$

Note that the third tableau is not ordered by increasing row and column (4 is larger than 3). Another application of this formula is to take the following matrix:

$$M = \begin{pmatrix} vs1 & vt1 & vq1 & vx1 \\ vs3 & vt3 & vq3 & vx3 \end{pmatrix}. \tag{A.8}$$

In the $D = 6$ case, we find that 2×2 minors made of the above matrix are vanishing, $[12] = [13] = [14] = 0$ for $\Delta_\phi = 2, \Delta_4 = 8$, and $\Delta_6 = 10$. Therefore they satisfy the relation of Eq. (A.3). In this $D = 6$ case, we also find that, for such values of $\Delta_\phi = 2,\ \Delta_4 = 8,\ \Delta_6 = 10$, the other three sets of minors are vanishing at $D = 6$:

$$[34] = \det \begin{pmatrix} vq1 & vx1 \\ vq3 & vx3 \end{pmatrix} = 0, \quad [24] = \det \begin{pmatrix} vt1 & vx1 \\ vt3 & vx3 \end{pmatrix} = 0,$$
$$[23] = \det \begin{pmatrix} vt1 & vq1 \\ vt3 & vq3 \end{pmatrix} = 0. \tag{A.9}$$

Relation 3 (3 × 6 matrix)

(i) The 3×6 matrix in the bootstrap minor method is

$$\begin{pmatrix} vs1 & vs2 & vs3 & vs4 & vs5 & vs6 \\ vt1 & vt2 & vt3 & vt4 & vt5 & vt6 \\ vq1 & vq2 & vq3 & vq4 & vq5 & vq6 \end{pmatrix}. \tag{A.10}$$

The Plüker relation is obtained by a mutual exchange of numbers:

$$[146][235] + [124][356] - [134][256] + [126][345] - [136][245] + [123][456] = 0 \tag{A.11}$$

This identity is obtained from the first term to the second term by $(2 \leftrightarrow 6)$, and from the second to third by $(2 \leftrightarrow 3)$, etc. with the sign. We can also use the following relations (obtained similarly):

$$[126][345] - [123][456] + [124][356] - [125][346] = 0$$
$$[136][245] + [123][456] + [134][256] - [135][246] = 0; \tag{A.12}$$

from these equations, Eq. (A.11) becomes

$$[146][235] = -3[123][456] - [125][346] + [135][246]. \tag{A.13}$$

In our minor examples, we have

$$d_{146} d_{235} = -3 d_{123} d_{456} - d_{125} d_{346} + d_{135} d_{246}. \tag{A.14}$$

The terms on the right-hand side are ordered by increasing row and column.

(ii) Another application can be taken as

$$\begin{pmatrix} vs1 & vt1 & vq1 & vx1 & vs1' & vs1'' \\ vs2 & vt2 & vq2 & vx2 & vs2' & vs2'' \\ vs3 & vt3 & vq3 & vx3 & vs3' & vs3'' \end{pmatrix}. \tag{A.15}$$

The Plüker relation is the same as before. In a tableau, Eq. (A.13) is expressed as

$$\begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 3 & 5 \\ \hline \end{array} = -3 \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array} - \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline \end{array} . \tag{A.16}$$

The left-hand side is not ordered by increasing column, but the right-hand side is a combination of ordering by both increasing row and column.

The Plüker relation is expressed by a two-row tableau [27], and for an $m \times n$ matrix, the formula becomes

$$\sum_{i_1 < \cdots < i_t, i_{t+1} \cdots < i_s} \sigma(i_1, \ldots, i_s)[a_1, \ldots, a_k, c_{i_1}, \ldots, c_{i_t}][c_{i_{t+1}}, \ldots, c_{i_s}, b_l, \ldots, b_m] = 0, \tag{A.17}$$

where $(1, \ldots, s) = (i_1, \ldots, i_s), a_j, b_{j'}, c_{j''} \in (1, \ldots, n)$, and $s = m - k + l - 1 > m$, $t = m - k > 0$. These tableau representations suggest algebraic structures such as character expansion.

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