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Isometric deformations of unstretchable material surfaces, a spatial variational treatment

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ABSTRACT

The stored energy of an unstretchable material surface is assumed to depend only upon the curvature tensor. By control of its edge(s), the surface is deformed isometrically from its planar undistorted reference configuration into an equilibrium shape. That shape is to be determined from a suitably constrained variational problem as a state of relative minimal potential energy. We pose the variational problem as one of relative minimum potential energy in a spatial form, wherein the deformation of a flat, undistorted region $\mathcal{D}$ in $\mathbb{E}^2$ to its distorted form $\mathcal{S}$ in $\mathbb{E}^3$ is assumed specified. We then apply the principle that the first variation of the potential energy, expressed as a functional over $\mathcal{S} \cup \partial\mathcal{S}$, must vanish for all admissible variations that correspond to isometric deformations from the distorted configuration $\mathcal{S}$ and that also contain the essence of flatness that characterizes the reference configuration $\mathcal{D}$, but is not covered by the single statement that the variation of $\mathcal{S}$ correspond to an isometric deformation. We emphasize the commonly overlooked condition that the spatial expression of the variational problem requires an additional variational constraint of zero Gaussian curvature to ensure that variations from $\mathcal{S}$ that are isometric deformations also contain the notion of flatness. In this context, it is particularly revealing to observe that the two constraints produce distinct, but essential and complementary, conditions on the first variation of $\mathcal{S}$. The resulting first variation integral condition, together with the constraints, may be applied, for example, to the case of a flat, undistorted, rectangular strip $\mathcal{D}$ that is deformed isometrically into a closed ring $\mathcal{S}$ by connecting its short edges and specifying that its long edges are free of loading and, therefore, subject to zero traction and couple traction. The elementary example of a closed ring without twist as a state of relative minimum potential energy is discussed in detail, and the bending of the strip by opposing specific bending moments on its short edges is treated as a particular case. Finally, the constrained variational problem, with the introduction of appropriate constraint reactions as Lagrangian multipliers to account for the requirements that the deformation from $\mathcal{D}$ to $\mathcal{S}$ is isometric and that $\mathcal{D}$ is flat, is formulated in the spatial form, and the associated Euler–Lagrange equations are derived. We then solve the Euler–Lagrange equations for two representative problems in which a planar undistorted rectangular material strip is isometrically deformed by applied edge tractions and couple tractions (i.e., specific edge moments) into (*i*) a bent and twisted circular cylindrical helical state, and (*ii*) a state conformal with the surface of a right circular conical form.

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1. Introduction

In mechanics, it is impossible to overlook the importance of material surfaces which bend easily but substantially resist stretching (and contracting). Everyday examples of such objects include photocopy paper and certain textiles. Our concern here is with the idealized situation in which the notion of unstretchability is treated strictly. Specifically, we consider material surfaces that are capable only of sustaining isometric deformations in a manner analogous to considering a nearly incompressible three-dimensional material, the conventional example being rubber at room temperature, as being capable only of sustaining isochoric deformations. For simplicity and clarity, we restrict attention to material surfaces that are intrinsically flat and, thus, are deformed, in pure bendings, from planar undistorted reference configurations.

In differential geometry, a mapping between two surfaces is an isometry if it preserves the lengths of curves on those surfaces (Kreyszig, 1968). Although an appreciation for the difference between an isometric deformation of a material surface and an isometry between two surfaces is implicitly evident in recent works of Guven and Müller (2008), Hornung (2011), and Freddi et al. (2016), those notions are still frequently confused. In a contemporary series of papers, we (Chen and Fried, 2016; Chen et al., 2015; Chen et al., 2017; Chen et al., 2018) clarify their differences and highlight what can go wrong if those differences are ignored.

Building on the perspective provided by our previous works, we herein pressent a framework for determining the equilibrium configurations of an unstretchable material surface that is deformed isometrically from a planar undistorted reference configuration identified with a region $\mathcal{D}$ in two-dimensional Euclidean point space $\mathbb{E}^2$ to a surface $\mathcal{S}$, oriented by a unit normal field $\boldsymbol{n}$, in three-dimensional Euclidean point space $\mathbb{E}^3$. We allow the stored energy density W of the material surface to be a generic frame indifferent function of the curvature tensor $\boldsymbol{L} = -\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{n}$ of $\mathcal{S}$, so that its total stored energy $\mathcal{E}$ is given by

$$\mathcal{E} = \int_{\mathcal{S}} W(\boldsymbol{L})\,\mathrm{d}a,$$

where $\mathrm{d}a$ denotes the area element on $\mathcal{S}$.[1] We formulate the problem in a spatial variational setting, accounting for the work performed by a system of dead loads including a traction $\boldsymbol{t}$ and a couple traction $\boldsymbol{c}$ acting on some portion $\partial_2\mathcal{S}$ of the boundary $\partial\mathcal{S}$ of $\mathcal{S}$.

We thus consider the problem of minimizing the functional

$$\mathcal{F}(\mathcal{S}_\epsilon) := \int_{\mathcal{S}} W(\boldsymbol{L}_\epsilon)\,\mathrm{d}a - \int_{\partial_2\mathcal{S}} (\boldsymbol{t}\cdot(\boldsymbol{z}_\varepsilon - \boldsymbol{y}) + \boldsymbol{c}\cdot(\boldsymbol{n}_\varepsilon - \boldsymbol{n}))\,\mathrm{d}s,$$

where $\mathcal{S}_\epsilon$, with $0 \le \epsilon \le 1$, is a variation of $\mathcal{S}$ induced by a variation $\boldsymbol{z}_\epsilon$ of position $\boldsymbol{y}$ on $\mathcal{S}$ satisfying $\boldsymbol{z}_\epsilon(\boldsymbol{y})|_{\varepsilon=0} = \boldsymbol{y}$ and $\boldsymbol{n}_\epsilon$ and $\boldsymbol{L}_\epsilon$ are the associated variations of $\boldsymbol{n}$ and $\boldsymbol{L}$. We determine the implications of varying $\mathcal{F}$ subject to the requirement that $\boldsymbol{z}_\epsilon$ is constrained consistent with the requirement that the material surface be unstretchable while maintaining its flatness and, thus, be capable only of sustaining variations that correspond to isometric deformations of zero Gaussian curvature. In so doing, we must ensure that three admissibility conditions, namely

$$\boldsymbol{L}_\varepsilon \boldsymbol{n}_\varepsilon = \boldsymbol{0}, \qquad (\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{z}_\varepsilon(\boldsymbol{y}))^{\top}\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{z}_\varepsilon(\boldsymbol{y}) - \boldsymbol{1}_{\mathcal{S}} = \boldsymbol{0}, \qquad \text{and} \qquad K_\varepsilon = 0,$$

where K_ϵ denotes the variation of the Gaussian curvature K of $\mathcal{S}$, are satisfied for each choice of ϵ.[2] A particular consequence of our results is that for the pure traction problem, in which $\partial_2\mathcal{S} = \partial\mathcal{S}$, the resultants of the applied traction and the applied couple traction must be force and moment balanced. More generally, we derive the first variation condition

$$\int_{\mathcal{S}} (\mathrm{div}_{\mathcal{S}}\,\mathrm{div}_{\mathcal{S}}\,W'(\boldsymbol{L}) - 2(W(\boldsymbol{L}) - 2W'(\boldsymbol{L})\cdot\boldsymbol{L})H)U\,\mathrm{d}a$$
$$= \int_{\partial\mathcal{S}} (U\mathrm{div}_{\mathcal{S}}\,W'(\boldsymbol{L}) - W'(\boldsymbol{L})\mathrm{grad}_{\mathcal{S}}\,U - W(\boldsymbol{L})\boldsymbol{u}^{\mathrm{tan}})\cdot\boldsymbol{\nu}\,\mathrm{d}s + \int_{\partial_2\mathcal{S}} (\boldsymbol{t}\cdot\boldsymbol{u} - \boldsymbol{c}\cdot(\mathrm{grad}_{\mathcal{S}}\,U + \boldsymbol{L}\boldsymbol{u}^{\mathrm{tan}}))\,\mathrm{d}s,$$

where $\boldsymbol{\nu}$ denotes the tangent normal to $\partial\mathcal{S}$ and the variation $\boldsymbol{u} = \boldsymbol{u}^{\mathrm{tan}} + U\boldsymbol{n}$. The foregoing result can be directly specialized to the important case where W has the quadratic form $W(\boldsymbol{L}) = \frac{1}{2}\mu(\mathrm{tr}\,\boldsymbol{L})^2$, involving a single bending stiffness $\mu > 0$, and on this basis we consider two simple but illustrative example problems in which a flat, undistorted, rectangular strip is deformed isometrically into a closed loop by connecting its short edges, its long edges are free of loading and, therefore, subject to zero traction and couple traction. We discuss in detail the elementary example of a closed ring without twist as a state of relative minimum potential energy, and consider as an example the bending of the strip by opposing specific bending moments on its short edges.

We also formulate a constrained version of the variational problem, with the introduction of appropriate constraint reactions as Lagrangian multipliers to account for the isometric constraint, and derive the associated Euler–Lagrange equations.

[1] Due to the isometry of the underlying deformation, $\mathrm{d}a$ is equal to its counterpart on $\mathcal{D}$.

[2] In this spatial setting, it is necessary to state explicitly that the Gaussian curvature of $\mathcal{S}_\epsilon$ has the same zero value as the Gaussian curvature of $\mathcal{S}$. This requirement, alone, guarantees an isometry relation between the surfaces, but does not enforce the deformation to be an isometric deformation. The second equation in the line above guarantees an isometric deformation, but does not carry any information about the Gaussian curvature of $\mathcal{S}$.

The reactions consist of a symmetric tensor field $\boldsymbol{T}$ and a scalar field p. The Euler–Lagrange equations include partial differential equations that hold on the interior of $\mathcal{S}$ and natural boundary conditions that hold on the dead-loaded portion $\partial_2\mathcal{S}$ of $\mathcal{S}$. The partial differential equations that apply on $\mathcal{S}$, namely

$$\operatorname{div}_{\mathcal{S}}\operatorname{div}_{\mathcal{S}} W'(\boldsymbol{L}) - 2(W(\boldsymbol{L}) - 2W'(\boldsymbol{L})\cdot\boldsymbol{L})H + \boldsymbol{T}\cdot\boldsymbol{L} - (2H\boldsymbol{1}_{\mathcal{S}} - \boldsymbol{L})\cdot\operatorname{grad}_{\mathcal{S}}\operatorname{grad}_{\mathcal{S}} p = 0$$

and

$$\boldsymbol{1}_{\mathcal{S}}\operatorname{div}_{\mathcal{S}}\boldsymbol{T} = \boldsymbol{0},$$

respectively express the components of force balance in directions normal and tangential on $\mathcal{S}$. The natural boundary conditions that apply on $\partial_2\mathcal{S}$, namely

$$(\operatorname{div}_{\mathcal{S}} W'(\boldsymbol{L}) - (2H\boldsymbol{1}_{\mathcal{S}} - \boldsymbol{L})\operatorname{grad}_{\mathcal{S}} p)\cdot\boldsymbol{\nu} + \frac{\partial}{\partial s}(\boldsymbol{\sigma}\cdot(W'(\boldsymbol{L})\boldsymbol{\nu} + p\boldsymbol{L}\boldsymbol{\nu} + \boldsymbol{c})) + \boldsymbol{t}\cdot\boldsymbol{n} = 0,$$

$$(W(\boldsymbol{L})\boldsymbol{1}_{\mathcal{S}} - \boldsymbol{T})\boldsymbol{\nu} + \boldsymbol{L}\boldsymbol{c} - \boldsymbol{t}^{\tan} = \boldsymbol{0},$$

and

$$(W'(\boldsymbol{L})\boldsymbol{\nu} - p(2H\boldsymbol{1}_{\mathcal{S}} - \boldsymbol{L})\boldsymbol{\nu} + \boldsymbol{c})\cdot\boldsymbol{\nu} = 0,$$

respectively express force balance normal to $\mathcal{S}$ on $\partial_2\mathcal{S}$, force balance tangential to $\mathcal{S}$ on $\partial_2\mathcal{S}$, and couple balance tangent-normal to $\mathcal{S}$ on $\partial_2\mathcal{S}$.

We conclude by considering example problems in which a planar undistorted rectangular material strip with stored energy density W of the quadratic form $W(\boldsymbol{L}) = \frac{1}{2}\mu(\operatorname{tr}\boldsymbol{L})^2$ is isometrically deformed by applied edge tractions and couple tractions (i.e., specific edge moments) into a bent and twisted circular cylindrical helical state and to a state conformal with the surface of a right circular conical surface.

The introduction of constraint reactions as Lagrangian multiplier fields and their inclusion into the potential energy functional to form a constrained variational problem appropriate for the study of the isometric deformation of planar, unstretchable material sheets subject to edge loading conditions is not common in the literature. A partial exception, though, is the interesting paper of Guven and Müller (2008), who introduce a spatial variational framework for describing the isometric deformation of a flat material sheet into a surface in $\mathbb{E}^3$. The constrained functional they pose differs from that considered in the present work in two important ways:

- the potential energy of possible applied edge tractions $\boldsymbol{t}$ and edge couples $\boldsymbol{c}$ is omitted, and
- the constraint reaction p, which represents a reaction to the second-order constraint that the Gaussian curvature is zero, is not included.

Although their formulation includes the first-order constraint reaction $\boldsymbol{T}$ that is associated with the isometric constraint is included in their spatial variational formulation, it does not include a specific statement that the variation refers to a surface of zero Gaussian curvature. Guven and Müller (2008) record the Euler–Lagrange equations that apply on $\mathcal{S}$. Modulo the absence of terms which involve the reaction p that appear in the equation that expresses the component of force balance normal to $\mathcal{S}$, those equations are consistent with equations we present, for the particular choice $W(\boldsymbol{L}) = \frac{1}{2}\mu(\operatorname{tr}\boldsymbol{L})^2$, in Section 9. As an application, Guven and Müller (2008) envision the isometric deformation of a planar disc into a generalized conical shape, wherein the apex angle need not be uniform as it is in the case of a right circular conical surface. They do not set out to determine a specific conical shape by integrating both of their Euler–Lagrange equations, but, rather, they partially integrate these equations to show explicitly how the constraint reaction field $\boldsymbol{T}$ on the deformed surface depends on the distance r from the apex of the conical shape. Singularities in $\boldsymbol{T}$ occur at the apex $r = 0$. The dependence of $\boldsymbol{T}$ on a coordinate transverse to the measurement of r is not determined. However, it is shown how such dependence is related to the total force and moment that is exerted on a closed curve $r =$ constant that cuts off a portion of the generalized conical shape including its apex. It is also noted that if the tip of the conical surface is subject to an applied load and moment then the variation of $\boldsymbol{T}$ along the $r =$ constant rim must accommodate a state of equilibrium with these applied actions at the tip.

The work presented herein, though fundamentally related to that of Guven and Müller (2008) because both are based on a spatial variational approach, has a distinctly different goal. Specifically, in Sections 10 and 11 we seek to determine through an appropriately constrained variational problem the conditions of external specific traction and specific moment that are applied to the edges of a flat rectangular strip so that it may sit in equilibrium in its isometrically deformed state in the form of a portion of a right circular cylindrical surface, or a right circular conical surface.

2. Setup of the variational problem

Consider an undistorted planar material surface identified with a subset $\mathcal{D}$ of two-dimensional Euclidean point space $\mathbb{E}^2$. Given an orthonormal basis $\{\boldsymbol{\imath}_1, \boldsymbol{\imath}_2\}$ for the translation space $\mathbb{V}^2$ of $\mathbb{E}^2$, let $\tilde{\boldsymbol{y}}$ be a deformation that takes each material point $\boldsymbol{x}$ of $\mathcal{D}$ to a point $\boldsymbol{y} = \tilde{\boldsymbol{y}}(\boldsymbol{x})$ on a surface $\mathcal{S}$ in three-dimensional Euclidean point space $\mathbb{E}^3$. The deformation gradient

$$\boldsymbol{F} := \nabla\tilde{\boldsymbol{y}} \tag{2.1}$$

is a linear transformation which maps vectors in the translation space $\mathbb{V}^3$ of $\mathbb{E}^3$ to vectors in the tangent space $\mathcal{S}^{\tan}$ of $\mathcal{S}$, with the property that

$$\boldsymbol{F}\boldsymbol{\imath}_3 = \boldsymbol{0}, \qquad \boldsymbol{\imath}_3 := \boldsymbol{\imath}_1 \times \boldsymbol{\imath}_2. \tag{2.2}$$

The stretch of a material fiber $\mathrm{d}\boldsymbol{x}$ with unit orientation $\boldsymbol{e}$ in $\mathbb{V}^2$ is given by

$$\lambda := \frac{|\mathrm{d}\boldsymbol{y}|}{|\mathrm{d}\boldsymbol{x}|} = \sqrt{\boldsymbol{e}\cdot\boldsymbol{F}^{\top}\boldsymbol{F}\boldsymbol{e}} = |\boldsymbol{F}\boldsymbol{e}|, \tag{2.3}$$

and the material surface $\mathcal{D}$ is said to be unstretchable if its admissible deformations satisfy $\lambda = 1$ for all choices of $\boldsymbol{e}$. In this case,

$$\boldsymbol{F}^{\top}\boldsymbol{F} = \boldsymbol{1}_{\mathcal{D}}, \tag{2.4}$$

where $\boldsymbol{1}_{\mathcal{D}} = \boldsymbol{\imath}_1 \otimes \boldsymbol{\imath}_1 + \boldsymbol{\imath}_2 \otimes \boldsymbol{\imath}_2$ denotes the identity linear transformation on $\mathbb{V}^2$, and the only possible deformations of $\mathcal{D}$ to $\mathcal{S}$ are isometric. As consequences of (2.4), we have

$$|\boldsymbol{F}\boldsymbol{\imath}_1| = |\boldsymbol{F}\boldsymbol{\imath}_2| = 1 \qquad \text{and} \qquad \boldsymbol{F}\boldsymbol{\imath}_1 \cdot \boldsymbol{F}\boldsymbol{\imath}_2 = 0. \tag{2.5}$$

A unit normal $\boldsymbol{n}$ to $\mathcal{S}$ can be defined as

$$\boldsymbol{n} := \frac{\boldsymbol{F}\boldsymbol{\imath}_1 \times \boldsymbol{F}\boldsymbol{\imath}_2}{|\boldsymbol{F}\boldsymbol{\imath}_1 \times \boldsymbol{F}\boldsymbol{\imath}_2|} = \frac{\boldsymbol{F}^{\mathrm{C}}\boldsymbol{\imath}_3}{|\boldsymbol{F}^{\mathrm{C}}\boldsymbol{\imath}_3|} = \boldsymbol{F}^{\mathrm{C}}\boldsymbol{\imath}_3, \tag{2.6}$$

where $\boldsymbol{F}^{\mathrm{C}}$ denotes the cofactor of $\boldsymbol{F}$ and we have used the identity $\boldsymbol{F}\boldsymbol{\imath}_1 \times \boldsymbol{F}\boldsymbol{\imath}_2 = \boldsymbol{F}^{\mathrm{C}}\boldsymbol{\imath}_3$, so that, by (2.4), $|\boldsymbol{F}^{\mathrm{C}}\boldsymbol{\imath}_3| = 1$. The curvature tensor of $\mathcal{S}$ is given by

$$\boldsymbol{L} := -\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{n}, \tag{2.7}$$

where '$\mathrm{grad}_{\mathcal{S}}$' denotes the surface gradient on $\mathcal{S}$. It is well known that $\boldsymbol{L}$ is a symmetric linear transformation of $\mathbb{V}^3$ to itself and that $\boldsymbol{L}$ annihilates $\boldsymbol{n}$:

$$\boldsymbol{L}^{\top} = \boldsymbol{L}, \qquad \boldsymbol{L}\boldsymbol{n} = \boldsymbol{0}. \tag{2.8}$$

Thus, $\boldsymbol{L}$ can also be viewed as a symmetric linear transformation of $\mathcal{S}^{\tan}$ to itself and in this sense its two eigenvalues represent the principal curvatures of $\mathcal{S}$. The mean curvature H of $\mathcal{S}$ is the average of the principal curvatures, which may be written as

$$H := \frac{1}{2}\mathrm{tr}\,\boldsymbol{L} = -\frac{1}{2}\mathrm{div}_{\mathcal{S}}\,\boldsymbol{n}, \tag{2.9}$$

and the Gaussian curvature K is the product of the principal curvatures, which may be written as

$$K := \det\boldsymbol{L} = \frac{1}{2}((\mathrm{tr}\,\boldsymbol{L})^2 - \mathrm{tr}\,(\boldsymbol{L}^2)). \tag{2.10}$$

Throughout this work, we let $\boldsymbol{1}$ denote the identity map on $\mathbb{V}^3$ and, later, we shall use

$$\boldsymbol{1}_{\mathcal{S}} := \boldsymbol{1} - \boldsymbol{n}\otimes\boldsymbol{n} \tag{2.11}$$

to denote the projection of $\mathbb{V}^3$ to $\mathcal{S}^{\tan}$. Formally, the inclusion map $\boldsymbol{I}$ is a linear transformation of $\mathcal{S}^{\tan}$ to $\mathbb{E}^3$. Thus, any element $\boldsymbol{\tau}$ of $\mathcal{S}^{\tan}$ is identified through $\boldsymbol{I}$ as an element $\boldsymbol{I}\boldsymbol{\tau}$ of $\mathbb{E}^3$. In this case, the identity linear transformation on $\mathcal{S}^{\tan}$ is given by $\boldsymbol{I}_{\mathcal{S}} := \boldsymbol{1}_{\mathcal{S}}\boldsymbol{I}$ and this distinguishes the projection of $\mathbb{E}^3$ to $\mathcal{S}^{\tan}$ from the identity on $\mathcal{S}^{\tan}$. Because this distinction is not needed in the present work, we shall identify $\boldsymbol{I}_{\mathcal{S}}$ with $\boldsymbol{1}_{\mathcal{S}}$.

In this development, we shall be concerned with unstretchable surfaces and we shall assume that the stored energy W of $\mathcal{S}$, measured per unit area of $\mathcal{S}$, is due only to bending and depends upon the deformation through the curvature tensor $\boldsymbol{L}$. Thus, W is defined on

$$\mathrm{Sym}_0 := \bigcup_{|\boldsymbol{n}|=1}\mathrm{Sym}\,(\boldsymbol{n}), \qquad \mathrm{Sym}\,(\boldsymbol{n}) := \{\boldsymbol{B}\in\mathrm{Sym}\,|\,\boldsymbol{B}\boldsymbol{n} = \boldsymbol{0}\}, \tag{2.12}$$

where Sym is the collection of symmetric linear transformations on $\mathbb{V}^3$. It follows from this that the derivative of W, denoted as W', is a symmetric linear transformation of $\mathbb{V}^3$ to itself which annihilates $\boldsymbol{n}$; thus, $W'(\boldsymbol{L})\boldsymbol{n} = \boldsymbol{0}$ for $\boldsymbol{L} = -\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{n}$. Of course, $W'(\boldsymbol{L})$ is a symmetric linear transformation of $\mathcal{S}^{\tan}$ to itself for each $\boldsymbol{L}$ in Sym_0.

In this work we assume that the deformation $\tilde{\boldsymbol{y}}$ defines an equilibrium configuration of $\mathcal{S}$ subject to the constraint (2.4) and the kinematical (or essential) and loading boundary conditions which specify, respectively, $\boldsymbol{y}$ and $\boldsymbol{n}$ on part $\partial_1\mathcal{S}$ of the boundary $\partial\mathcal{S}$, and the traction $\boldsymbol{t}$ and the couple traction $\boldsymbol{c}$ or, equivalently, the associated specific moment

$$\boldsymbol{m} := \boldsymbol{n}\times\boldsymbol{c}, \tag{2.13}$$

on the remainder $\partial_2\mathcal{S} = \partial\mathcal{S}\setminus\partial_1\mathcal{S}$ of $\partial\mathcal{S}$. We seek to characterize this equilibrium configuration as a constrained relative minimum of the potential energy of $\mathcal{S}$ and determine the corresponding Euler–Lagrange equations, the constraint in question stemming from the stipulation that $\mathcal{D}$ be unstretchable.

Remark 2.1. Recall that the constraint (2.4) for an isometric deformation requires as a necessary condition that the Gaussian curvature of $\mathcal{S}$ must vanish:

$$K = 0. \tag{2.14}$$

Granted that the deformation is three times continuously differentiable, a direct proof of this can be found in the work of Chen and Fried (2016). Also, this is proven, under the same smoothness hypothesis, by Chen et al. (2015). By the Cayley–Hamilton theorem, $\boldsymbol{L}$ satisfies

$$\boldsymbol{L}^2 - 2H\boldsymbol{L} + K\boldsymbol{1}_{\mathcal{S}} = \boldsymbol{0}. \tag{2.15}$$

However, since the assumed equilibrium deformation is isometric, we see from (2.14) and (2.15) that

$$\boldsymbol{L}^2 = 2H\boldsymbol{L}. \tag{2.16}$$

Conversely, if $\boldsymbol{L}$ satisfies (2.16) then we see from (2.10) that K vanishes. □

Remark 2.2. Before we consider particular examples, we shall avoid being more specific concerning boundary conditions. In this regard, it suffices to mention that our considerations restrict the deformation to be isometric and include the possibility wherein the reference configuration $\mathcal{D}$ is a planar undistorted rectangular strip with one 'short' end fixed and the other, together denoted as $\partial_1\mathcal{S}$, brought to the same fixed position as the first so that the distorted configuration $\mathcal{S}$ is a smooth loop. In this case, the 'long' ends, together denoted as $\partial_2\mathcal{S}$, are free of traction and couple traction, and the equilibrium configuration $\mathcal{S}$ may contain a twist equal to a multiple of π. □

We suppose that the mechanical response of the material surface is characterized by a stored energy density W that depends on the curvature tensor $\boldsymbol{L} = -\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{n}$ of $\mathcal{S}$ in any way consistent with the provision

$$W(\boldsymbol{QLQ}^{\top}) = W(\boldsymbol{L}), \qquad \boldsymbol{Q}^{\top}\boldsymbol{Q} = \boldsymbol{1}, \qquad \boldsymbol{Qn} = \boldsymbol{n}, \tag{2.17}$$

of frame indifference. The total stored energy $\mathcal{E}$ of $\mathcal{S}$ is

$$\mathcal{E}(\mathcal{S}) = \int_{\mathcal{S}} W(\boldsymbol{L})\,\mathrm{d}a. \tag{2.18}$$

To define the relevant potential energy functional, we first introduce a variation $\boldsymbol{z} := \boldsymbol{z}_{\varepsilon}$ defined on $\mathcal{S}$ and for $0 \le \varepsilon \le 1$ such that

$$\boldsymbol{z}_0(\boldsymbol{y}) := \boldsymbol{z}_{\varepsilon}(\boldsymbol{y})|_{\varepsilon=0} = \boldsymbol{y} \qquad \text{and} \qquad \dot{\boldsymbol{z}}_{\varepsilon}|_{\varepsilon=0} =: \boldsymbol{u}, \tag{2.19}$$

where a superposed dot denotes differentiation with respect to ε. We also use the notation

$$\mathcal{S}_{\varepsilon} := \boldsymbol{z}_{\varepsilon}(\mathcal{S}), \qquad \text{with} \qquad \mathcal{S}_0 = \mathcal{S}, \tag{2.20}$$

to denote the surface under the variation $\boldsymbol{z}_{\varepsilon}$, with $\boldsymbol{n}_{\varepsilon}$ its normal field and $\boldsymbol{L}_{\varepsilon} := -\mathrm{grad}_{\mathcal{S}_{\varepsilon}}\boldsymbol{n}_{\varepsilon}$ its curvature tensor, and introduce the one parameter family of potential energy functionals

$$\mathcal{F}(\mathcal{S}_{\varepsilon}) := \int_{\mathcal{S}_{\varepsilon}} W(\boldsymbol{L}_{\varepsilon})\,\mathrm{d}a_{\varepsilon} - \int_{\partial_2\mathcal{S}} (\boldsymbol{t}\cdot(\boldsymbol{z}_{\varepsilon} - \boldsymbol{y}) + \boldsymbol{c}\cdot(\boldsymbol{n}_{\varepsilon} - \boldsymbol{n}))\,\mathrm{d}s, \tag{2.21}$$

where we have assumed that the applied traction $\boldsymbol{t}$ and the applied couple traction $\boldsymbol{c}$, prescribed for all $\boldsymbol{y}$ belonging to $\partial_2\mathcal{S}$, are 'dead' loads — namely, loads that are independent of ε when considered, respectively, as an applied force and an applied couple measured per unit length on $\partial\mathcal{S}$ — during the variation. In the second integral in (2.21), it is implicit that $\boldsymbol{n}_{\varepsilon}$ is considered as the composition $\boldsymbol{n}_{\varepsilon}\circ\,\boldsymbol{z}_{\varepsilon}$. When, during a deformation from $\mathcal{S}$ to $\mathcal{S}_{\varepsilon}$, the traction and couple traction are prescribed as 'dead' loads on $\partial_2\mathcal{S}$ this integral represents the work done by these loads. Since $\boldsymbol{y}$ and $\boldsymbol{n}$ are supposed to be prescribed for all $\boldsymbol{y}$ belonging to $\partial_1\mathcal{S}$, these kinematical conditions are fixed during the deformation from $\mathcal{S}$ to $\mathcal{S}_{\varepsilon}$ and the integral over $\partial_1\mathcal{S}$ does not appear in (2.21).

The admissibility conditions[3]

$$\boldsymbol{L}_{\varepsilon}\boldsymbol{n}_{\varepsilon} = \boldsymbol{0}, \qquad (\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{z}_{\varepsilon}(\boldsymbol{y}))^{\top}\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{z}_{\varepsilon}(\boldsymbol{y}) - \boldsymbol{1}_{\mathcal{S}} = \boldsymbol{0}, \qquad \text{and} \qquad K_{\varepsilon} = 0. \tag{2.22}$$

must be respected when evaluating $\mathcal{F}$ at $\mathcal{S}_{\varepsilon}$ for $0 \le \varepsilon \le 1$. A material element in $\mathcal{S}_{\varepsilon}^{\mathrm{tan}}$ is given by $\mathrm{d}\boldsymbol{z} = (\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{z}_{\varepsilon})\mathrm{d}\boldsymbol{y}$, where $\mathrm{d}\boldsymbol{y}$ is the corresponding material element in $\mathcal{S}^{\mathrm{tan}}$ with length $\mathrm{d}s$. The element of material area $\mathrm{d}a_{\varepsilon}$ on $\mathcal{S}_{\varepsilon}$ may thus be defined through

$$\mathrm{d}a_{\varepsilon} = |(\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{z}_{\varepsilon})\mathrm{d}\boldsymbol{y}_1 \times (\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{z}_{\varepsilon})\mathrm{d}\boldsymbol{y}_2|, \tag{2.23}$$

[3] The third condition in (2.22) states that the Gaussian curvature of the surface $\mathcal{S}_{\varepsilon}$ is zero. This can be proved from the second condition, in (2.22) which states that the mapping $\boldsymbol{z}_{\varepsilon}$ of $\mathcal{S}$ to $\mathcal{S}_{\varepsilon}$ is isometric, together with the hypothesis that the equilibrium deformation $\tilde{\boldsymbol{y}}$ is an isometric mapping of a planar region $\mathcal{D}$ to the surface $\mathcal{S}$. Of course, the proof of this fact is a confirmation of Theorema egregium of Gauss (1827). It is important to note that these admissibility conditions are spatial in form. Therefore, without including $K_{\varepsilon} = 0$ in (2.22), there is no mention that the surface $\mathcal{S}_{\varepsilon}$ under the variation $\boldsymbol{z}_{\varepsilon}$ originates from a surface $\mathcal{S}$ of zero Gaussian curvature. This needs to be included and expressed in a way that emphasizes the spatial variational nature of the admissibility conditions. It is well known that $K_{\varepsilon} = 0$ is sufficient for the surfaces $\mathcal{S}$ and $\mathcal{S}_{\varepsilon}$ to form an isometry relation, but this does not ensure that the deformation $\boldsymbol{z}_{\varepsilon}$ is an isometric deformation. Thus, by including $K_{\varepsilon} = 0$ in (2.22) we ensure that the information of zero Gaussian curvature is convected with the variation of $\mathcal{S}$.

where $\mathrm{d}\boldsymbol{y}_1$ and $\mathrm{d}\boldsymbol{y}_2$ are two material elements in $\mathcal{S}^{\tan}$. Since $(2.22)_2$ readily implies that $|(\mathrm{grad}_{\mathcal{S}}\boldsymbol{z}_\varepsilon)\mathrm{d}\boldsymbol{y}_1| = |\mathrm{d}\boldsymbol{y}_1|$, $|(\mathrm{grad}_{\mathcal{S}}\boldsymbol{z}_\varepsilon)\mathrm{d}\boldsymbol{y}_2| = |\mathrm{d}\boldsymbol{y}_2|$, and $(\mathrm{grad}_{\mathcal{S}}\boldsymbol{z}_\varepsilon)\mathrm{d}\boldsymbol{y}_1 \cdot (\mathrm{grad}_{\mathcal{S}}\boldsymbol{z}_\varepsilon)\mathrm{d}\boldsymbol{y}_2 = \mathrm{d}\boldsymbol{y}_1 \cdot \mathrm{d}\boldsymbol{y}_2$, we see however that $\mathrm{d}a_\varepsilon = \mathrm{d}a$. With the change of variables from $\boldsymbol{z}$ to $\boldsymbol{y}$ via $\boldsymbol{z}_\varepsilon$, we may thus rewrite (2.21) as

$$\mathcal{F}(\mathcal{S}_\varepsilon) := \int_{\mathcal{S}} W(\boldsymbol{L}_\varepsilon)\,\mathrm{d}a - \int_{\partial_2\mathcal{S}} (\boldsymbol{t}\cdot(\boldsymbol{z}_\varepsilon - \boldsymbol{y}) + \boldsymbol{c}\cdot(\boldsymbol{n}_\varepsilon - \boldsymbol{n}))\,\mathrm{d}s, \tag{2.24}$$

where in the integral over $\mathcal{S}$ it is implicit that

$$\boldsymbol{L}_\varepsilon = -(\mathrm{grad}_{\mathcal{S}_\varepsilon}\boldsymbol{n}_\varepsilon(\boldsymbol{z}))|_{\boldsymbol{z}=\boldsymbol{z}_\varepsilon}. \tag{2.25}$$

3. First variation considerations

Our immediate aim is to investigate the first variation condition

$$\delta\mathcal{F}(\mathcal{S})[\boldsymbol{u}] := \dot{\overline{\mathcal{F}(\mathcal{S}_\varepsilon)}}|_{\varepsilon=0} = 0, \tag{3.1}$$

with $\mathcal{F}$ given by (2.24) subject to the admissibility conditions (2.22) and the kinematical boundary conditions $\boldsymbol{u} = \boldsymbol{0}$ and $\dot{\boldsymbol{n}}_\varepsilon|_{\varepsilon=0} = \boldsymbol{0}$ on $\partial_1\mathcal{S}$. Since,

$$\dot{\overline{\mathcal{F}(\mathcal{S}_\varepsilon)}}|_{\varepsilon=0} = \int_{\mathcal{S}} W'(\boldsymbol{L})\cdot\dot{\boldsymbol{L}}_\varepsilon|_{\varepsilon=0}\,\mathrm{d}a - \int_{\partial_2\mathcal{S}} (\boldsymbol{t}\cdot\boldsymbol{u} + \boldsymbol{c}\cdot\dot{\boldsymbol{n}}_\varepsilon|_{\varepsilon=0})\,\mathrm{d}s, \tag{3.2}$$

this aim hinges on understanding how the first variations $\dot{\boldsymbol{n}}_\varepsilon|_{\varepsilon=0}$ and $\dot{\boldsymbol{L}}_\varepsilon|_{\varepsilon=0}$ of $\boldsymbol{n}$ and $\boldsymbol{L}$ depend on the variation $\dot{\boldsymbol{z}}_\varepsilon|_{\varepsilon=0} = \boldsymbol{u}$ of $\boldsymbol{z}$. Representations for those variations in terms of $\boldsymbol{n}$, $\boldsymbol{L}$, and $\boldsymbol{u}$ are established in two lemmas that we next state and prove.

Lemma 1. (Variation of $\boldsymbol{n}$.) *The first variation $\dot{\boldsymbol{n}}_\varepsilon|_{\varepsilon=0}$ of the unit normal field $\boldsymbol{n}$ to $\mathcal{S}$ is given by*

$$\dot{\boldsymbol{n}}_\varepsilon|_{\varepsilon=0} = -(\mathrm{grad}_{\mathcal{S}}\boldsymbol{u})^{\top}\boldsymbol{n} = -\boldsymbol{L}\boldsymbol{u}^{\tan} - \mathrm{grad}_{\mathcal{S}}U, \tag{3.3}$$

where we have used the decomposition

$$\boldsymbol{u} = U\boldsymbol{n} + \boldsymbol{u}^{\tan}, \qquad U = \boldsymbol{u}\cdot\boldsymbol{n}, \qquad \boldsymbol{u}^{\tan} = \boldsymbol{1}_{\mathcal{S}}\boldsymbol{u}. \tag{3.4}$$

Proof of Lemma 1. Since $\mathrm{grad}_{\mathcal{S}}\boldsymbol{z}_\varepsilon\,d\boldsymbol{y}_\alpha$ belongs to $\mathcal{S}_\varepsilon^{\tan}$ for any two material fibers $\mathrm{d}\boldsymbol{y}_\alpha$, $\alpha = 1, 2$, of $\mathcal{S}^{\tan}$, we see that $\boldsymbol{n}_\varepsilon\cdot\mathrm{grad}_{\mathcal{S}}\boldsymbol{z}_\varepsilon\,\mathrm{d}\boldsymbol{y}_\alpha = 0$. Differentiating this last equation with respect to ε and evaluating the resulting expression at $\varepsilon = 0$, we thus find that

$$(\dot{\boldsymbol{n}}_\varepsilon|_{\varepsilon=0} + (\mathrm{grad}_{\mathcal{S}}\boldsymbol{u})^{\top}\boldsymbol{n})\cdot d\boldsymbol{y}_\alpha = 0, \qquad \alpha = 1, 2. \tag{3.5}$$

But, also, because $\boldsymbol{n}_\varepsilon$ is unit vector valued, we know that $\boldsymbol{n}_\varepsilon\cdot\dot{\boldsymbol{n}}_\varepsilon = 0$. Then, evaluating at $\varepsilon = 0$ and noting that $\boldsymbol{n}\cdot(\mathrm{grad}_{\mathcal{S}}\boldsymbol{u})^{\top}\boldsymbol{n} = \boldsymbol{n}\cdot(\mathrm{grad}_{\mathcal{S}}\boldsymbol{u})\boldsymbol{n} = 0$, we see that

$$(\dot{\boldsymbol{n}}_\varepsilon|_{\varepsilon=0} + (\mathrm{grad}_{\mathcal{S}}\boldsymbol{u})^{\top}\boldsymbol{n})\cdot\boldsymbol{n} = 0. \tag{3.6}$$

Because $\{\mathrm{d}\boldsymbol{y}_1, \mathrm{d}\boldsymbol{y}_2, \boldsymbol{n}\}$ is a basis for $\mathbb{V}^3$, we obtain the first equality in (3.3). The second equality in (3.3) then follows from the symmetry of $\boldsymbol{L}$. $\square$

Remark 3.1. A useful consequence of Lemma 1 is that the kinematical boundary conditions $\boldsymbol{u} = \boldsymbol{0}$ and $\dot{\boldsymbol{n}}_\varepsilon|_{\varepsilon=0} = \boldsymbol{0}$ on $\partial_1\mathcal{S}$ are equivalent to the conditions $\boldsymbol{u} = \boldsymbol{0}$ and $\boldsymbol{\nu}\cdot\mathrm{grad}_{\mathcal{S}}U = 0$ on $\partial_1\mathcal{S}$. To see this, we first note from (3.3) that $\mathrm{grad}_{\mathcal{S}}U = \boldsymbol{0}$ if $\boldsymbol{u} = \boldsymbol{0}$ and $\dot{\boldsymbol{n}}_\varepsilon|_{\varepsilon=0} = \boldsymbol{0}$. Since $\boldsymbol{n}\cdot\mathrm{grad}_{\mathcal{S}}\varphi = 0$ for any smooth scalar field φ on $\mathcal{S}$, we must however have $\boldsymbol{n}\cdot\mathrm{grad}_{\mathcal{S}}U = 0$ on $\partial_1\mathcal{S}$. Introducing the positively oriented unit tangent $\boldsymbol{\sigma} := \boldsymbol{n}\times\boldsymbol{\nu}$ to $\mathcal{S}$ on $\partial_1\mathcal{S}$ and noticing that if $\boldsymbol{u} = \boldsymbol{0}$ on $\partial_1\mathcal{S}$ then $\boldsymbol{\sigma}\cdot\mathrm{grad}_{\mathcal{S}}U = 0$ on $\partial_1\mathcal{S}$, we thus confirm that the condition $\boldsymbol{\nu}\cdot\mathrm{grad}_{\mathcal{S}}U = 0$ must hold on $\partial_1\mathcal{S}$. $\square$

Lemma 2. (Variation of $\boldsymbol{L}$.) *The first variation $\dot{\boldsymbol{L}}_\varepsilon|_{\varepsilon=0}$ of the curvature tensor field $\boldsymbol{L}$ for $\mathcal{S}$ is given by*

$$\dot{\boldsymbol{L}}_\varepsilon|_{\varepsilon=0} = \boldsymbol{L}(\mathrm{grad}_{\mathcal{S}}\boldsymbol{u})^{\top}\boldsymbol{n}\otimes\boldsymbol{n} + \mathrm{grad}_{\mathcal{S}}((\mathrm{grad}_{\mathcal{S}}\boldsymbol{u})^{\top}\boldsymbol{n}) - \boldsymbol{L}\,\mathrm{grad}_{\mathcal{S}}\boldsymbol{u}. \tag{3.7}$$

Proof of Lemma 2. We first note that

$$\mathrm{grad}_{\mathcal{S}}\boldsymbol{n}_\varepsilon = (\mathrm{grad}_{\mathcal{S}_\varepsilon}\boldsymbol{n}_\varepsilon)\mathrm{grad}_{\mathcal{S}}\boldsymbol{z}_\varepsilon = -\boldsymbol{L}_\varepsilon\,\mathrm{grad}_{\mathcal{S}}\boldsymbol{z}_\varepsilon. \tag{3.8}$$

Thus,

$$\dot{\overline{\mathrm{grad}_{\mathcal{S}}\boldsymbol{n}_\varepsilon}} = \mathrm{grad}_{\mathcal{S}}\dot{\boldsymbol{n}}_\varepsilon = -\dot{\boldsymbol{L}}_\varepsilon\,\mathrm{grad}_{\mathcal{S}}\boldsymbol{z}_\varepsilon - \boldsymbol{L}_\varepsilon\,\mathrm{grad}_{\mathcal{S}}\dot{\boldsymbol{z}}_\varepsilon, \tag{3.9}$$

and, with the conditions (2.19), we see that

$$\mathrm{grad}_{\mathcal{S}}\dot{\boldsymbol{n}}_\varepsilon|_{\varepsilon=0} = -\dot{\boldsymbol{L}}_\varepsilon|_{\varepsilon=0}\boldsymbol{1}_{\mathcal{S}} - \boldsymbol{L}\,\mathrm{grad}_{\mathcal{S}}\boldsymbol{u}. \tag{3.10}$$

Now, recalling the condition $(2.22)_1$, we see further that

$$\dot{\overline{\boldsymbol{L}_\varepsilon\boldsymbol{n}_\varepsilon}}|_{\varepsilon=0} = \dot{\boldsymbol{L}}_\varepsilon|_{\varepsilon=0}\boldsymbol{n} + \boldsymbol{L}\dot{\boldsymbol{n}}_\varepsilon|_{\varepsilon=0} = \boldsymbol{0}, \tag{3.11}$$

which implies that

$$\dot{\boldsymbol{L}}_\varepsilon|_{\varepsilon=0} = \dot{\boldsymbol{L}}_\varepsilon|_{\varepsilon=0}\boldsymbol{1}_{\mathcal{S}} = \dot{\boldsymbol{L}}_\varepsilon|_{\varepsilon=0} + \boldsymbol{L}\dot{\boldsymbol{n}}_\varepsilon|_{\varepsilon=0} \otimes \boldsymbol{n}. \tag{3.12}$$

Thus,

$$\operatorname{grad}_{\mathcal{S}}\dot{\boldsymbol{n}}_\varepsilon|_{\varepsilon=0} = -\dot{\boldsymbol{L}}_\varepsilon|_{\varepsilon=0} - \boldsymbol{L}\dot{\boldsymbol{n}}_\varepsilon|_{\varepsilon=0} \otimes \boldsymbol{n} - \boldsymbol{L}\operatorname{grad}_{\mathcal{S}}\boldsymbol{u}, \tag{3.13}$$

or

$$\dot{\boldsymbol{L}}_\varepsilon|_{\varepsilon=0} = -\operatorname{grad}_{\mathcal{S}}\dot{\boldsymbol{n}}_\varepsilon|_{\varepsilon=0} - \boldsymbol{L}\dot{\boldsymbol{n}}_\varepsilon|_{\varepsilon=0} \otimes \boldsymbol{n} - \boldsymbol{L}\operatorname{grad}_{\mathcal{S}}\boldsymbol{u}. \tag{3.14}$$

Augmenting (3.14) with the expression (3.3) for the variation $\dot{\boldsymbol{n}}_\varepsilon|_{\varepsilon=0}$ of $\boldsymbol{n}$, we finally arrive at (3.7). □

4. Special results for rigid variations of $\mathcal{S}$

Consider the special subclass of the general class of variations introduced in (2.19) that transform $\mathcal{S}$ rigidly to $\mathcal{S}_\varepsilon$. These are admissible variations for the first variation condition (3.1), for the pure traction problem wherein the traction $\boldsymbol{t}$ and the couple traction $\boldsymbol{c}$ are prescribed on the complete boundary $\partial\mathcal{S}$, so that $\partial_1\mathcal{S} = \varnothing$ and $\partial_2\mathcal{S} = \partial\mathcal{S}$. Any variation belonging to this class admits a representation of the form

$$\boldsymbol{z}_\varepsilon(\boldsymbol{y}) = \boldsymbol{Q}_\varepsilon(\boldsymbol{y} - \boldsymbol{o}) + \boldsymbol{c}_\varepsilon, \tag{4.1}$$

where $\boldsymbol{o}$ is a fixed point in $\mathbb{E}^3$, $\boldsymbol{Q}_\varepsilon$ satisfying $\boldsymbol{Q}_0 = \boldsymbol{1}$ belongs to the collection Orth^+ of proper orthogonal linear transformations on $\mathbb{V}^3$, and $\boldsymbol{c}_\varepsilon$ satisfying $\boldsymbol{c}_0 = \boldsymbol{o}$ is in $\mathbb{E}^3$. In this case, we readily find, according to (2.19), that

$$\dot{\boldsymbol{z}}_\varepsilon(\boldsymbol{y})|_{\varepsilon=0} =: \boldsymbol{u}(\boldsymbol{y}) = \dot{\boldsymbol{Q}}_0(\boldsymbol{y} - \boldsymbol{o}) + \dot{\boldsymbol{c}}_0 = \dot{\boldsymbol{q}}_0 \times (\boldsymbol{y} - \boldsymbol{o}) + \dot{\boldsymbol{c}}_0, \tag{4.2}$$

where $\dot{\boldsymbol{Q}}_0$ belongs to the collection Skew of skew linear transformations on $\mathbb{V}^3$, $\dot{\boldsymbol{q}}_0$ belonging to $\mathbb{V}^3$ is the axial vector of $\dot{\boldsymbol{Q}}_0$, and $\dot{\boldsymbol{c}}_0$ belongs to $\mathbb{V}^3$. Consequently, it follows that $\operatorname{grad}_{\mathcal{S}}\boldsymbol{u} = \dot{\boldsymbol{Q}}_0\boldsymbol{1}_{\mathcal{S}}$ and from (3.3) we see that

$$\dot{\boldsymbol{n}}_\varepsilon|_{\varepsilon=0} = -\boldsymbol{1}_{\mathcal{S}}\dot{\boldsymbol{Q}}_0^{\top}\boldsymbol{n} = \boldsymbol{1}_{\mathcal{S}}\dot{\boldsymbol{Q}}_0\boldsymbol{n} = \boldsymbol{1}_{\mathcal{S}}(\dot{\boldsymbol{q}}_0 \times \boldsymbol{n}). \tag{4.3}$$

In the first variation condition (3.1) for the pure traction problem we may thus make the replacements

$$\boldsymbol{c} \cdot \dot{\boldsymbol{n}}_\varepsilon|_{\varepsilon=0} = \boldsymbol{c} \cdot (\dot{\boldsymbol{q}}_0 \times \boldsymbol{n}) = \dot{\boldsymbol{q}}_0 \cdot (\boldsymbol{n} \times \boldsymbol{c}) = \dot{\boldsymbol{q}}_0 \cdot \boldsymbol{m} \tag{4.4a}$$

and

$$\boldsymbol{t} \cdot \boldsymbol{u} = \boldsymbol{t} \cdot \dot{\boldsymbol{c}}_0 + ((\boldsymbol{y} - \boldsymbol{o}) \times \boldsymbol{t}) \cdot \dot{\boldsymbol{q}}_0. \tag{4.4b}$$

Since $\dot{\boldsymbol{Q}}_0$ belongs to Skew, we see in addition from (3.7) that, for $\mathcal{S}_\varepsilon$ a rigid transformation of $\mathcal{S}$,

$$\dot{\boldsymbol{L}}_\varepsilon|_{\varepsilon=0} = \boldsymbol{L}\dot{\boldsymbol{Q}}_0^{\top}\boldsymbol{n} \otimes \boldsymbol{n} - \boldsymbol{L}\dot{\boldsymbol{Q}}_0\boldsymbol{1}_{\mathcal{S}} - \boldsymbol{1}_{\mathcal{S}}\dot{\boldsymbol{Q}}_0\boldsymbol{L} = \boldsymbol{L}\dot{\boldsymbol{Q}}_0^{\top} + \boldsymbol{1}_{\mathcal{S}}\dot{\boldsymbol{Q}}_0^{\top}\boldsymbol{L}. \tag{4.5}$$

Recalling once again that $W'(\boldsymbol{L})$ is a symmetric linear transformation of $\mathcal{S}^{\tan}$ to itself for each $\boldsymbol{L}$ in Sym_0, we thus infer that

$$\begin{aligned}
\overline{W(\boldsymbol{L}_\varepsilon)}^{\boldsymbol{\cdot}}|_{\varepsilon=0} &= (W'(\boldsymbol{L}_\varepsilon) \cdot \dot{\boldsymbol{L}}_\varepsilon)|_{\varepsilon=0} \\
&= W'(\boldsymbol{L}) \cdot \dot{\boldsymbol{L}}_\varepsilon|_{\varepsilon=0} \\
&= \operatorname{tr}(W'(\boldsymbol{L})(\boldsymbol{L}\dot{\boldsymbol{Q}}_0^{\top} + \boldsymbol{1}_{\mathcal{S}}\dot{\boldsymbol{Q}}_0^{\top}\boldsymbol{L})) \\
&= \operatorname{tr}(W'(\boldsymbol{L})\boldsymbol{L}\dot{\boldsymbol{Q}}_0^{\top} + W'(\boldsymbol{L})\dot{\boldsymbol{Q}}_0^{\top}\boldsymbol{L}) \\
&= \operatorname{tr}((W'(\boldsymbol{L})\boldsymbol{L} + \boldsymbol{L}W'(\boldsymbol{L}))\dot{\boldsymbol{Q}}_0^{\top}) \\
&= 0,
\end{aligned} \tag{4.6}$$

the last equality being a consequence of the symmetry of the sum $W'(\boldsymbol{L})\boldsymbol{L} + \boldsymbol{L}W'(\boldsymbol{L})$. The identity (4.6), of course, expresses a particular consequence of the invariance, ensured by (2.17), of the stored energy density W under a rigid transformation of $\mathcal{S}$.

Remark 4.1. From the above discussion, we see that for the pure traction variational problem, for which $\partial_2\mathcal{S} = \partial\mathcal{S}$ with $\boldsymbol{t}$ and $\boldsymbol{c}$ (or, equivalently, $\boldsymbol{m}$) prescribed on the complete boundary $\partial\mathcal{S}$, the condition

$$\int_{\partial\mathcal{S}} (\boldsymbol{t} \cdot \dot{\boldsymbol{c}}_0 + ((\boldsymbol{y} - \boldsymbol{o}) \times \boldsymbol{t} + \boldsymbol{m}) \cdot \dot{\boldsymbol{q}}_0)\,\mathrm{d}s = 0 \tag{4.7}$$

must be satisfied for all choices of $\dot{\boldsymbol{c}}_0$ and $\dot{\boldsymbol{q}}_0$ in $\mathbb{V}^3$, which implies that the applied traction and applied couple traction must be force and moment balanced, namely that the conditions

$$\int_{\partial\mathcal{S}} \boldsymbol{t}\,\mathrm{d}s = \boldsymbol{0} \qquad \text{and} \qquad \int_{\partial\mathcal{S}} ((\boldsymbol{y} - \boldsymbol{o}) \times \boldsymbol{t} + \boldsymbol{m})\,\mathrm{d}s = \boldsymbol{0} \tag{4.8}$$

must hold. In Section 8, we assume that the stored energy depends quadratically on the mean curvature and deduce the implications of the first variation condition (3.1) for two particular cases of the pure traction version of our variational problem. □

5. Variational implications of the first order isometric admissibility condition

Now, let us turn to the first order isometric admissibility condition (2.22)$_2$. Calculating the first variation of that condition, we find that

$$\overline{((\mathrm{grad}_{\mathcal{S}}\boldsymbol{z}_\varepsilon)^\top \mathrm{grad}_{\mathcal{S}}\,\boldsymbol{z}_\varepsilon - \boldsymbol{1}_{\mathcal{S}})}^{\boldsymbol{\cdot}}|_{\varepsilon=0} = (\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{u})^\top \boldsymbol{1}_{\mathcal{S}} + \boldsymbol{1}_{\mathcal{S}}\,\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{u} = \boldsymbol{0} \tag{5.1}$$

and thus that $\boldsymbol{u}$ must obey[4]

$$\mathrm{div}_{\mathcal{S}}\,\boldsymbol{u} = 0, \qquad \boldsymbol{L}\cdot \mathrm{grad}_{\mathcal{S}}\,\boldsymbol{u} = 0, \qquad \text{and} \qquad \boldsymbol{L}^2\cdot \mathrm{grad}_{\mathcal{S}}\,\boldsymbol{u} = 0. \tag{5.2}$$

Moreover, using the decomposition (3.4) of $\boldsymbol{u}$ and noting that

$$\boldsymbol{1}_{\mathcal{S}}(\boldsymbol{n}\otimes \mathrm{grad}_{\mathcal{S}}\,U) = \boldsymbol{0}, \qquad (\boldsymbol{n}\otimes \mathrm{grad}_{\mathcal{S}}\,U)^\top \boldsymbol{1}_{\mathcal{S}} = \boldsymbol{0}, \tag{5.3}$$

and that,

$$\mathrm{grad}_{\mathcal{S}}\,(U\boldsymbol{n}) = \mathrm{grad}_{\mathcal{S}}\,(\boldsymbol{u} - \boldsymbol{u}^{\tan}) = \boldsymbol{n}\otimes \mathrm{grad}_{\mathcal{S}}\,U - U\boldsymbol{L}, \tag{5.4}$$

we may rewrite (5.1) in the equivalent form[5]

$$\overline{((\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{z}_\varepsilon)^\top \mathrm{grad}_{\mathcal{S}}\,\boldsymbol{z}_\varepsilon - \boldsymbol{1}_{\mathcal{S}})}^{\boldsymbol{\cdot}}|_{\varepsilon=0} = \boldsymbol{1}_{\mathcal{S}}\,\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{u}^{\tan} + (\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{u}^{\tan})^\top \boldsymbol{1}_{\mathcal{S}} - 2U\boldsymbol{L} = \boldsymbol{0}, \tag{5.5}$$

which, in fact, implies the conditions (5.2)$_{1,\,2}$ and, thus, of course, (5.2)$_3$, that $\boldsymbol{u}$ must satisfy. Additionally, taking the trace of (5.5) we find that[6]

$$\mathrm{div}_{\mathcal{S}}\,\boldsymbol{u}^{\tan} = 2HU. \tag{5.6}$$

We therefore infer that (5.5) represents the general first variation condition arising from the isometric admissibility condition (2.22)$_2$.

From (5.5) and Footnote 5, we see that for a single-valued tangential vector field $\boldsymbol{u}^{\tan}$ on $\mathcal{S}$ to exist, the quantity $U\boldsymbol{L}$ in (5.5) must satisfy the classical compatibility condition[7]

$$\mathrm{curl}_{\mathcal{S}}\mathrm{curl}_{\mathcal{S}}(U\boldsymbol{L}) = \boldsymbol{0}. \tag{5.7}$$

Using (2.8)$_1$, the condition (5.7) works out to be

$$\boldsymbol{L}\cdot \mathrm{grad}_{\mathcal{S}}\,\mathrm{grad}_{\mathcal{S}}\,U + 2H\Delta_{\mathcal{S}}U - 2\,\mathrm{grad}_{\mathcal{S}}\,U\cdot \mathrm{grad}_{\mathcal{S}}\,H + \mathrm{grad}_{\mathcal{S}}\,U\cdot \mathrm{div}_{\mathcal{S}}\,\boldsymbol{L} = 0, \tag{5.8}$$

where $\Delta_{\mathcal{S}}$ denotes the Laplace operator on $\mathcal{S}$. Toward simplifying (5.8), we first observe that, by (2.8), (2.14), and (2.16), $\mathrm{div}_{\mathcal{S}}\,\boldsymbol{L}$ can be written as

$$\begin{aligned}\mathrm{div}_{\mathcal{S}}\,\boldsymbol{L} &= \boldsymbol{1}_{\mathcal{S}}\,\mathrm{div}_{\mathcal{S}}\,\boldsymbol{L} + (\boldsymbol{n}\cdot \mathrm{div}_{\mathcal{S}}\,\boldsymbol{L})\boldsymbol{n}\\ &= -\mathrm{grad}_{\mathcal{S}}\,\mathrm{div}_{\mathcal{S}}\,\boldsymbol{n} + (\mathrm{div}_{\mathcal{S}}\,(\boldsymbol{L}^\top\boldsymbol{n}) + |\boldsymbol{L}|^2)\boldsymbol{n}\\ &= 2\,\mathrm{grad}_{\mathcal{S}}\,H + (\mathrm{div}_{\mathcal{S}}\,(\boldsymbol{L}\boldsymbol{n}) + |\boldsymbol{L}|^2)\boldsymbol{n}\\ &= 2(\mathrm{grad}_{\mathcal{S}}\,H + 2H^2\boldsymbol{n}).\end{aligned} \tag{5.9}$$

Hence, since $\boldsymbol{n}\cdot \mathrm{grad}_{\mathcal{S}}\,\varphi = 0$ for any smooth scalar field φ on $\mathcal{S}$, we find that the last two terms on the left-hand side of the expanded version (5.8) of the compatibility condition (5.7) cancel. Invoking the identity

$$\Delta_{\mathcal{S}}U = \boldsymbol{1}_{\mathcal{S}}\cdot \mathrm{grad}_{\mathcal{S}}\mathrm{grad}_{\mathcal{S}}\,U, \tag{5.10}$$

we may thus equivalently express (5.8) in the alternative form

$$(2H\boldsymbol{1}_{\mathcal{S}} - \boldsymbol{L})\cdot \mathrm{grad}_{\mathcal{S}}\mathrm{grad}_{\mathcal{S}}\,U = 0. \tag{5.11}$$

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[4] By (2.16), (5.2)$_2$ implies (5.2)$_3$ and vice versa.

[5] If $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is a basis in $\mathcal{S}^{\tan}$ on $\mathcal{S}$, with corresponding dual basis $\{\boldsymbol{e}^1, \boldsymbol{e}^2\}$, we may use the represention $u_\alpha\boldsymbol{e}^\alpha$ of $\boldsymbol{u}^{\tan}$ to show that

$$\boldsymbol{1}_{\mathcal{S}}\,\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{u}^{\tan} = (\boldsymbol{1} - \boldsymbol{n}\otimes\boldsymbol{n})\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{u}^{\tan} = u_{\alpha;\beta}\boldsymbol{e}^\alpha\otimes\boldsymbol{e}^\beta,$$

where a semicolon is used to denote covariant differentiation and, thus, $u_{\alpha;\,\beta}$ denotes the covariant components of the projection of the surface gradient of $\boldsymbol{u}^{\tan}$ onto $\mathcal{S}^{\tan}$. The condition (5.5) may therefore be written as

$$\frac{1}{2}(u_{\alpha;\beta} + u_{\beta;\alpha})\boldsymbol{e}^\alpha\otimes\boldsymbol{e}^\beta = -U\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{n} = UL_{\alpha\beta}\boldsymbol{e}^\alpha\otimes\boldsymbol{e}^\beta,$$

from which we deduce the identity $u_{\alpha;\beta} + u_{\beta;\alpha} = 2UL_{\alpha\beta}$.

[6] The identity (5.6) also follows independently from (5.2)$_1$ and a decomposition of the variation $\boldsymbol{u}$ that we subsequently introduce in (7.3).

[7] Expressed in terms of components, (5.7) takes the form $\varepsilon^{\alpha\beta}\varepsilon^{\lambda\gamma}(UL_{\alpha\lambda})_{;\beta\gamma} = 0$, where $\varepsilon^{\alpha\beta}$ is the contravariant two-dimensional alternator symbol.

From the consequence

$$\mathrm{div}_{\mathcal{S}}\,(2H\mathbf{1}_{\mathcal{S}} - \mathbf{L}) = \mathbf{0}, \tag{5.12}$$

of (5.9) and the elementary relation $\mathrm{div}_{\mathcal{S}}\,\mathbf{1}_{\mathcal{S}} = 2H\mathbf{n}$, we find that

$$(2H\mathbf{1}_{\mathcal{S}} - \mathbf{L}) \cdot \mathrm{grad}_{\mathcal{S}}\mathrm{grad}_{\mathcal{S}}\,U = \mathrm{div}_{\mathcal{S}}\,((2H\mathbf{1}_{\mathcal{S}} - \mathbf{L})\mathrm{grad}_{\mathcal{S}}\,U) \tag{5.13}$$

and, thus, with reference to (5.11), we obtain another useful alternative

$$\mathrm{div}_{\mathcal{S}}\,((2H\mathbf{1}_{\mathcal{S}} - \mathbf{L})\mathrm{grad}_{\mathcal{S}}\,U) = 0 \tag{5.14}$$

to the compatibility condition (5.7).

Remark 5.1. If the compatibility condition holds and the surface $\mathcal{S}$ is simply connected, then there exists a single valued tangential vector field $\mathbf{u}^{\mathrm{tan}}$ that depends upon the choice of the normal component U of the first variation $\mathbf{u}$ of $\mathbf{z}_{\varepsilon}$ on $\mathcal{S}$. Moreover, $\mathbf{u}^{\mathrm{tan}}$ is unique up to an additive tangential vector field $\hat{\mathbf{u}}^{\mathrm{tan}}$ that satisfies the condition

$$\mathbf{1}_{\mathcal{S}}\,\mathrm{grad}_{\mathcal{S}}\,\hat{\mathbf{u}}^{\mathrm{tan}} + (\mathrm{grad}_{\mathcal{S}}\,\hat{\mathbf{u}}^{\mathrm{tan}})^{\top}\mathbf{1}_{\mathcal{S}} = \mathbf{0}. \quad \square \tag{5.15}$$

6. Variational implications of the compatibility condition: Second order isometric admissibility condition

Since the curvature tensor $\mathbf{L}_{\varepsilon}$ of $\mathcal{S}_{\varepsilon}$ is a symmetric linear transformation of $\mathbb{V}^3$ to itself that annihilates $\mathbf{n}_{\varepsilon}$, it also is a symmetric linear transformation of $\mathcal{S}_{\varepsilon}^{\mathrm{tan}}$ to itself and satisfies the Cayley–Hamilton equation

$$\mathbf{L}_{\varepsilon}^2 - 2H_{\varepsilon}\mathbf{L}_{\varepsilon} + K_{\varepsilon}\mathbf{1}_{\mathcal{S}_{\varepsilon}} = \mathbf{0}, \tag{6.1}$$

where $H_{\varepsilon} := \mathrm{tr}\,\mathbf{L}_{\varepsilon}/2$ and $K_{\varepsilon} := \det\mathbf{L}_{\varepsilon}$ denote the mean and Gaussian curvatures of $\mathcal{S}_{\varepsilon}$ and $\mathbf{1}_{\mathcal{S}_{\varepsilon}} := \mathbf{1} - \mathbf{n}_{\varepsilon}\otimes\mathbf{n}_{\varepsilon}$ denotes the projection of $\mathbb{V}^3$ to $\mathcal{S}_{\varepsilon}^{\mathrm{tan}}$. Taking the trace of (6.1) and calculating the first variation of the resulting expression, we thus obtain

$$\dot{K}_{\epsilon}|_{\varepsilon=0} = \overline{\det\mathbf{L}_{\varepsilon}}^{\,\boldsymbol{\cdot}}|_{\varepsilon=0} = (2H\mathbf{1}_{\mathcal{S}} - \mathbf{L}) \cdot \dot{\mathbf{L}}_{\varepsilon}|_{\varepsilon=0}. \tag{6.2}$$

We now show that when the compatibility condition (5.11) is satisfied then the right-hand side of (6.2) vanishes and, consequently, that the compatibility condition (5.11) yields a second order isometric admissibility condition

$$\dot{K}_{\varepsilon}|_{\varepsilon=0} = 0, \tag{6.3}$$

which, though arrived at independently, is to be expected because of the Theorema egregium of Gauss. We also establish the converse, namely that (6.3) implies (5.11). Hence, we conclude that the compatibility condition (5.11) and the secondary isometric admissibility condition (6.3) are equivalent.

To confirm the foregoing assertion, we first observe from (2.8), (3.7), and (5.9) that

$$\begin{aligned} 2H\mathbf{1}_{\mathcal{S}} \cdot \dot{\mathbf{L}}_{\varepsilon}|_{\varepsilon=0} &= 2H\mathbf{1}_{\mathcal{S}} \cdot (\mathrm{grad}_{\mathcal{S}}\,((\mathrm{grad}_{\mathcal{S}}\,\mathbf{u})^{\top}\mathbf{n}) - \mathbf{L}\,\mathrm{grad}_{\mathcal{S}}\,\mathbf{u}) \\ &= 2H(\mathbf{1}_{\mathcal{S}} \cdot \mathrm{grad}_{\mathcal{S}}\mathrm{grad}_{\mathcal{S}}\,U + \mathrm{div}_{\mathcal{S}}\,(\mathbf{L}\mathbf{u}) - \mathbf{L} \cdot \mathrm{grad}_{\mathcal{S}}\,\mathbf{u}) \\ &= 2H\mathbf{1}_{\mathcal{S}} \cdot \mathrm{grad}_{\mathcal{S}}\mathrm{grad}_{\mathcal{S}}\,U + 2H\mathbf{u} \cdot \mathrm{div}_{\mathcal{S}}\,\mathbf{L}. \end{aligned} \tag{6.4}$$

In a similar development, using (2.8), (2.16), and (5.9), we observe further that

$$\begin{aligned} \mathbf{L} \cdot \dot{\mathbf{L}}_{\varepsilon}|_{\varepsilon=0} &= \mathbf{L} \cdot \mathrm{grad}_{\mathcal{S}}((\mathrm{grad}_{\mathcal{S}}\,\mathbf{u})^{\top}\mathbf{n}) - \mathbf{L} \cdot \mathbf{L}\,\mathrm{grad}_{\mathcal{S}}\,\mathbf{u} \\ &= \mathbf{L} \cdot (\mathrm{grad}_{\mathcal{S}}\mathrm{grad}_{\mathcal{S}}\,U + \mathrm{grad}_{\mathcal{S}}\,(\mathbf{L}\mathbf{u})) - \mathbf{L}^2 \cdot \mathrm{grad}_{\mathcal{S}}\,\mathbf{u} \\ &= \mathbf{L} \cdot \mathrm{grad}_{\mathcal{S}}\mathrm{grad}_{\mathcal{S}}\,U + \mathrm{div}_{\mathcal{S}}\,(\mathbf{L}^2\mathbf{u}) - \mathbf{L}\mathbf{u} \cdot \mathrm{div}_{\mathcal{S}}\,\mathbf{L} - 2H\mathbf{L} \cdot \mathrm{grad}_{\mathcal{S}}\,\mathbf{u} \\ &= \mathbf{L} \cdot \mathrm{grad}_{\mathcal{S}}\mathrm{grad}_{\mathcal{S}}\,U + 2(\mathrm{div}_{\mathcal{S}}\,(H\mathbf{L}\mathbf{u}) - \mathbf{L}\mathbf{u} \cdot \mathrm{grad}_{\mathcal{S}}H - H\mathbf{L} \cdot \mathrm{grad}_{\mathcal{S}}\,\mathbf{u}) \\ &= \mathbf{L} \cdot \mathrm{grad}_{\mathcal{S}}\mathrm{grad}_{\mathcal{S}}\,U + 2H(\mathrm{div}_{\mathcal{S}}\,(\mathbf{L}\mathbf{u}) - \mathbf{L} \cdot \mathrm{grad}_{\mathcal{S}}\,\mathbf{u}) \\ &= \mathbf{L} \cdot \mathrm{grad}_{\mathcal{S}}\mathrm{grad}_{\mathcal{S}}\,U + 2H\mathbf{u} \cdot \mathrm{div}_{\mathcal{S}}\,\mathbf{L}. \end{aligned} \tag{6.5}$$

Finally, using (6.4) and (6.5) in (6.2), we deduce the relation

$$\dot{K}_{\varepsilon}|_{\varepsilon=0} = (2H\mathbf{1}_{\mathcal{S}} - \mathbf{L}) \cdot \mathrm{grad}_{\mathcal{S}}\mathrm{grad}_{\mathcal{S}}\,U \tag{6.6}$$

and thereby conclude, as claimed, that the compatibility condition (5.11) holds if and only if (6.3) holds.

7. General variational problem

We now use the representations (3.3) and (3.7) for $\dot{\boldsymbol{n}}_\varepsilon|_{\varepsilon=0}$ and $\dot{\boldsymbol{L}}_\varepsilon|_{\varepsilon=0}$ in (3.2) to derive a more explicit version of the first variation condition (3.1). In this regard, it is first convenient to consider separately each contribution to the integrand $W'(\boldsymbol{L})\cdot\dot{\boldsymbol{L}}_\varepsilon|_{\varepsilon=0}$ of the first term on the right-hand side of (3.2). The relevant calculations rely repeatedly on the status of $W'(\boldsymbol{L})$ as a symmetric linear transformation of $\mathcal{S}^{\tan}$ to itself for each $\boldsymbol{L}$ in Sym_0. First, we find that

$$W'(\boldsymbol{L})\cdot(\boldsymbol{L}(\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{u})^{\top}\boldsymbol{n}\otimes\boldsymbol{n}) = W'(\boldsymbol{L})\boldsymbol{n}\cdot(\boldsymbol{L}(\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{u})^{\top}\boldsymbol{n}) = 0. \tag{7.1}$$

Taking note of the identities

$$(\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{u})^{\top}\boldsymbol{n} = \mathrm{grad}_{\mathcal{S}}\,U + \boldsymbol{L}\boldsymbol{u}^{\tan} \tag{7.2}$$

and

$$\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{u} = \boldsymbol{n}\otimes\mathrm{grad}_{\mathcal{S}}\,U - U\boldsymbol{L} + \mathrm{grad}_{\mathcal{S}}\,\boldsymbol{u}^{\tan}, \tag{7.3}$$

both of which stem from (3.4), we next find that

$$W'(\boldsymbol{L})\cdot(\mathrm{grad}_{\mathcal{S}}((\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{u})^{\top}\boldsymbol{n})) = W'(\boldsymbol{L})\cdot(\mathrm{grad}_{\mathcal{S}}\,\mathrm{grad}_{\mathcal{S}}\,U + \mathrm{grad}_{\mathcal{S}}(\boldsymbol{L}\boldsymbol{u}^{\tan})) \tag{7.4}$$

and, referring to (2.16), that

$$-W'(\boldsymbol{L})\cdot(\boldsymbol{L}\,\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{u}) = W'(\boldsymbol{L})\cdot(2HU\boldsymbol{L} - \boldsymbol{L}\,\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{u}^{\tan}). \tag{7.5}$$

Moreover, since

$$W'(\boldsymbol{L})\cdot(\mathrm{grad}_{\mathcal{S}}\,\mathrm{grad}_{\mathcal{S}}\,U) = U\mathrm{div}_{\mathcal{S}}\,\mathrm{div}_{\mathcal{S}}\,W'(\boldsymbol{L}) - \mathrm{div}_{\mathcal{S}}\,(U\mathrm{div}_{\mathcal{S}}\,W'(\boldsymbol{L}) - W'(\boldsymbol{L})\mathrm{grad}_{\mathcal{S}}\,U), \tag{7.6}$$

and, with reference to (5.6),

$$W'(\boldsymbol{L})\cdot(\mathrm{grad}_{\mathcal{S}}(\boldsymbol{L}\boldsymbol{u}^{\tan}) - \boldsymbol{L}\,\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{u}^{\tan}) = -2HW(\boldsymbol{L})U + \mathrm{div}_{\mathcal{S}}\,(W(\boldsymbol{L})\boldsymbol{u}^{\tan}), \tag{7.7}$$

we see from (7.1), (7.4), and (7.5) that

$$W'(\boldsymbol{L})\cdot\dot{\boldsymbol{L}}_\varepsilon|_{\varepsilon=0} = (\mathrm{div}_{\mathcal{S}}\,\mathrm{div}_{\mathcal{S}}\,W'(\boldsymbol{L}) - 2(W(\boldsymbol{L}) - W'(\boldsymbol{L})\cdot\boldsymbol{L})H)U - \mathrm{div}_{\mathcal{S}}\,(U\mathrm{div}_{\mathcal{S}}\,W'(\boldsymbol{L}) - W'(\boldsymbol{L})\mathrm{grad}_{\mathcal{S}}\,U - W(\boldsymbol{L})\boldsymbol{u}^{\tan}). \tag{7.8}$$

Using (3.3) and (7.8) in (3.2) and invoking the surface divergence theorem,[8] we find that the first variation condition (3.1) can be expressed as

$$\begin{aligned}&\int_{\mathcal{S}}(\mathrm{div}_{\mathcal{S}}\,\mathrm{div}_{\mathcal{S}}\,W'(\boldsymbol{L}) - 2(W(\boldsymbol{L}) - 2W'(\boldsymbol{L})\cdot\boldsymbol{L})H)U\,\mathrm{d}a\\ &\quad= \int_{\partial\mathcal{S}}(U\mathrm{div}_{\mathcal{S}}\,W'(\boldsymbol{L}) - W'(\boldsymbol{L})\mathrm{grad}_{\mathcal{S}}\,U - W(\boldsymbol{L})\boldsymbol{u}^{\tan})\cdot\boldsymbol{\nu}\,\mathrm{d}s + \int_{\partial_2\mathcal{S}}(\boldsymbol{t}\cdot\boldsymbol{u} - \boldsymbol{c}\cdot(\mathrm{grad}_{\mathcal{S}}\,U + \boldsymbol{L}\boldsymbol{u}^{\tan}))\,\mathrm{d}s,\end{aligned} \tag{7.9}$$

where we recall that $\boldsymbol{\nu}$ denotes the tangent normal to $\mathcal{S}$ and that the subset $\partial_2\mathcal{S}$ of $\partial\mathcal{S}$ and the traction $\boldsymbol{t}$ and couple traction $\boldsymbol{c}$ remain to be prescribed. The general first variation condition (7.9) must hold for all admissible variations $\boldsymbol{u} = \boldsymbol{u}^{\tan} + U\boldsymbol{n}$ that satisfy (5.5) and the compatibility condition (5.11) in conjunction with the homogeneous boundary conditions on $\partial_1\mathcal{S}$, as noted in Remark 3.1.

8. Variational problem for a material surface with quadratic stored energy density: Results for the pure traction version of the variational problem

To be definite, let us assume that the stored energy density W has the quadratic form

$$W(\boldsymbol{L}) = \frac{1}{2}\mu(\mathrm{tr}\,\boldsymbol{L})^2, \tag{8.1}$$

where $\mu > 0$, the bending stiffness, carries the dimensions of force × length and is a material constant. It is readily confirmed that the particular choice (8.1) of W is frame indifferent in the sense of (2.17).

[8] In going from (7.8) to (7.9), it is essential to be cognizant that, since $W'(\boldsymbol{L})$ is a symmetric linear transformation of $\mathcal{S}^{\tan}$ to itself for each $\boldsymbol{L}$ in Sym_0, it follows that

$$(U\mathrm{div}_{\mathcal{S}}\,W'(\boldsymbol{L}) - W'(\boldsymbol{L})\mathrm{grad}_{\mathcal{S}}\,U - W(\boldsymbol{L})\boldsymbol{u}^{\tan})\cdot\boldsymbol{n} = UW'(\boldsymbol{L})\cdot\boldsymbol{L}$$

and, thus, that

$$\int_{\mathcal{S}}\mathrm{div}_{\mathcal{S}}\,(U\mathrm{div}_{\mathcal{S}}\,W'(\boldsymbol{L}) - W'(\boldsymbol{L})\mathrm{grad}_{\mathcal{S}}\,U - W(\boldsymbol{L})\boldsymbol{u}^{\tan})\,\mathrm{d}a = \int_{\partial\mathcal{S}}(U\mathrm{div}_{\mathcal{S}}\,W'(\boldsymbol{L}) - W'(\boldsymbol{L})\mathrm{grad}_{\mathcal{S}}\,U - W(\boldsymbol{L})\boldsymbol{u}^{\tan})\cdot\boldsymbol{\nu}\,\mathrm{d}s - \int_{\mathcal{S}}2H(W'(\boldsymbol{L})\cdot\boldsymbol{L})U\,\mathrm{d}a.$$

Remark 8.1. The tradition of using (8.1) as a model for the stored energy density of an unstretchable material surface dates at least as far back to Sadowsky's (1929, 1930a, 1930b) works on the equilibrium shape of a Möbius band.[9] See also the related works of Wunderlich (1962),[10] Mahadevan and Keller (1993), Starostin and van der Heijden (2007, 2015), Kirby and Fried (2015), and Shen et al. (2015). □

For the particular choice (8.1) of W, we see, from (5.9), that

$$\operatorname{div}_{\mathcal{S}} \operatorname{div}_{\mathcal{S}} W'(\boldsymbol{L}) - 2(W(\boldsymbol{L}) - 2W'(\boldsymbol{L}) \cdot \boldsymbol{L})H = 2\mu(\Delta_{\mathcal{S}} H + 2H^3) \tag{8.2}$$

and that

$$U\operatorname{div}_{\mathcal{S}} W'(\boldsymbol{L}) - W'(\boldsymbol{L})\operatorname{grad}_{\mathcal{S}} U - W(\boldsymbol{L})\boldsymbol{u}^{\tan} = 2\mu(U\operatorname{grad}_{\mathcal{S}} H - H\operatorname{grad}_{\mathcal{S}} U - H^2\boldsymbol{u}^{\tan}). \tag{8.3}$$

Hence, the general first variation condition (7.9) specializes to

$$\int_{\mathcal{S}} 2\mu(\Delta_{\mathcal{S}} H + 2H^3)U \,\mathrm{d}a = \int_{\partial\mathcal{S}} 2\mu(U\operatorname{grad}_{\mathcal{S}} H - H\operatorname{grad}_{\mathcal{S}} U - H^2\boldsymbol{u}^{\tan}) \cdot \boldsymbol{\nu} \,\mathrm{d}s + \int_{\partial_2\mathcal{S}} (\boldsymbol{t} \cdot \boldsymbol{u} - \boldsymbol{c} \cdot (\operatorname{grad}_{\mathcal{S}} U + \boldsymbol{L}\boldsymbol{u}^{\tan}))\,\mathrm{d}s. \tag{8.4}$$

Toward obtaining some elementary consequences of the first variation condition, we integrate the compatibility condition (5.14) over $\mathcal{S}$ and use the surface divergence theorem and the symmetry condition $(2.8)_1$ to yield the identity

$$\int_{\partial\mathcal{S}} 2H\boldsymbol{\nu} \cdot \operatorname{grad}_{\mathcal{S}} U \,\mathrm{d}s = \int_{\partial\mathcal{S}} \boldsymbol{\nu} \cdot \boldsymbol{L}\operatorname{grad}_{\mathcal{S}} U \,\mathrm{d}s. \tag{8.5}$$

Using (8.5) in (8.4), we thus see that the first variation condition (8.4) for a material surface with quadratic stored energy density (8.1) can be written as

$$\int_{\mathcal{S}} 2\mu(\Delta_{\mathcal{S}} H + 2H^3)U \,\mathrm{d}a = \int_{\partial\mathcal{S}} \mu(2U\operatorname{grad}_{\mathcal{S}} H - \boldsymbol{L}\operatorname{grad}_{\mathcal{S}} U - 2H^2\boldsymbol{u}^{\tan}) \cdot \boldsymbol{\nu} \,\mathrm{d}s + \int_{\partial_2\mathcal{S}} (\boldsymbol{t} \cdot \boldsymbol{u} - \boldsymbol{c} \cdot (\boldsymbol{L}\boldsymbol{u}^{\tan} + \operatorname{grad}_{\mathcal{S}} U))\,\mathrm{d}s. \tag{8.6}$$

We next consider the isometric deformation of a flat undistorted rectangular strip into a smooth loop $\mathcal{S}$, as described in Remark 2.2. In that case, we must have $\boldsymbol{u} = \boldsymbol{0}$ and $\boldsymbol{\nu} \cdot \operatorname{grad}_{\mathcal{S}} U = 0$ on the two short edges of $\mathcal{S}$ that are brought together, which are denoted as $\partial_1\mathcal{S}$. In this example, the two long edges of $\mathcal{S}$, which are denoted as $\partial_2\mathcal{S}$, are also assumed to be free of loads, so that the traction $\boldsymbol{t}$ and the couple traction $\boldsymbol{c}$ both vanish on $\partial_2\mathcal{S}$ and $\boldsymbol{u}$ is unrestricted on these long edges.

8.1. Example 1

If the reference configuration $\mathcal{D}$ is a flat, undistorted rectangular strip of length ℓ and the two short ends are brought together smoothly without overlap or a twist to form a supposed equilibrium configuration $\mathcal{S}$, then, as noted in Remark 2.2, the variation $\boldsymbol{u}$ and the tangent-normal derivative $\boldsymbol{\nu} \cdot \operatorname{grad}_{\mathcal{S}} U$ of its normal component U must both vanish on the two short ends, denoted as $\partial_1\mathcal{S}$, of $\mathcal{S}$, and the edge integral over $\partial\mathcal{S}$ in (8.6) reduces to an integral over $\partial_2\mathcal{S}$. In this example, the long edges are assumed to be free of loads — so that $\boldsymbol{t} = \boldsymbol{c} = \boldsymbol{0}$ on $\partial_2\mathcal{S}$.

Let us suppose now that $H = \text{constant} \neq 0$. Then, with the aid of (5.6) and the surface divergence theorem, it follows that the left hand side of (8.6) may be written as

$$\int_{\mathcal{S}} 2\mu(\Delta_{\mathcal{S}} H + 2H^3)U \,\mathrm{d}a = 2\mu H^2 \int_{\partial_2\mathcal{S}} \boldsymbol{u}^{\tan} \cdot \boldsymbol{\nu} \,\mathrm{d}s. \tag{8.7}$$

Next, we observe that because the deformation $\tilde{\boldsymbol{y}}$ is isometric and, consequently, since one of the principal curvatures must vanish everywhere on $\mathcal{S}$, the deformed configuration $\mathcal{S}$ must conform to a circular cylindrical surface with generators parallel to a fixed unit vector $\boldsymbol{e}$. Then, by choosing first $\boldsymbol{u} = \boldsymbol{0}$ and $\boldsymbol{\nu} \cdot \operatorname{grad}_{\mathcal{S}} U$ arbitrarily on $\partial_2\mathcal{S}$, we may infer from (8.6) and (8.7) that $\boldsymbol{\nu} \cdot \boldsymbol{L}\boldsymbol{\nu} = 0$ on $\partial_2\mathcal{S}$, which implies that the curvature of $\mathcal{S}$ vanishes in the direction of $\boldsymbol{\nu}$ on $\partial_2\mathcal{S}$. Consequently, $\boldsymbol{\nu}$ must be parallel to $\boldsymbol{e}$ on $\partial_2\mathcal{S}$ and $\mathcal{S}$ must have the form of a right circular cylinder. Moreover, because the two short ends of the strip are brought together without overlap, the radius of the resulting right circular cylinder $\mathcal{S}$ must be $R = \ell/2\pi$. As a result, noting (8.7), the variational condition (8.6) reduces to

$$4\mu H^2 \int_{\partial_2\mathcal{S}} \boldsymbol{u}^{\tan} \cdot \boldsymbol{\nu} \,\mathrm{d}s = 0 \tag{8.8}$$

for all admissible $\boldsymbol{u}$. But, we know from (5.5) that $\boldsymbol{e} \cdot (\operatorname{grad}_{\mathcal{S}} \boldsymbol{u}^{\tan})\boldsymbol{e} = 0$, and this has the consequence that $\boldsymbol{u}^{\tan} \cdot \boldsymbol{e} = \text{constant}$ along each generator on $\mathcal{S}$. Since $\boldsymbol{\nu} = \pm\boldsymbol{e}$ are the outer unit normals to the long edges $\partial_2\mathcal{S}$, we hence see that the integral above vanishes and that, indeed, $\delta\mathcal{F}(\mathcal{S})[\boldsymbol{u}] = 0$ holds. We therefore conclude that the right circular cylindrical form for $\mathcal{S}$ satisfies the necessary first variation condition for an energy extremal.

[9] See Hinz and Fried (2015a, 2015b, 2015c) for English translations of these landmark papers.
[10] See Todres (2015) for an English translation of this work.

8.2. Example 2

Suppose that, as an alternative boundary condition for the strip, the complete boundary of the deformed strip $\partial\mathcal{S}$ is loaded as follows: the traction $\boldsymbol{t}$ vanishes on $\partial\mathcal{S}$; the couple traction $\boldsymbol{c}$ vanishes on the long edges, herein denoted as $\partial_l\mathcal{S}$; the couple traction $\boldsymbol{c}$ is given by $\boldsymbol{c} = c\boldsymbol{\nu}$ on the short edges, denoted herein as $\partial_s\mathcal{S}$, where c is a prescribed constant. Assuming that $H = \text{constant} \neq 0$ and that the deformation of $\mathcal{D}$ to $\mathcal{S}$ is isometric, we once again see that $\mathcal{S}$ must lie on a circular cylindrical surface with generators parallel to a fixed unit vector $\boldsymbol{e}$. Also, again noting, similar to (8.7), that in this case the left hand side of (8.6) may be written as

$$\int_{\mathcal{S}} 2\mu(\Delta_{\mathcal{S}} H + 2H^3)U \, \mathrm{d}a = 2\mu H^2 \int_{\partial\mathcal{S}} \boldsymbol{u}^{\tan} \cdot \boldsymbol{\nu} \, \mathrm{d}s, \tag{8.9}$$

we infer that (8.6) has the form

$$\int_{\partial_l\mathcal{S}} \mu(\boldsymbol{L}\operatorname{grad}_{\mathcal{S}} U + 4H^2\boldsymbol{u}^{\tan}) \cdot \boldsymbol{\nu} \, \mathrm{d}s + \int_{\partial_s\mathcal{S}} ((c\boldsymbol{1}_{\mathcal{S}} + \mu\boldsymbol{L})\operatorname{grad}_{\mathcal{S}} U + (4\mu H^2\boldsymbol{1}_{\mathcal{S}} + c\boldsymbol{L})\boldsymbol{u}^{\tan}) \cdot \boldsymbol{\nu} \, \mathrm{d}s = 0. \tag{8.10}$$

Proceeding as in the previous example, we first choose $\boldsymbol{u} = \boldsymbol{0}$ on $\partial_l\mathcal{S} \cup \partial_s\mathcal{S}$ and $\boldsymbol{\nu} \cdot \operatorname{grad}_{\mathcal{S}} U = 0$ on $\partial_s\mathcal{S}$, and note that $\boldsymbol{\nu} \cdot \operatorname{grad}_{\mathcal{S}} U$ can be chosen arbitrarily on $\partial_l\mathcal{S}$. Then, from (8.10) we deduce that $\boldsymbol{\nu} \cdot \boldsymbol{L}\boldsymbol{\nu} = 0$ on $\partial_l\mathcal{S}$ and, thus, that $\boldsymbol{\nu}$ must be parallel to $\boldsymbol{e}$ on $\partial_l\mathcal{S}$. We again find that $\mathcal{S}$ must have the form of a right circular cylinder. Additionally, we infer that the curvature tensor $\boldsymbol{L}$ must have the form $\boldsymbol{L} = k\,\boldsymbol{d} \otimes \boldsymbol{d}$, where k is constant and $\boldsymbol{d} := \boldsymbol{e} \times \boldsymbol{n}$ is chosen so that $\{\boldsymbol{e}, \boldsymbol{n}, \boldsymbol{d}\}$ is a positively oriented orthonormal basis on $\mathcal{S}$.

Returning to (8.10) and stipulating $\boldsymbol{u}$ vanishes on $\partial_l\mathcal{S}$ and that both U and $\boldsymbol{\nu} \cdot \operatorname{grad}_{\mathcal{S}} U$ vanish on $\partial_s\mathcal{S}$, we may use the arbitrariness of $\boldsymbol{u}^{\tan}$ on $\partial_s\mathcal{S}$ to conclude that $4\mu H^2\boldsymbol{\nu} + c\boldsymbol{L}\boldsymbol{\nu} = \boldsymbol{0}$ on $\partial_s\mathcal{S}$. Thus, the outer unit normal $\boldsymbol{\nu}$ on $\partial_s\mathcal{S}$, which is parallel or antiparallel to $\boldsymbol{d}$, is an eigenvector of $\boldsymbol{L}$ on $\partial_s\mathcal{S}$, with corresponding eigenvalue $-4\mu H^2/c$. From this, we see that the edges of $\partial_l\mathcal{S}$ and $\partial_s\mathcal{S}$ are orthogonal where they meet and, recalling that $k = 2H$, that $k = -\mu k^2/c$, or $c = -\mu k$.

From what we have found so far, we see that the first term of the integrand of the integral on the first line of (8.10) vanishes identically, as does the entire integrand of the integral on the second line of (8.10). We are consequently left with (8.8), wherein $\partial_2\mathcal{S}$ is replaced with $\partial_l\mathcal{S}$, which, recapitulating the argument in the lines following (8.8), we may dismiss as being identically satisfied.

Letting R denote the radius of the right cylindrical form of $\mathcal{S}$, so that the curvature k is given by $k = -1/R$, we conclude that $c = \mu/R > 0$, where we recall that $\boldsymbol{c} = c\boldsymbol{\nu}$ is the couple traction applied to the short ends $\partial_s\mathcal{S}$. In terms of the specific (bending) moment $\boldsymbol{m}$ applied to the short ends $\partial_s\mathcal{S}$, we have $\boldsymbol{m} = \boldsymbol{n} \times \boldsymbol{c} = c\boldsymbol{\sigma}$. Clearly, if the length ℓ of the strip is too large in the sense that $\ell > 2\pi R = 2\pi\mu/c$, then the short ends of the strip will overlap, and this will happen for any fixed length ℓ if the magnitude of the specific bending moment c is sufficiently large.

9. Lagrange multiplier investigation

To investigate the Lagrange Multiplier method, we introduce the variation $\boldsymbol{z}_\varepsilon$ of $\tilde{\boldsymbol{y}}$ from (2.19) and return to the variational problem

$$\delta\mathcal{F}(\mathcal{S})[\boldsymbol{u}] := \int_{\mathcal{S}} W'(\boldsymbol{L}) \cdot \dot{\boldsymbol{L}}_\varepsilon|_{\varepsilon=0} \, \mathrm{d}a - \int_{\partial_2\mathcal{S}} (\boldsymbol{t} \cdot \boldsymbol{u} + \boldsymbol{c} \cdot \dot{\boldsymbol{n}}_\varepsilon|_{\varepsilon=0}) \, \mathrm{d}s = 0, \tag{9.1}$$

subject to the first order isometric deformation constraint

$$(\operatorname{grad}_{\mathcal{S}} \boldsymbol{z}_\varepsilon)^\top \operatorname{grad}_{\mathcal{S}} \boldsymbol{z}_\varepsilon - \boldsymbol{1}_{\mathcal{S}} = \boldsymbol{0} \tag{9.2}$$

recorded in $(2.22)_2$, and the second order Gaussian curvature constraint

$$K_\varepsilon = 0 \tag{9.3}$$

recorded in $(2.22)_3$.

Remark 9.1. Throughout this work, it is assumed that the equilibrium deformation $\tilde{\boldsymbol{y}}$, which maps the planar, undistorted region $\mathcal{D}$ to the surface $\mathcal{S}$, is isometric so that (2.4) holds and, consequently, the Gaussian curvature K of $\mathcal{S}$ is identically zero. With this provision, the first order isometric deformation constraint (9.2) is equivalent to the hypothesis that the composition $\boldsymbol{z}_\varepsilon \circ \tilde{\boldsymbol{y}}$ that deforms $\mathcal{D}$ to $\mathcal{S}_\varepsilon$ is isometric and that the Gaussian curvature K_ε of $\mathcal{S}_\varepsilon$ is everywhere zero, namely that the second order isometric constraint (9.3) holds. However, the variational problem (9.1) is spatial and has as its base the spatially deformed surface $\mathcal{S}$, without explicit recognition of the reference configuration $\mathcal{D}$. As such, the constraint (9.2) that the deformation $\boldsymbol{z}_\varepsilon$ from $\mathcal{S}$ be isometric does not contain any information about the Gaussian curvature K of $\mathcal{S}$ — in particular that it vanishes. The second order Gaussian curvature constraint (9.3) ensures that the surfaces $\mathcal{S}_\varepsilon$ and $\mathcal{S}$ share an isometry relationship that convects with the variation, but this, alone, does not guarantee that the deformation $\boldsymbol{z}_\varepsilon$ is isometric. The Lagrange multiplier method for this spatial variational problem requires that the underlying constraint be expressed in a spatial form, and we do so in recognizing both (9.2) and (9.3). Throughout the remainder of this work, we regard (9.2) and (9.3) as two explicit variational constraints on the geometry of the deformation $\tilde{\boldsymbol{y}}$ that is presumed to solve the variational problem (9.1). In support of this, it is noteworthy to recall from the contents of Sections 5 and 6, in particular the last summary sentence of Section 6 including (6.6), that the two constraints produce distinct and essential, complementary conditions on the variation of $\mathcal{S}$. See Footnote 2 for additional discussion of this issue. □

In the Lagrangian multiplier method, the constraints (9.2) and (9.3) are relaxed and we introduce the Lagrangian multiplier fields $\boldsymbol{T} = \boldsymbol{T}^\top$ and p, defined for all points $\boldsymbol{y}$ of $\mathcal{S}$ (and, thus, implicitly, through the deformation $\tilde{\boldsymbol{y}}$, all material points $\boldsymbol{x}$ of $\mathcal{D}$), as, respectively, a symmetric linear transformation field from $\mathcal{S}^{\tan}$ to itself and a scalar field. These fields are interpreted as constraint reaction fields internal to $\mathcal{S}$, which carry, respectively, the dimensions of force/length and force $\times$ length, fields that are compatible with and faithful to the condition that the deformation $\tilde{\boldsymbol{y}}$ be an isometric deformation from the flat, undistorted reference configuration $\mathcal{D}$ to the surface $\mathcal{S}$. On this basis, we define the augmented energy functional[11]

$$\mathcal{L}(\mathcal{S}_\varepsilon) := \int_{\mathcal{S}_\varepsilon} W(\boldsymbol{L}_\varepsilon)\,\mathrm{d}a_\varepsilon - \int_{\partial_2\mathcal{S}} \big(\boldsymbol{t}\cdot(\boldsymbol{z}_\varepsilon - \boldsymbol{y}) + \boldsymbol{c}\cdot(\boldsymbol{n}_\varepsilon - \boldsymbol{n})\big)\,\mathrm{d}s - \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{T}\cdot\big((\mathrm{grad}_{\mathcal{S}}\boldsymbol{z}_\varepsilon)^\top \mathrm{grad}_{\mathcal{S}}\boldsymbol{z}_\varepsilon - \boldsymbol{1}_{\mathcal{S}}\big)\,\mathrm{d}a - \int_{\mathcal{S}} pK_\varepsilon\,\mathrm{d}a, \tag{9.4}$$

which we may rewrite as

$$\mathcal{L}(\mathcal{S}_\varepsilon) := \int_{\mathcal{S}} W(\boldsymbol{L}_\varepsilon)\gamma_\varepsilon\,\mathrm{d}a - \int_{\partial_2\mathcal{S}} \big((\boldsymbol{t}\cdot(\boldsymbol{z}_\varepsilon - \boldsymbol{y}) + \boldsymbol{c}\cdot(\boldsymbol{n}_\varepsilon - \boldsymbol{n})\big)\,\mathrm{d}s - \frac{1}{2}\int_{\mathcal{S}} \boldsymbol{T}\cdot\big((\mathrm{grad}_{\mathcal{S}}\boldsymbol{z}_\varepsilon)^\top \mathrm{grad}_{\mathcal{S}}\boldsymbol{z}_\varepsilon - \boldsymbol{1}_{\mathcal{S}}\big)\,\mathrm{d}a - \int_{\mathcal{S}} pK_\varepsilon\,\mathrm{d}a, \tag{9.5}$$

where we have used the relation $\mathrm{d}a_\varepsilon = \gamma_\varepsilon\,\mathrm{d}a$ in conjunction with the definition

$$\gamma_\varepsilon := |(\mathrm{grad}_{\mathcal{S}}\boldsymbol{z}_\varepsilon)^{\mathrm{c}}\boldsymbol{n}|. \tag{9.6}$$

Noting the identities

$$\gamma_\varepsilon|_{\varepsilon=0} = 1 \qquad \text{and} \qquad \dot{\gamma}_\varepsilon|_{\varepsilon=0} = \mathrm{div}_{\mathcal{S}}\,\boldsymbol{u} = \mathrm{div}_{\mathcal{S}}\,\boldsymbol{u}^{\tan} - 2HU, \tag{9.7}$$

we see that

$$\begin{aligned}\overline{W(\boldsymbol{L}_\varepsilon)\gamma_\varepsilon}^{\cdot}|_{\varepsilon=0} &= W'(\boldsymbol{L})\cdot\dot{\boldsymbol{L}}_\varepsilon\gamma_\varepsilon|_{\varepsilon=0} + W(\boldsymbol{L})\dot{\gamma}_\varepsilon|_{\varepsilon=0}\\ &= W'(\boldsymbol{L})\cdot\dot{\boldsymbol{L}}_\varepsilon|_{\varepsilon=0} + W(\boldsymbol{L})(\mathrm{div}_{\mathcal{S}}\,\boldsymbol{u}^{\tan} - 2HU),\end{aligned} \tag{9.8}$$

Thus, proceeding as in the derivation of (7.9) from (3.1) and using the expression (7.8) for the first term on the right-hand side of the second line of (9.8) and the expressions (5.5) and (6.6) for the first variations of the first and second order isometric admissibility constraints (9.2) and (9.3), we find that for each admissible $\boldsymbol{u}$ the first variation condition $\delta\mathcal{L}(\mathcal{S})[\boldsymbol{u}] = 0$ associated with (9.5) takes the form

$$\begin{aligned}&\int_{\mathcal{S}}(\mathrm{div}_{\mathcal{S}}\,\mathrm{div}_{\mathcal{S}}\,W'(\boldsymbol{L}) - 2(W(\boldsymbol{L}) - 2W'(\boldsymbol{L})\cdot\boldsymbol{L})H)U\,\mathrm{d}a\\ &\quad -\frac{1}{2}\int_{\mathcal{S}}\boldsymbol{T}\cdot(\boldsymbol{1}_{\mathcal{S}}\,\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{u}^{\tan} + (\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{u}^{\tan})^\top\boldsymbol{1}_{\mathcal{S}} - 2U\boldsymbol{L})\,\mathrm{d}a - \int_{\mathcal{S}} p(2H\boldsymbol{1}_{\mathcal{S}} - \boldsymbol{L})\cdot\mathrm{grad}_{\mathcal{S}}\,\mathrm{grad}_{\mathcal{S}}\,U\,\mathrm{d}a\\ &\quad = \int_{\partial\mathcal{S}}(U\mathrm{div}_{\mathcal{S}}\,W'(\boldsymbol{L}) - W'(\boldsymbol{L})\mathrm{grad}_{\mathcal{S}}\,U - W(\boldsymbol{L})\boldsymbol{u}^{\tan})\cdot\boldsymbol{\nu}\,\mathrm{d}s + \int_{\partial_2\mathcal{S}}(\boldsymbol{t}\cdot\boldsymbol{u} - \boldsymbol{c}\cdot(\mathrm{grad}_{\mathcal{S}}\,U + \boldsymbol{L}\boldsymbol{u}^{\tan})\,\mathrm{d}s = 0.\end{aligned} \tag{9.9}$$

The deformation $\tilde{\boldsymbol{y}}$ that satisfies the variational requirement (9.9) must, of course, comply with the constraint equations

$$\boldsymbol{F}^\top\boldsymbol{F} = \boldsymbol{1}_{\mathcal{D}} \qquad \text{and} \qquad K = 0. \tag{9.10}$$

Remark 9.2. For the constrained variational problem, admissibility requires only that $\boldsymbol{u}$ and $\boldsymbol{\nu}\cdot\mathrm{grad}_{\mathcal{S}}U$ vanish on $\partial_1\mathcal{S}$, namely the part of the boundary of $\mathcal{S}$ where $\boldsymbol{y}$ and $\boldsymbol{n}$ are prescribed. Recall that the traction $\boldsymbol{t}$ and couple traction $\boldsymbol{c}$ are prescribed for all $\boldsymbol{y}\in\partial_2\mathcal{S} = \partial\mathcal{S}\setminus\partial_1\mathcal{S}$. If $\partial_2\mathcal{S} = \partial\mathcal{S}$, so that $\boldsymbol{t}$ and $\boldsymbol{c}$ are prescribed on the complete boundary $\partial\mathcal{S}$, then the force and moment balance conditions (4.8) must hold. If the undistorted reference configuration is a flat strip and its distorted placement is a smooth closed loop as described in Remark 2.2, then $\boldsymbol{u}$ and $\boldsymbol{\nu}\cdot\mathrm{grad}_{\mathcal{S}}U$ must vanish on the short edges that are connected to one another. In this regard, consult the final sentence of the paragraph containing (7.9). □

Further observations are needed before drawing any conclusions from the first variation condition (9.9). Invoking the symmetry of $\boldsymbol{T}$ and the surface divergence theorem, we next see that the second line of (9.9) can be written as

$$\frac{1}{2}\int_{\mathcal{S}}\boldsymbol{T}\cdot(\boldsymbol{1}_{\mathcal{S}}\,\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{u}^{\tan} + (\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{u}^{\tan})^\top\boldsymbol{1}_{\mathcal{S}} - 2U\boldsymbol{L})\,\mathrm{d}a = -\int_{\mathcal{S}}(U\boldsymbol{T}\cdot\boldsymbol{L} + \boldsymbol{u}^{\tan}\cdot\mathrm{div}_{\mathcal{S}}\,\boldsymbol{T})\,\mathrm{d}a + \int_{\partial\mathcal{S}}\boldsymbol{T}\boldsymbol{\nu}\cdot\boldsymbol{u}^{\tan}\,\mathrm{d}s. \tag{9.11}$$

In a similar development that relies on (5.12) and the surface divergence theorem, we find that the third line of (9.9) can be written as

$$\int_{\mathcal{S}} p(2H\boldsymbol{1}_{\mathcal{S}} - \boldsymbol{L})\cdot\mathrm{grad}_{\mathcal{S}}\,\mathrm{grad}_{\mathcal{S}}\,U\,\mathrm{d}a = \int_{\mathcal{S}} U(2H\boldsymbol{1}_{\mathcal{S}} - \boldsymbol{L})\cdot\mathrm{grad}_{\mathcal{S}}\,\mathrm{grad}_{\mathcal{S}}\,p\,\mathrm{d}a - \int_{\partial\mathcal{S}}(2H\boldsymbol{1}_{\mathcal{S}} - \boldsymbol{L})(U\mathrm{grad}_{\mathcal{S}}\,p - p\mathrm{grad}_{\mathcal{S}}\,U)\cdot\boldsymbol{\nu}\,\mathrm{d}s. \tag{9.12}$$

[11] The factor of 1/2 in the third term on the right-hand side of (9.4) is a matter of convention and is introduced so that the constraint reaction $\boldsymbol{T}$ multiplies an appropriate measure of surface strain.

Using (9.11) and (9.12) in (9.9) and invoking, once again, the symmetry of $\boldsymbol{T}$, we deduce that

$$\begin{aligned}&\int_{\mathcal{S}}(\mathrm{div}_{\mathcal{S}}\,\mathrm{div}_{\mathcal{S}}\,W'(\boldsymbol{L}) - 2(W(\boldsymbol{L}) - 2W'(\boldsymbol{L})\cdot\boldsymbol{L})H)U\,\mathrm{d}a + \int_{\mathcal{S}}((\boldsymbol{T}\cdot\boldsymbol{L} - (2H\boldsymbol{1}_{\mathcal{S}} - \boldsymbol{L})\cdot\mathrm{grad}_{\mathcal{S}}\,\mathrm{grad}_{\mathcal{S}}\,p)U + \boldsymbol{u}^{\tan}\cdot\mathrm{div}_{\mathcal{S}}\,\boldsymbol{T})\,\mathrm{d}a\\ &\quad= \int_{\partial\mathcal{S}}(U\mathrm{div}_{\mathcal{S}}\,W'(\boldsymbol{L}) - W'(\boldsymbol{L})\mathrm{grad}_{\mathcal{S}}\,U - W(\boldsymbol{L})\boldsymbol{u}^{\tan})\cdot\boldsymbol{\nu}\,\mathrm{d}s\\ &\qquad - \int_{\partial\mathcal{S}}((2H\boldsymbol{1}_{\mathcal{S}} - \boldsymbol{L})(U\mathrm{grad}_{\mathcal{S}}\,p - p\mathrm{grad}_{\mathcal{S}}\,U) - \boldsymbol{T}\boldsymbol{u}^{\tan})\cdot\boldsymbol{\nu}\,\mathrm{d}s + \int_{\partial_2\mathcal{S}}(\boldsymbol{t}\cdot\boldsymbol{u} - \boldsymbol{c}\cdot(\mathrm{grad}_{\mathcal{S}}\,U + \boldsymbol{L}\boldsymbol{u}^{\tan})\,\mathrm{d}s.\end{aligned} \tag{9.13}$$

Next, introducing a unit tangent vector $\boldsymbol{\sigma} = \boldsymbol{n}\times\boldsymbol{\nu}$ to $\partial\mathcal{S}$ and using the notation $\partial U/\partial s = \boldsymbol{\sigma}\cdot\mathrm{grad}_{\mathcal{S}}\,U$ and $\partial U/\partial\nu = \boldsymbol{\nu}\cdot\mathrm{grad}_{\mathcal{S}}\,U$,[12] we observe that

$$\mathrm{grad}_{\mathcal{S}}\,U = \frac{\partial U}{\partial s}\boldsymbol{\sigma} + \frac{\partial U}{\partial\nu}\boldsymbol{\nu}, \tag{9.14}$$

on $\partial\mathcal{S}$. Thus, we see that, on $\partial\mathcal{S}$,

$$\boldsymbol{\nu}\cdot W'(\boldsymbol{L})\mathrm{grad}_{\mathcal{S}}\,U = \frac{\partial}{\partial s}(U\boldsymbol{\sigma}\cdot W'(\boldsymbol{L})\boldsymbol{\nu}) - U\frac{\partial}{\partial s}(\boldsymbol{\sigma}\cdot W'(\boldsymbol{L})\boldsymbol{\nu}) + (\boldsymbol{\nu}\cdot W'(\boldsymbol{L})\boldsymbol{\nu})\frac{\partial U}{\partial\nu}, \tag{9.15}$$

where we have used the symmetry of $W'(\boldsymbol{L})$ for each $\boldsymbol{L}$ in Sym_0, and

$$p(2H\boldsymbol{1}_{\mathcal{S}} - \boldsymbol{L})\boldsymbol{\nu}\cdot\mathrm{grad}_{\mathcal{S}}\,U = U\frac{\partial}{\partial s}(p\boldsymbol{\sigma}\cdot\boldsymbol{L}\boldsymbol{\nu}) - \frac{\partial}{\partial s}(pU\boldsymbol{\sigma}\cdot\boldsymbol{L}\boldsymbol{\nu}) + p(2H - \boldsymbol{\nu}\cdot\boldsymbol{L}\boldsymbol{\nu})\frac{\partial U}{\partial\nu}, \tag{9.16}$$

while, on $\partial_2\mathcal{S}$,

$$\boldsymbol{c}\cdot\mathrm{grad}_{\mathcal{S}}\,U = \frac{\partial}{\partial s}(U\boldsymbol{c}\cdot\boldsymbol{\sigma}) - U\frac{\partial}{\partial s}(\boldsymbol{c}\cdot\boldsymbol{\sigma}) + (\boldsymbol{c}\cdot\boldsymbol{\nu})\frac{\partial U}{\partial\nu}. \tag{9.17}$$

Using (9.16) and (9.17) in (9.13), we find that the first variation condition (9.9) can be written as

$$\begin{aligned}&\int_{\mathcal{S}}(\mathrm{div}_{\mathcal{S}}\,\mathrm{div}_{\mathcal{S}}\,W'(\boldsymbol{L}) - 2(W(\boldsymbol{L}) - 2W'(\boldsymbol{L})\cdot\boldsymbol{L})H)U\,\mathrm{d}a + \int_{\mathcal{S}}((\boldsymbol{T}\cdot\boldsymbol{L} - (2H\boldsymbol{1}_{\mathcal{S}} - \boldsymbol{L})\cdot\mathrm{grad}_{\mathcal{S}}\,\mathrm{grad}_{\mathcal{S}}\,p)U + \boldsymbol{u}^{\tan}\cdot\mathrm{div}_{\mathcal{S}}\,\boldsymbol{T})\,\mathrm{d}a\\ &\quad= \int_{\partial_2\mathcal{S}}(\mathrm{div}_{\mathcal{S}}\,W'(\boldsymbol{L}) - (2H\boldsymbol{1}_{\mathcal{S}} - \boldsymbol{L})\mathrm{grad}_{\mathcal{S}}\,p)\cdot\boldsymbol{\nu}U\,\mathrm{d}s + \int_{\partial_2\mathcal{S}}\frac{\partial}{\partial s}(\boldsymbol{\sigma}\cdot(W'(\boldsymbol{L})\boldsymbol{\nu} + p\boldsymbol{L}\boldsymbol{\nu} + \boldsymbol{c}))U\,\mathrm{d}s\\ &\qquad - \int_{\partial_2\mathcal{S}}(W(\boldsymbol{L})\boldsymbol{\nu} - \boldsymbol{T}\boldsymbol{\nu} - \boldsymbol{t}^{\tan} + \boldsymbol{L}\boldsymbol{c})\cdot\boldsymbol{u}^{\tan}\,\mathrm{d}s - \int_{\partial_2\mathcal{S}}(W'(\boldsymbol{L})\boldsymbol{\nu} - p(2H\boldsymbol{1}_{\mathcal{S}} - \boldsymbol{L})\boldsymbol{\nu} + \boldsymbol{c})\cdot\boldsymbol{\nu}\frac{\partial U}{\partial\nu}\,\mathrm{d}s,\end{aligned} \tag{9.18}$$

where we have introduced the tangential component

$$\boldsymbol{t}^{\tan} = \boldsymbol{1}_{\mathcal{S}}\boldsymbol{t} \tag{9.19}$$

of $\boldsymbol{t}$.

Since (9.18) must hold all admissible $\boldsymbol{u}$, we may use the arbitrariness of $\boldsymbol{u}$ in $\mathcal{S}$, and $\boldsymbol{u}$ and $\mathrm{grad}_{\mathcal{S}}\,U\cdot\boldsymbol{\nu}$ on $\partial_2\mathcal{S}$ to derive Euler–Lagrange equations from (9.18). The resulting system consists of conditions

$$\mathrm{div}_{\mathcal{S}}\,\mathrm{div}_{\mathcal{S}}\,W'(\boldsymbol{L}) - 2(W(\boldsymbol{L}) - 2W'(\boldsymbol{L})\cdot\boldsymbol{L})H + \boldsymbol{T}\cdot\boldsymbol{L} - (2H\boldsymbol{1}_{\mathcal{S}} - \boldsymbol{L})\cdot\mathrm{grad}_{\mathcal{S}}\,\mathrm{grad}_{\mathcal{S}}\,p = 0, \tag{9.20a}$$

and[13]

$$\boldsymbol{1}_{\mathcal{S}}\,\mathrm{div}_{\mathcal{S}}\,\boldsymbol{T} = \boldsymbol{0} \tag{9.20b}$$

that apply on $\mathcal{S}$ augmented by equations

$$(\mathrm{div}_{\mathcal{S}}\,W'(\boldsymbol{L}) - (2H\boldsymbol{1}_{\mathcal{S}} - \boldsymbol{L})\mathrm{grad}_{\mathcal{S}}\,p)\cdot\boldsymbol{\nu} + \frac{\partial}{\partial s}(\boldsymbol{\sigma}\cdot(W'(\boldsymbol{L})\boldsymbol{\nu} + p\boldsymbol{L}\boldsymbol{\nu} + \boldsymbol{c})) + \boldsymbol{t}\cdot\boldsymbol{n} = 0, \tag{9.20c}$$

$$(W(\boldsymbol{L})\boldsymbol{1}_{\mathcal{S}} - \boldsymbol{T})\boldsymbol{\nu} + \boldsymbol{L}\boldsymbol{c} - \boldsymbol{t}^{\tan} = \boldsymbol{0}, \tag{9.20d}$$

and

$$(W'(\boldsymbol{L})\boldsymbol{\nu} - p(2H\boldsymbol{1}_{\mathcal{S}} - \boldsymbol{L})\boldsymbol{\nu} + \boldsymbol{c})\cdot\boldsymbol{\nu} = 0 \tag{9.20e}$$

that apply on the loaded portion $\partial_2\mathcal{S}$ of the boundary $\partial\mathcal{S}$ of $\mathcal{S}$.

Remark 9.3. The unknown fields that are supposed to satisfy these equations, together with the boundary conditions that $\boldsymbol{y}$ and $\boldsymbol{n}$ are given on $\partial_1\mathcal{S}$ and the traction $\boldsymbol{t}$ and couple traction $\boldsymbol{c}$ are given on $\partial_2\mathcal{S} = \partial\mathcal{S}\setminus\partial_1\mathcal{S}$, and the constraint equations (9.10), may be listed in various ways. We choose the listing $\{\boldsymbol{L}(\boldsymbol{y}), \boldsymbol{T}(\boldsymbol{y}), p(\boldsymbol{y})\}$ which, because $\boldsymbol{L}$ and $\boldsymbol{T}$ are symmetric transformations from $\mathcal{S}^{\tan}$ to $\mathcal{S}^{\tan}$, consists of seven unknown fields. The governing field equations are (9.20 a), (9.20 b), $(9.10)_1$,

[12] With $\boldsymbol{\sigma}$ as defined, $\{\boldsymbol{\sigma}, \boldsymbol{n}, \boldsymbol{\nu}\}$ is the (positively oriented) Darboux frame for $\partial\mathcal{S}$.

[13] We may alternatively write (9.20 b) as $\mathrm{div}_{\mathcal{S}}\,\boldsymbol{T} = (\boldsymbol{T}\cdot\boldsymbol{L})\boldsymbol{n}$.

and $(9.10)_2$, which amount also to seven equations. However, it is known that $(9.10)_1$ implies $(9.10)_2$ (see, e.g., Chen et al., 2015 and Chen and Fried, 2016), so the system of independent field equations is reduced to six. Thus, it is not expected that the Lagrangian multiplier fields $\boldsymbol{T}$ and p would be uniquely determined by the system (9.20), $(9.10)_1$, and $(9.10)_2$, and we shall see this to be so for the two problems considered in Sections 10 and 11. These fields merely represent reactions to the constraint that the ribbon is unstretchable — reactions internal to $\mathcal{S}$ that allow $\mathcal{S}$ to have sustained a particular isometric deformation from its flat undistorted configuration $\mathcal{D}$. □

Remark 9.4. An interesting and important question that we do not address in the present work concerns the construction of universal solutions to the system consisting of (9.10) and the Euler–Lagrange equations (9.20) in the case where $\partial_1\mathcal{S}=\varnothing$ so that $\partial_2\mathcal{S}=\partial\mathcal{S}$ and, thus, where (9.20 c–e) apply on the entirety of $\partial\mathcal{S}$. A universal solution would satisfy the system for all choices of W consistent with the requirement (2.17) of frame indifference. In particular, it seems reasonable to expect that any such solution should be made up of pieces of planes, cylinders and cones. □

Remark 9.5. For the quadratic stored energy density (8.1), the Euler–Lagrange equations specialize to

$$2\mu(\Delta_{\mathcal{S}}H+2H^3)-2H\Delta_{\mathcal{S}}p+(\boldsymbol{T}+\operatorname{grad}_{\mathcal{S}}\operatorname{grad}_{\mathcal{S}}p)\cdot\boldsymbol{L}=0, \tag{9.21a}$$

$$\boldsymbol{1}_{\mathcal{S}}\operatorname{div}_{\mathcal{S}}\boldsymbol{T}=\boldsymbol{0}, \tag{9.21b}$$

$$(2\mu\operatorname{grad}_{\mathcal{S}}H-2H\operatorname{grad}_{\mathcal{S}}p+\boldsymbol{L}\operatorname{grad}_{\mathcal{S}}p)\cdot\boldsymbol{\nu}+\frac{\partial}{\partial s}(\boldsymbol{\sigma}\cdot(p\boldsymbol{L}\boldsymbol{\nu}+\boldsymbol{c}))+\boldsymbol{t}\cdot\boldsymbol{n}=0, \tag{9.21c}$$

$$(2\mu H^2\boldsymbol{1}_{\mathcal{S}}-\boldsymbol{T})\boldsymbol{\nu}+\boldsymbol{L}\boldsymbol{c}-\boldsymbol{t}^{\tan}=\boldsymbol{0}, \tag{9.21d}$$

and

$$2\mu H-p\boldsymbol{\sigma}\cdot\boldsymbol{L}\boldsymbol{\sigma}+\boldsymbol{c}\cdot\boldsymbol{\nu}=0, \tag{9.21e}$$

where (9.21 a) and (9.21 b) apply on $\mathcal{S}$ and (9.21 c–e) apply on the loaded portion $\partial_2\mathcal{S}$ of the boundary $\partial\mathcal{S}$ of $\mathcal{S}$. In deriving (9.21 e) from (9.20 e), we have used the representation $H=(\boldsymbol{\sigma}\cdot\boldsymbol{L}\boldsymbol{\sigma}+\boldsymbol{\nu}\cdot\boldsymbol{L}\boldsymbol{\nu})/2$ for H on $\partial\mathcal{S}$. □

Remark 9.6. In Sections 10 and 11, we consider two example solutions to the specialization (9.21) of the general constrained variational problem (9.20) that arises on specializing the stored energy density W to be of the quadratic form (8.1) in the case where $\partial_1\mathcal{S}=\varnothing$ so that $\partial_2\mathcal{S}=\partial\mathcal{S}$; respectively, these problems concern the isometric deformation of an undistorted rectangular strip $\mathcal{D}$ to a helical band and to a conical band with $\boldsymbol{t}=\boldsymbol{0}$ and $\boldsymbol{m}$ balanced on $\partial\mathcal{S}$ according to (4.8). In both cases, the Lagrangian multiplier field $\boldsymbol{T}$ is determined, while in each case we find that p is only partially determined, even though the deformation is fixed and the configuration $\mathcal{S}$ is balanced in equilibrium. Consequently, as we shall see, the applied specific moment $\boldsymbol{m}$ on $\partial\mathcal{S}$ is determined, modulo the indeterminacy in p. The constraint that the ribbon is unstretchable, and its consequence that $\mathcal{D}$ may only undergo an isometric deformation, is responsible for this indeterminacy in the applied loading. □

Remark 9.7. For an open surface $\mathcal{S}$, the stored energy of the Willmore (1965) problem can be expressed as

$$\int_{\mathcal{S}}(2\mu H^2-\bar{\mu}K)\,\mathrm{d}a, \tag{9.22}$$

where $\bar{\mu}\neq 0$ is a constant material parameter. If however, the deformation from $\mathcal{D}$ to $\mathcal{S}$ is isometric, then we see from (2.14) that (9.22) coincides with the specialization of (2.18) for the quadratic stored energy density (8.1). From this perspective, the Euler–Lagrange equations (9.21) are the equilibrium conditions for the isometrically constrained version of the Willmore problem for an open surface $\mathcal{S}$ subject to dead loads $\boldsymbol{t}$ and $\boldsymbol{c}$ on some portion $\partial\mathcal{S}_2$ of its boundary. □

10. Solution of the constrained variational problem for the quadratic stored energy density: Rectangular strip to a circular helical band

Consider a flat, undistorted rectangular material strip $\mathcal{D}$ with stored energy density W of the quadratic form (8.1) that is isometrically deformed so that the traction $\boldsymbol{t}$ and the couple traction $\boldsymbol{c}$ are given everywhere on $\partial\mathcal{S}$, in which case $\partial_2\mathcal{S}$ corresponds to the complete boundary $\partial\mathcal{S}$ of the deformed strip and the force and moment balance conditions (4.8) must hold. In addition, we set $\partial\mathcal{S}=\partial_{\mathrm{l}}\mathcal{S}\cup\partial_{\mathrm{s}}\mathcal{S}$, where $\partial_{\mathrm{l}}\mathcal{S}$ denotes the two long edges of $\mathcal{S}$ and $\partial_{\mathrm{s}}\mathcal{S}$ denotes the two short edges of $\mathcal{S}$. We then consider the following loading system:

i. The traction $\boldsymbol{t}$ satisfies $\boldsymbol{t}=\boldsymbol{0}$ on $\partial\mathcal{S}$.
ii. The couple traction $\boldsymbol{c}$ on the long edges $\partial_{\mathrm{l}}\mathcal{S}$ of $\partial\mathcal{S}$ satisfies $\boldsymbol{c}=c_{\mathrm{l}}^{\sigma}\boldsymbol{\sigma}+c_{\mathrm{l}}^{\nu}\boldsymbol{\nu}$, where c_{l}^{σ} and c_{l}^{ν} are presumed given.
iii. The couple traction $\boldsymbol{c}$ on the short edges $\partial_{\mathrm{s}}\mathcal{S}$ of $\partial\mathcal{S}$ satisfies $\boldsymbol{c}=c_{\mathrm{s}}^{\sigma}\boldsymbol{\sigma}+c_{\mathrm{s}}^{\nu}\boldsymbol{\nu}$, where c_{s}^{σ} and c_{s}^{ν} are presumed given.

Then, granted that the mean curvature H satisfies $H = \text{constant} \neq 0$ and that the deformation $\tilde{\boldsymbol{y}}$ of $\mathcal{D}$ to $\mathcal{S}$ is isometric, we again see that $\mathcal{S}$ must lie on a circular cylindrical surface whose axis and generators are parallel to a fixed unit vector $\boldsymbol{e}$. In addition, we know that $\boldsymbol{L} = k\boldsymbol{d} \otimes \boldsymbol{d}$ where $k = 2H = -1/R$ is the constant curvature of $\mathcal{S}$, R is the radius of the cylinder $\mathcal{T}_c$ on which $\mathcal{S}$ lies, and $\boldsymbol{d} := \boldsymbol{e} \times \boldsymbol{n}$ so that $\{\boldsymbol{e}, \boldsymbol{n}, \boldsymbol{d}\}$ is a positively oriented orthornormal basis on $\mathcal{S}$.

Elsewhere, Chen et al. (2018) have proven that for the above conditions $\mathcal{S}$ must have the form of a circular helical band whose long edges $\partial_l\mathcal{S}$, for positive chirality relative to $\boldsymbol{e}$, are at a constant angle $\theta \in [0, \pi/2]$ with $\boldsymbol{e}$. If $\theta = \pi/2$, then the short edges are coincident with the generators of the cylinder and the long edges are wrapped around the cylinder. If $\theta = 0$, then the long edges are coincident with the generators of the cylinder and the short edges are the ones that are wrapped. To avoid tedious qualifications which eliminate the possible overlap (i.e., self-intersection) of $\mathcal{S}$, qualifications that are not essential to the goal of determining a radius $R > 0$ and an angle θ satisfying $0 \leq \theta \leq \pi/2$ such that equations (9.21) hold for $\boldsymbol{t} = \boldsymbol{0}$ and for any prescribed $\boldsymbol{c}$ such that the applied specific moment $\boldsymbol{m} = \boldsymbol{n} \times \boldsymbol{c}$ is balanced (namely that $(4.8)_2$ holds consistent with the stipulation that $\boldsymbol{t} = \boldsymbol{0}$), we shall proceed with tacit awareness of this issue.

In Fig. 1, we show the reference configuration of the rectangular strip $\mathcal{D}$ and in Fig. 2 we depict a possible spatial configuration of the isometrically deformed strip $\mathcal{S}$.

10.1. Solution of the Euler–Lagrange equations

We now develop a solution of the Euler–Lagrange equations (9.21), first considering the tangential component (9.21 b) of force balance on $\mathcal{S}$ and writing

$$\boldsymbol{T} = T_{dd}\boldsymbol{d} \otimes \boldsymbol{d} + T_{de}(\boldsymbol{d} \otimes \boldsymbol{e} + \boldsymbol{e} \otimes \boldsymbol{d}) + T_{ee}\boldsymbol{e} \otimes \boldsymbol{e}. \tag{10.1}$$

Then, as shown in the Appendix A.1, we have that

$$\operatorname{div}_{\mathcal{S}} \boldsymbol{T} = (\boldsymbol{d} \cdot \operatorname{grad}_{\mathcal{S}} T_{dd} + \boldsymbol{e} \cdot \operatorname{grad}_{\mathcal{S}} T_{de})\boldsymbol{d} + (\boldsymbol{d} \cdot \operatorname{grad}_{\mathcal{S}} T_{de} + \boldsymbol{e} \cdot \operatorname{grad}_{\mathcal{S}} T_{ee})\boldsymbol{e} + 2HT_{dd}\,\boldsymbol{n}. \tag{10.2}$$

and, thus, by (2.11), that

$$\boldsymbol{1}_{\mathcal{S}} \operatorname{div}_{\mathcal{S}} \boldsymbol{T} = (\boldsymbol{d} \cdot \operatorname{grad}_{\mathcal{S}} T_{dd} + \boldsymbol{e} \cdot \operatorname{grad}_{\mathcal{S}} T_{de})\boldsymbol{d} + (\boldsymbol{d} \cdot \operatorname{grad}_{\mathcal{S}} T_{de} + \boldsymbol{e} \cdot \operatorname{grad}_{\mathcal{S}} T_{ee})\boldsymbol{e}. \tag{10.3}$$

To satisfy (9.21 b) it therefore suffices to take T_{dd}, T_{de}, and T_{ee} to be constant, which we shall do.

We next consider the tangent-normal component (9.21 e) of the couple balance on $\partial\mathcal{S}$ and observe that, because $\boldsymbol{L} = k\boldsymbol{d} \otimes \boldsymbol{d}$,

$$\mu k - kp(\boldsymbol{d} \cdot \boldsymbol{\sigma})^2 + \boldsymbol{c} \cdot \boldsymbol{\nu} = 0 \quad \text{on } \partial\mathcal{S}. \tag{10.4}$$

More specifically, since $\boldsymbol{c} \cdot \boldsymbol{\nu} = c_l^{\nu}$ and $(\boldsymbol{d} \cdot \boldsymbol{\sigma})^2 = \sin^2\theta$ on $\partial_l\mathcal{S}$, and $\boldsymbol{c} \cdot \boldsymbol{\nu} = c_s^{\nu}$ and $(\boldsymbol{d} \cdot \boldsymbol{\sigma})^2 = \cos^2\theta$ on $\partial_s\mathcal{S}$, we may conclude that (9.21 e) yields

$$c_l^{\nu} = -\mu k + kp\sin^2\theta \quad \text{on } \partial_l\mathcal{S} \tag{10.5a}$$

and

$$c_s^{\nu} = -\mu k + kp\cos^2\theta \quad \text{on } \partial_s\mathcal{S}. \tag{10.5b}$$

To continue, knowing that $\boldsymbol{L} = k\boldsymbol{d} \otimes \boldsymbol{d}$ and that $\boldsymbol{t} = \boldsymbol{0}$, we may express the tangential component (9.21 d) of force balance on $\partial\mathcal{S}$ as

$$\boldsymbol{T}\boldsymbol{\nu} - \frac{\mu k^2}{2}\boldsymbol{\nu} - k(\boldsymbol{d} \cdot \boldsymbol{c})\boldsymbol{d} = \boldsymbol{0} \quad \text{on } \partial\mathcal{S}. \tag{10.6}$$

Then, referring to Fig. 2, because $\boldsymbol{\nu} = \pm(\cos\theta\boldsymbol{d} - \sin\theta\boldsymbol{e})$ on $\partial_l\mathcal{S}^{\pm}$, and because $\boldsymbol{d} \cdot \boldsymbol{c} = \pm(c_l^{\nu}\cos\theta + c_l^{\sigma}\sin\theta)$ on $\partial_l\mathcal{S}^{\pm}$, we find that

$$\boldsymbol{T}\boldsymbol{d}\cos\theta - \boldsymbol{T}\boldsymbol{e}\sin\theta - \frac{\mu k^2}{2}(\cos\theta\boldsymbol{d} - \sin\theta\boldsymbol{e}) - k(c_l^{\nu}\cos\theta + c_l^{\sigma}\sin\theta)\boldsymbol{d} = \boldsymbol{0} \quad \text{on } \partial_l\mathcal{S} = \partial_l\mathcal{S}^- \cup \partial_l\mathcal{S}^+. \tag{10.7}$$

Similarly, because $\boldsymbol{\nu} = \pm(\sin\theta\boldsymbol{d} + \cos\theta\boldsymbol{e})$, and $\boldsymbol{d} \cdot \boldsymbol{c} = \pm(c_s^{\nu}\sin\theta - c_s^{\sigma}\cos\theta)$ on $\partial_s\mathcal{S}^{\pm}$, we find that

$$\boldsymbol{T}\boldsymbol{d}\sin\theta + \boldsymbol{T}\boldsymbol{e}\cos\theta - \frac{\mu k^2}{2}(\sin\theta\boldsymbol{d} + \cos\theta\boldsymbol{e}) - k(c_s^{\nu}\sin\theta - c_s^{\sigma}\cos\theta)\boldsymbol{d} = \boldsymbol{0} \quad \text{on } \partial_s\mathcal{S} = \partial_s\mathcal{S}^- \cup \partial_s\mathcal{S}^+. \tag{10.8}$$

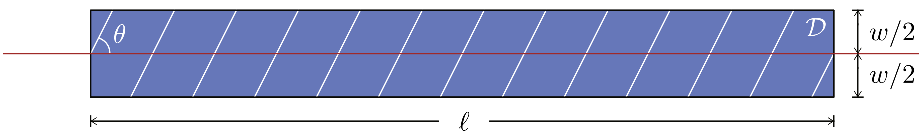

Fig. 1. The material strip $\mathcal{D}$ showing some members of the family of straight parallel lines which become the lines of zero principal curvature of the isometrically deformed strip $\mathcal{S}$, which lies on the right circular cylindrical surface $\mathcal{T}_c$ in Fig. 2.

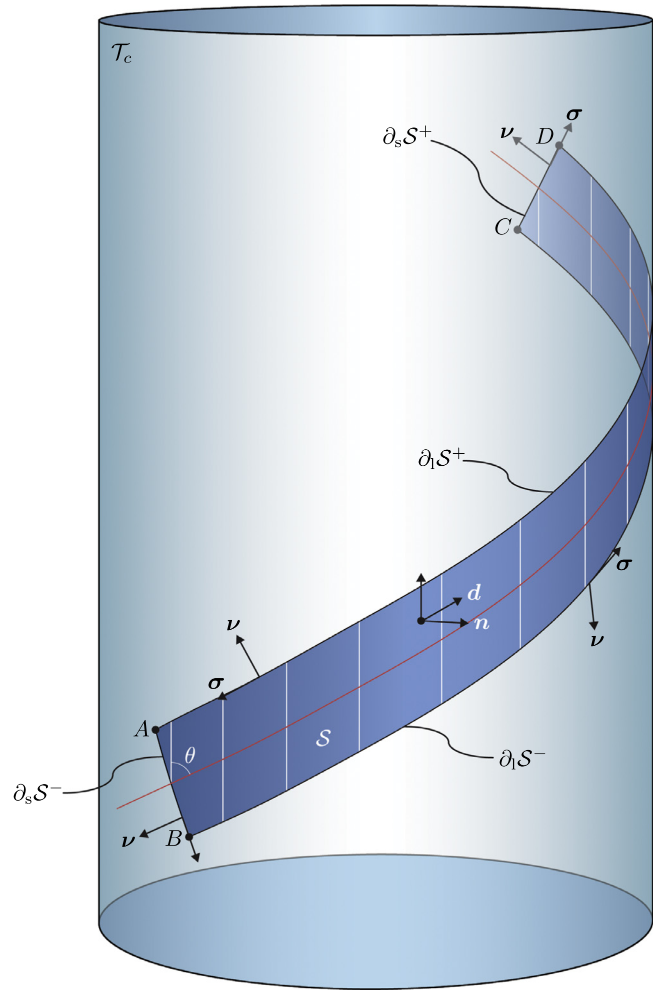

Fig. 2. Spatial configuration $\mathcal{S}$ of an isometrically deformed material strip $\mathcal{D}$ of length ℓ and width w lying on a cylinder $\mathcal{T}_c$ of radius R. The boundary $\partial\mathcal{S}$ is composed of long sides $\partial_l\mathcal{S} = \partial_l\mathcal{S}^+ \cup \partial_l\mathcal{S}^-$ and short sides $\partial_s\mathcal{S} = \partial_s\mathcal{S}^+ \cup \partial_s\mathcal{S}^-$ having oriented unit tanget $\boldsymbol{\sigma}$ and tangent-normal $\boldsymbol{\nu} := \boldsymbol{\sigma} \times \boldsymbol{n}$, where $\boldsymbol{n}$ is the unit normal to $\mathcal{S}$ oriented outward to $\mathcal{T}_c$. The generators of $\mathcal{S}$ are parallel to the unit vector $\boldsymbol{e}$ and $\boldsymbol{d} := \boldsymbol{e} \times \boldsymbol{n}$ is orthogonal to the generators so that $\{\boldsymbol{e}, \boldsymbol{n}, \boldsymbol{d}\}$ is a positively oriented basis for $\mathbb{V}^3$. The constant angle θ, shown here and in Fig. 1, defines the angle between the midline of $\mathcal{S}$ and its generators. (Note: To increase legibility, this figure is not to scale relative to Fig. 1.).

For $\theta = 0$ we see from (10.7) that $T_{de} = \boldsymbol{d} \cdot \boldsymbol{T}\boldsymbol{e} = 0$ on $\partial_l\mathcal{S}$ and from (10.8) that $T_{ee} = \boldsymbol{e} \cdot \boldsymbol{T}\boldsymbol{e} = \mu k^2/2$ on $\partial_s\mathcal{S}$. Similarly, for $\theta = \pi/2$, we see from (10.8) that $T_{de} = 0$ on $\partial_s\mathcal{S}$ and from (10.7) that $T_{ee} = \mu k^2/2$ on $\partial_l\mathcal{S}$. Moreover, since T_{dd}, T_{de}, and T_{ee} are constant in $\mathcal{S}$, we may compute the dot products of (10.7) and (10.8) with $\boldsymbol{e}$, multiply the first of the ensuing identities by $\cos\theta$ and the remaining identity by $\sin\theta$, and add the resulting equations to find that $T_{de} = 0$, for $\theta \neq 0$ and $\theta \neq \pi/2$. For each θ satisfying $0 \le \theta \le \pi/2$, we thus have

$$T_{de} = 0 \quad \text{in } \mathcal{S}. \tag{10.9}$$

With this, if we compute the dot product of either (10.7) or (10.8) with $\boldsymbol{e}$ and recall that $k = -1/R$, we obtain

$$T_{ee} = \frac{\mu k^2}{2} = \frac{\mu}{2R^2} \quad \text{in } \mathcal{S}. \tag{10.10}$$

Finally, since $T_{dd} = \boldsymbol{d} \cdot \boldsymbol{T}\boldsymbol{d}$ is constant in $\mathcal{S}$, we infer that (10.7) and (10.8) are satisfied if their $\boldsymbol{d}$ components vanish, so that

$$-T_{dd}\cos\theta + \frac{\mu k^2 \cos\theta}{2} + k(c_l^\nu \cos\theta + c_l^\sigma \sin\theta) = 0 \tag{10.11a}$$

and

$$-T_{dd}\sin\theta + \frac{\mu k^2 \sin\theta}{2} + k(c_s^\nu \sin\theta - c_s^\sigma \cos\theta) = 0. \tag{10.11b}$$

With the aid of (10.5a), we see from (10.11) that, for $\theta = 0$,

$$T_{dd} = -\frac{\mu k^2}{2}, \qquad c_s^\sigma = 0, \qquad c_l^\sigma \text{ undetermined}, \tag{10.12a}$$

while, for $\theta = \pi/2$,

$$T_{dd} = -\frac{\mu k^2}{2}, \qquad c_l^\sigma = 0, \qquad c_s^\sigma \text{ undetermined}. \tag{10.12b}$$

Furthermore, for $0 < \theta < \pi/2$, (10.5a) and (10.11) yield, respectively,

$$c_l^\sigma = \frac{T_{dd} + \mu k^2/2}{k} \cot\theta - kp\cos\theta\sin\theta \quad \text{on } \partial_l\mathcal{S} \tag{10.13a}$$

and

$$c_s^\sigma = -\frac{T_{dd} + \mu k^2/2}{k} \tan\theta + kp\cos\theta\sin\theta \quad \text{on } \partial_s\mathcal{S}. \tag{10.13b}$$

We now need to determine p for $0 \le \theta \le \pi/2$ and T_{dd} for $0 < \theta < \pi/2$. To do this, we first turn to the normal component (9.21 a) of force balance on $\mathcal{S}$. Because $\boldsymbol{L} = k\boldsymbol{d}\otimes\boldsymbol{d}$, $k = 2H$, and $\Delta_{\mathcal{S}} p = \boldsymbol{1}_{\mathcal{S}} \cdot \operatorname{grad}_{\mathcal{S}} \operatorname{grad}_{\mathcal{S}} p$, we readily see that (9.21 a) may be written as

$$\boldsymbol{e} \cdot (\operatorname{grad}_{\mathcal{S}} \operatorname{grad}_{\mathcal{S}} p)\boldsymbol{e} = T_{dd} + \frac{\mu k^2}{2} \quad \text{on } \mathcal{S}, \tag{10.14}$$

which, because T_{dd} is constant on $\mathcal{S}$, leads us to conclude that the gradient of $\boldsymbol{e} \cdot \operatorname{grad}_{\mathcal{S}} p$ along each generator of $\mathcal{S}$ must be constant.

Next, let us, again, recall that $\boldsymbol{L} = k\boldsymbol{d}\otimes\boldsymbol{d}$, and observe, using the geometry shown in Figs. 1 and 2, that for each θ satisfying $0 \le \theta \le \pi/2$ we have

$$p\boldsymbol{\sigma} \cdot \boldsymbol{L}\boldsymbol{\nu} = pk(\boldsymbol{\nu}\cdot\boldsymbol{d})(\boldsymbol{\sigma}\cdot\boldsymbol{d}) = \begin{cases} pk\cos\theta\sin\theta & \text{on } \partial_l\mathcal{S}, \\ -pk\cos\theta\sin\theta & \text{on } \partial_s\mathcal{S}. \end{cases} \tag{10.15}$$

We next consider the normal component (9.21 c) of force balance on $\partial\mathcal{S}$, first considering $0 < \theta < \pi/2$ and leaving for later the special cases $\theta = 0$ and $\theta = \pi/2$. For $0 < \theta < \pi/2$, we see from (10.13) and (10.15) that

$$\boldsymbol{\sigma} \cdot (p\boldsymbol{L}\boldsymbol{\nu} + \boldsymbol{c}) = \begin{cases} \dfrac{T_{dd} + \mu k^2/2}{k} \cot\theta & \text{on } \partial_l\mathcal{S}, \\[2ex] -\dfrac{T_{dd} + \mu k^2/2}{k} \cot\theta & \text{on } \partial_s\mathcal{S}. \end{cases} \tag{10.16}$$

Since the alternatives on the right-hand side of (10.16) are constant, it follows that $\partial(\boldsymbol{\sigma}\cdot(p\boldsymbol{L}\boldsymbol{\nu} + \boldsymbol{c}))/\partial s = 0$ in (9.21 c). In addition, since $\boldsymbol{t} = \boldsymbol{0}$ on $\partial\mathcal{S}$, $\boldsymbol{L} = k\boldsymbol{d}\otimes\boldsymbol{d}$ with constant $k = 2H \neq 0$, and $2H\boldsymbol{1}_{\mathcal{S}} - \boldsymbol{L} = k\boldsymbol{e}\otimes\boldsymbol{e}$, we see that (9.21 c) reduces to

$$\boldsymbol{e} \cdot \operatorname{grad}_{\mathcal{S}} p = 0 \quad \text{on } \partial\mathcal{S}. \tag{10.17}$$

But, we showed, from (10.14), that the gradient of $\boldsymbol{e} \cdot \operatorname{grad}_{\mathcal{S}} p$ must be constant along each generator of $\mathcal{S}$. Thus, it follows that, for $0 < \theta < \pi/2$,

$$\boldsymbol{e} \cdot \operatorname{grad}_{\mathcal{S}} p = 0 \quad \text{on } \mathcal{S} \tag{10.18}$$

and, thus, that

$$p = \hat{p}(\varphi) \quad \text{on } \mathcal{S}, \tag{10.19}$$

where φ denotes the polar angle measured positively from any fixed point on the cylinder $\mathcal{T}_c$ in Fig. 2, in the right-hand sense relative to the axis $\boldsymbol{e}$. Consequently, from (10.14) and the relation $k = -1/R$, we find that

$$T_{dd} = -\frac{\mu k^2}{2} = -\frac{\mu}{2R^2} \quad \text{on } \mathcal{S}. \tag{10.20}$$

Thus, for $0 < \theta < \pi/2$, we see from (10.13) that

$$\left.\begin{aligned} c_l^\sigma &= -k\hat{p}(\varphi)\cos\theta\sin\theta \quad \text{on } \partial_l\mathcal{S}, \\ c_s^\sigma &= k\hat{p}(\varphi)\cos\theta\sin\theta \quad \text{on } \partial_s\mathcal{S}, \end{aligned}\right\} \tag{10.21}$$

and from (10.5a) that

$$\left.\begin{aligned} c_l^\nu &= -k(\mu - \hat{p}(\varphi)\sin^2\theta) \quad \text{on } \partial_l\mathcal{S}, \\ c_s^\nu &= -k(\mu - \hat{p}(\varphi)\cos^2\theta) \quad \text{on } \partial_s\mathcal{S}. \end{aligned}\right\} \tag{10.22}$$

10.2. Global moment balance

We have satisfied all of the Euler–Lagrange equations (9.21) for an arbitrary choice of the function $\hat{p}$ determining p in terms of φ. Although the global force balance $(4.8)_1$ holds trivially for $\boldsymbol{t} = \boldsymbol{0}$, it still remains to ensure that the global moment balance $(4.8)_2$ is satisfied. For $\boldsymbol{t} = \boldsymbol{0}$, that condition reduces to

$$\int_{\partial\mathcal{S}} \boldsymbol{m}\,\mathrm{d}s = \boldsymbol{0}. \tag{10.23}$$

Now, from the definition (2.13) of the specific moment $\boldsymbol{m}$, we have that

$$\boldsymbol{m} = \begin{cases} \boldsymbol{m}_\mathrm{l} := -c_\mathrm{l}^\sigma\,\boldsymbol{\nu} + c_\mathrm{l}^\nu\boldsymbol{\sigma} & \text{on } \partial_\mathrm{l}\mathcal{S}, \\ \boldsymbol{m}_\mathrm{s} := -c_\mathrm{s}^\sigma\,\boldsymbol{\nu} + c_\mathrm{s}^\nu\boldsymbol{\sigma} & \text{on } \partial_\mathrm{s}\mathcal{S}, \end{cases} \tag{10.24}$$

where we may substitute (10.21) and (10.22) as needed. To proceed further, it is convenient to express $\boldsymbol{m}$ in terms of its components in the directions of $\boldsymbol{d}$ and $\boldsymbol{e}$; with this in mind we see, with the aid of the geometry expressed in Figs. 1 and 2, that

$$\left.\begin{aligned} \boldsymbol{m}_\mathrm{l}\cdot\boldsymbol{d} &= \pm(c_\mathrm{l}^\sigma\cos\theta - c_\mathrm{l}^\nu\sin\theta) = \pm k(\mu - \hat{p}(\varphi))\sin\theta \qquad \text{on } \partial_\mathrm{l}\mathcal{S}^\pm, \\ \boldsymbol{m}_\mathrm{s}\cdot\boldsymbol{d} &= \pm(-c_\mathrm{s}^\sigma\sin\theta - c_\mathrm{s}^\nu\cos\theta) = \pm k(\mu - \hat{p}(\varphi))\cos\theta \quad \text{on } \partial_\mathrm{s}\mathcal{S}^\pm, \end{aligned}\right\} \tag{10.25}$$

and that

$$\left.\begin{aligned} \boldsymbol{m}_\mathrm{l}\cdot\boldsymbol{e} &= \pm\mu k\cos\theta \quad \text{on } \partial_\mathrm{l}\mathcal{S}^\pm, \\ \boldsymbol{m}_\mathrm{s}\cdot\boldsymbol{e} &= \pm\mu k\sin\theta \quad \text{on } \partial_\mathrm{s}\mathcal{S}^\pm. \end{aligned}\right\} \tag{10.26}$$

Thus, we may write

$$\begin{aligned} \int_{\partial\mathcal{S}} \boldsymbol{m}\,\mathrm{d}s &= \int_{\partial_\mathrm{l}\mathcal{S}} \boldsymbol{m}_\mathrm{l}\,\mathrm{d}s + \int_{\partial_\mathrm{s}\mathcal{S}} \boldsymbol{m}_\mathrm{s}\,\mathrm{d}s \\ &= \int_{\partial_\mathrm{l}\mathcal{S}} \big((\boldsymbol{m}_\mathrm{l}\cdot\boldsymbol{e})\boldsymbol{e} + (\boldsymbol{m}_\mathrm{l}\cdot\boldsymbol{d})\boldsymbol{d}\big)\,\mathrm{d}s + \int_{\partial_\mathrm{s}\mathcal{S}} \big((\boldsymbol{m}_\mathrm{s}\cdot\boldsymbol{e})\boldsymbol{e} + \boldsymbol{m}_\mathrm{s}\cdot\boldsymbol{d})\boldsymbol{d}\big)\,\mathrm{d}s \\ &= \int_{\partial_\mathrm{l}\mathcal{S}} (\boldsymbol{m}_\mathrm{l}\cdot\boldsymbol{d})\boldsymbol{d}\,\mathrm{d}s + \int_{\partial_\mathrm{s}\mathcal{S}} (\boldsymbol{m}_\mathrm{s}\cdot\boldsymbol{d})\boldsymbol{d}\,\mathrm{d}s, \end{aligned} \tag{10.27}$$

the latter simplification arising because, by (10.26) and the constancy of $\boldsymbol{e}$ on $\partial\mathcal{S}$,

$$\int_{\partial_\mathrm{l}\mathcal{S}} (\boldsymbol{m}_\mathrm{l}\cdot\boldsymbol{e})\boldsymbol{e}\,\mathrm{d}s = \int_{\partial_\mathrm{s}\mathcal{S}} (\boldsymbol{m}_\mathrm{s}\cdot\boldsymbol{e})\boldsymbol{e}\,\mathrm{d}s = \boldsymbol{0}.$$

Moreover, by (10.25) and because $\hat{p}$ depends only on φ, we see that

$$\begin{aligned} \int_{\partial\mathcal{S}} \boldsymbol{m}\,\mathrm{d}s &= -k\sin\theta\Big(\int_{\partial_\mathrm{l}\mathcal{S}^-} (\mu - \hat{p}(\varphi))\boldsymbol{d}\,\mathrm{d}s - \int_{\partial_\mathrm{l}\mathcal{S}^+} (\mu - \hat{p}(\varphi))\boldsymbol{d}\,\mathrm{d}s\Big) \\ &\qquad - k\cos\theta\Big(\int_{\partial_\mathrm{s}\mathcal{S}^-} (\mu - \hat{p}(\varphi))\boldsymbol{d}\,\mathrm{d}s - \int_{\partial_\mathrm{s}\mathcal{S}^+} (\mu - \hat{p}(\varphi))\boldsymbol{d}\,\mathrm{d}s\Big) \\ &= -kR\Big(\int_{\varphi_B}^{\varphi_C} (\mu - \hat{p}(\varphi))\boldsymbol{d}\,\mathrm{d}\varphi - \int_{\varphi_A}^{\varphi_D} (\mu - \hat{p}(\varphi))\boldsymbol{d}\,\mathrm{d}\varphi\Big) \\ &\qquad - kR\Big(\int_{\varphi_A}^{\varphi_B} (\mu - \hat{p}(\varphi))\boldsymbol{d}\,\mathrm{d}\varphi - \int_{\varphi_D}^{\varphi_C} (\mu - \hat{p}(\varphi))\boldsymbol{d}\,\mathrm{d}\varphi\Big) \\ &= -kR\Big(\int_{\varphi_A}^{\varphi_C} (\mu - \hat{p}(\varphi))\boldsymbol{d}\,\mathrm{d}\varphi - \int_{\varphi_A}^{\varphi_C} (\mu - \hat{p}(\varphi))\boldsymbol{d}\,\mathrm{d}\varphi\Big) \\ &= \boldsymbol{0}, \end{aligned} \tag{10.28}$$

where R is the radius of the cylinder $\mathcal{T}_\mathrm{c}$ and we have used the identities $\sin\theta\,\mathrm{d}s = R\,\mathrm{d}\varphi$ on $\partial_\mathrm{l}\mathcal{S}$ and $\cos\theta\,\mathrm{d}s = R\,\mathrm{d}\varphi$ on $\partial_\mathrm{s}\mathcal{S}$. Also, we have identified the corners A, B, C, and D of $\mathcal{S}$ with the associated angles φ_A, φ_B, φ_C, and φ_D in Fig. 2. Thus, we conclude that the moment balance (10.23) is identically satisfied for any choice of the function $p = \hat{p}(\varphi)$.

We have completed the main calculations which result in a solution, parameterized by $\hat{p}$, of the Euler–Lagrange equations (9.21) for $0 < \theta < \pi/2$. Before summarizing and interpreting the results, let us briefly resolve the special cases $\theta = 0$ and $\theta = \pi/2$.

For $\theta = \pi/2$, the generators of $\mathcal{S}$, being parallel to the axis $\boldsymbol{e}$ of the cylinder $\mathcal{T}_\mathrm{c}$, are parallel to the short edges $\partial_\mathrm{s}\mathcal{S}$, which means that the long edges of the reference rectangular strip are being bent around the cylinder $\mathcal{T}_\mathrm{c}$ to form $\mathcal{S}$. As before, we find, following our earlier argument leading to (10.19), that p is a function $\hat{p}$ of φ. To go forward, recall from (10.12) that for $\theta = \pi/2$, $T_{dd} = -\mu k^2/2$, $c^\sigma_l = 0$, and c^σ_s is undetermined. Then, (10.5a) requires that

$$c^\nu_l = -k(\mu - \hat{p}(\varphi)), \qquad c^\nu_s = -\mu k, \tag{10.29}$$

and, consequently, from (10.24), we see that

$$\boldsymbol{m} = \begin{cases} \boldsymbol{m}_l := -k(\mu - \hat{p}(\varphi))\boldsymbol{\sigma} & \text{on } \partial_l\mathcal{S}, \\ \boldsymbol{m}_s := -c^\sigma_s\,\boldsymbol{\nu} - \mu k\boldsymbol{\sigma} & \text{on } \partial_s\mathcal{S}. \end{cases} \tag{10.30}$$

Further, to be consistent with $(10.21)_2$ in the limit $\theta \to \pi/2$ (recall, here, that $k = -1/R$), we determine that $c^\sigma_s = 0$, which implies that there is zero applied twisting moment on the short edges $\partial_s\mathcal{S}$. Thus, we find that

$$\boldsymbol{m} = \begin{cases} \boldsymbol{m}_l := -k(\mu - \hat{p}(\varphi))\boldsymbol{\sigma} & \text{on } \partial_l\mathcal{S}, \\ \boldsymbol{m}_s := -\mu k\boldsymbol{\sigma} & \text{on } \partial_s\mathcal{S}, \end{cases} \tag{10.31}$$

and, because $\boldsymbol{\sigma} = \pm\boldsymbol{d}$ on $\partial_l\mathcal{S}^{\mp}$, and because $\boldsymbol{\sigma} = \pm\boldsymbol{e}$ on $\partial_s\mathcal{S}^{\pm}$, we conclude that

$$\int_{\partial\mathcal{S}} \boldsymbol{m}\,\mathrm{d}s = \int_{\partial_l\mathcal{S}} \boldsymbol{m}_l\,\mathrm{d}s = -k\int_{\partial_l\mathcal{S}} (\mu - \hat{p}(\varphi))\boldsymbol{\sigma}\,\mathrm{d}s = \boldsymbol{0}, \tag{10.32}$$

the latter because the two integrals over $\partial_l\mathcal{S} = \partial_l\mathcal{S}^- \cup \partial_l\mathcal{S}^+$ mutually cancel. From this, we conclude that the moment balance (10.23) is, again, satisfied for $\theta = \pi/2$ for any choice of $\hat{p}$. Finally, for $\theta = \pi/2$, we observe that the boundary condition (9.20 e) is identically satisfied because $\boldsymbol{t} = \boldsymbol{0}$, $c^\sigma_s = c^\sigma_l = 0$ implies that $\boldsymbol{c}\cdot\boldsymbol{\sigma} = 0$, $\boldsymbol{L} = k\boldsymbol{d}\otimes\boldsymbol{d}$ with $k = $ constant, and (10.16) implies that $\boldsymbol{\sigma}\cdot(p\boldsymbol{L}\boldsymbol{\nu} + \boldsymbol{c}) = 0$ on $\partial\mathcal{S}$. Thus, we conclude that even in the case $\theta = \pi/2$, (10.20) correctly determines $T_{dd} = -\mu k^2/2$, (10.21) correctly determines $c^\sigma_l = c^\sigma_s = 0$, (10.22) correctly determines $c^\nu_s = -\mu k$ and $c^\nu_l = -k(\mu - \hat{p}(\varphi))$, and (10.24) yields (10.31) for the applied specific moment on $\partial_l\mathcal{S}$ and $\partial_s\mathcal{S}$. Moreover, the moment balance (10.23) is satisfied.

A similar analysis yields, again, the conclusion that (10.20), (10.21), (10.22), and (10.24) also apply for $\theta = 0$, and, moreover, that the moment balance (10.23) holds. Thus, we may apply (10.20), (10.21), (10.22), and (10.24), including (10.25) and (10.26), for all choices of θ satisfying $0 \le \theta \le \pi/2$. At this point, the Lagrangian multiplier $\boldsymbol{T}$ has the form

$$\boldsymbol{T} = -\frac{\mu}{2R^2}(\boldsymbol{d}\otimes\boldsymbol{d} - \boldsymbol{e}\otimes\boldsymbol{e}), \tag{10.33}$$

where the unit vector $\boldsymbol{e}$ defines the axis and generators of the cylinder $\mathcal{T}_c$ whose radius is R, and $\boldsymbol{d} := \boldsymbol{e}\times\boldsymbol{n}$, with $\boldsymbol{n}$ being the outward unit normal to $\mathcal{T}_c$, upon which $\mathcal{S}$ lies. However, the manner in which the Lagrangian multiplier $p = \hat{p}(\varphi)$ depends on φ remains indeterminate. Consequently, the applied specific moment $\boldsymbol{m}$ on $\partial\mathcal{S}$, given by (10.21), (10.22), and (10.24), is parameterized by this indeterminacy, and, within the class of isometric deformations, the deformed material ribbon $\mathcal{S}$ will remain in the same fixed cylindrical helical deformed shape for any applied specific moment on $\partial\mathcal{S}$ of this form. The material ribbon is unstretchable and this accounts for the indeterminacy of $\boldsymbol{m}$ within the parametrization $\hat{p}$. Stated another way, within this indeterminacy of $\boldsymbol{m}$, if the material ribbon may only deform isometrically from its configuration $\mathcal{S}$, it cannot deform at all.

In the case $\theta = \pi/2$, where the reference rectangular strip is bent around its long edges into a right cylindrical strip of radius R, we see from (10.31) that this bending may be accomplished by applying solely the uniquely determined opposing specific bending moments $\boldsymbol{m}_s$ on the short edges $\partial_s\mathcal{S}$ provided the specific bending moments $\boldsymbol{m}_l$ on the long edges $\partial_l\mathcal{S}$ vanish. Clearly, this requires that

$$p = \mu, \tag{10.34}$$

which is a possible (and natural) choice for $p = \hat{p}(\varphi)$.

More generally, for $0 \le \theta \le \pi/2$, we note that the bending of the rectangular strip $\mathcal{D}$ into the helical form $\mathcal{S} \subset \mathcal{T}_c$ is uniform and takes place around the generators $\boldsymbol{e}$ of $\mathcal{T}_c$. In accord with (10.25) and (10.26), all such uniform bendings can be accomplished by applying particular specific moments on $\partial\mathcal{S}$ that are parallel to the generators $\boldsymbol{e}$ of $\mathcal{T}_c$, as is natural, by choosing p in accord with (10.34). Specifically, for this requirement we see, from (10.21) and (10.22), that

$$\left.\begin{aligned} c^\nu_l &= -\mu k\cos^2\theta, \\ c^\nu_s &= -\mu k\sin^2\theta, \end{aligned}\right\} \tag{10.35a}$$

and

$$\left.\begin{aligned} c^\sigma_l &= -4\mu k\sin\theta\cos\theta, \\ c^\sigma_s &= \;\; 4\mu k\sin\theta\cos\theta. \end{aligned}\right\} \tag{10.35b}$$

Accordingly, after recalling that $k = -1/R$, we then determine from (10.24) that the applied specific moment $\boldsymbol{m}$ on $\partial\mathcal{S}$ has the form

$$\boldsymbol{m} = \begin{cases} \boldsymbol{m}_l := \dfrac{\mu\cos\theta}{R}(\boldsymbol{\sigma}\cos\theta - \boldsymbol{\nu}\sin\theta) & \text{on } \partial_l\mathcal{S}, \\[2ex] \boldsymbol{m}_s := \dfrac{\mu\sin\theta}{R}(\boldsymbol{\sigma}\sin\theta + \boldsymbol{\nu}\cos\theta) & \text{on } \partial_s\mathcal{S}, \end{cases} \tag{10.36}$$

which shows that the boundary $\partial\mathcal{S}$ is subject to a specific bending moment around the tangent $\boldsymbol{\sigma}$ and a specific twisting moment around the edge tangent-normal $\boldsymbol{\nu}$.

The isometric deformation of an undistorted planar rectangular strip $\mathcal{D}$ to the circular cylindrical helical form $\mathcal{S}$ that we have characterized in this section as a solution of the constrained variational problem (9.5) governed by the Euler–Lagrange equations (9.21) was previously described, from a purely kinematical perspective, by Chen et al. (2018, Section 7.1). For the particular choice (10.34) of $p = \hat{p}(\varphi)$, we have, by (10.36) and (10.26), that

$$|\boldsymbol{m}| = \begin{cases} |\boldsymbol{m}_{\rm l}| = |\boldsymbol{m}_{\rm l}\cdot\boldsymbol{e}| = \dfrac{\mu\cos\theta}{R} & \text{on } \partial_{\rm l}\mathcal{S}, \\ |\boldsymbol{m}_{\rm s}| = |\boldsymbol{m}_{\rm s}\cdot\boldsymbol{e}| = \dfrac{\mu\sin\theta}{R} & \text{on } \partial_{\rm s}\mathcal{S} \end{cases} \tag{10.37}$$

for each θ satisfying $0 \le \theta \le \pi/2$, and for $p = \mu$ we have, by (10.25), that

$$\boldsymbol{m}\cdot\boldsymbol{d} = 0 \quad \text{on } \partial\mathcal{S} = \partial_{\rm l}\mathcal{S}\cup\partial_{\rm s}\mathcal{S}, \tag{10.38}$$

which emphasizes that the applied specific moment $\boldsymbol{m}$ is colinear with $\boldsymbol{e}$ at all points on the boundary $\partial\mathcal{S}$ and that the deformed strip $\mathcal{S}$ is bent around its generators.

Finally, given $|\boldsymbol{m}_{\rm s}| \ge 0$ and $|\boldsymbol{m}_{\rm l}| \ge 0$, we note that the angle θ between the axis $\boldsymbol{e}$ and the midline of the helical wrap is determined by

$$\tan\theta = \frac{|\boldsymbol{m}_{\rm s}|}{|\boldsymbol{m}_{\rm l}|}, \qquad 0 \le \theta \le \frac{\pi}{2}, \tag{10.39}$$

and the radius $R > 0$ is determined by

$$R = \frac{\mu}{\sqrt{|\boldsymbol{m}_{\rm s}|^2 + |\boldsymbol{m}_{\rm l}|^2}}. \tag{10.40}$$

Thus, for $|\boldsymbol{m}_{\rm l}| = 0$, the angle θ must be $\theta = \pi/2$, while, for $|\boldsymbol{m}_{\rm s}| > 0$, the deformed strip $\mathcal{S}$ is circularly bent around the cylinder of radius $\mu/|\boldsymbol{m}_{\rm s}|$. As $|\boldsymbol{m}_{\rm s}|$ increases, the radius R decreases. Further, note that for fixed $|\boldsymbol{m}_{\rm s}|$, as $|\boldsymbol{m}_{\rm l}|$ is increased from 0 the angle θ decreases from $\pi/2$ and approaches 0 as $|\boldsymbol{m}_{\rm l}|$ approaches $+\infty$, with a corresponding decrease in the radius R. Clearly, $\mathcal{S}$ may overlap in this scenario. The tedious calculations to prevent such events by limiting the width w and length ℓ of the reference rectangle $\mathcal{D}$ relative to the specific moment that is applied to $\partial\mathcal{S}$ are, of course, possible, but are not considered of major importance to the overall variational problem.

11. Solution of the constrained variational problem for the quadratic stored energy density: Rectangular strip to a conical band

Consider a flat, undistorted rectangular material strip $\mathcal{D}$ with stored energy density W of the quadratic form (8.1) that is isometrically deformed to a portion $\mathcal{S}$ of a right circular conical surface $\mathcal{K}$ with tip angle 2φ, where $0 < \varphi < \pi/2$. The kinematics of this deformation have been described by Chen et al. (2018, Section 9.1), and we will call upon relevant details from that work in the following development. As in Section 10, we suppose that the traction $\boldsymbol{t}$ and the couple traction $\boldsymbol{c}$ are given everywhere on $\partial\mathcal{S}$. Thus, in the notation of (9.21), $\partial_2\mathcal{S}$ corresponds to the complete boundary $\partial\mathcal{S}$ of the deformed strip. As explained in Section 10, we set $\partial\mathcal{S} = \partial_{\rm l}\mathcal{S}\cup\partial_{\rm s}\mathcal{S}$ where $\partial_{\rm l}\mathcal{S}$ and $\partial_{\rm s}\mathcal{S}$ denote, respectively, the two long and two short edges of $\mathcal{S}$. We also consider the same loading system as described in Section 10.

In Fig. 3, we show the reference configuration of the rectangular strip $\mathcal{D}$ and in Fig. 4 we show the spatial configuration of a typical isometrically deformed surface from $\mathcal{D}$ to $\mathcal{S}\subset\mathcal{K}$. We assume throughout this investigation that θ_0 satisfies

$$0 < \theta_0 < \pi/2. \tag{11.1}$$

In Fig. 4, we show at each point on the conical surface $\mathcal{K}$ the positively oriented orthonormal basis $\{\boldsymbol{e}, \boldsymbol{n}, \boldsymbol{d}\}$ for $\mathbb{V}^3$, where $\boldsymbol{e}$ is directed along a generator of $\mathcal{K}$ pointing toward its apex, $\boldsymbol{n}$ is normal to $\mathcal{K}$, and $\boldsymbol{d} := \boldsymbol{e}\times\boldsymbol{n}$. Of course, at each point of $\mathcal{K}$, $\{\boldsymbol{d}, \boldsymbol{e}\}$ is an orthonormal basis for the tangent space in $\mathbb{E}^2$ associated with the conical surface $\mathcal{K}$. Note that at every point on $\mathcal{K}$ the curvature tensor for $\mathcal{K}$ may be written in the form $\boldsymbol{L} = -\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{n} = k\,\boldsymbol{d}\otimes\boldsymbol{d}$, where k is the nonzero principal curvature of $\mathcal{K}$. On the boundary $\partial\mathcal{S}$ of $\mathcal{S}$, we shall assume that the orthonormal triad $\{\boldsymbol{\sigma}, \boldsymbol{n}, \boldsymbol{\nu}\}$ is positively oriented, where $\boldsymbol{\sigma}$ is the oriented tangent to $\partial\mathcal{S}$ and $\boldsymbol{\nu} := \boldsymbol{\sigma}\times\boldsymbol{n}$ is the tangent-normal. Finally, we seek a solution to the Euler–Lagrange equations (9.21) such that the applied specific moment $\boldsymbol{m} := \boldsymbol{n}\times\boldsymbol{c}$ is balanced in the sense that $(4.8)_2$ holds with $\boldsymbol{t} = \boldsymbol{0}$.

Note that the rulings of $\mathcal{D}$ lie on a family of intersecting straight lines in the plane of $\mathcal{D}$ and that the image of each such ruling is a straight line in $\mathcal{S}$ that coincides with a generator of $\mathcal{K}$. These straight lines are the generators of $\mathcal{S}$ and its midline $\mathcal{C}$ corresponds to the deformed image of the midline of $\mathcal{D}$. Also, the apex of $\mathcal{K}$ corresponds to the mapped image of the point in the plane of $\mathcal{D}$ where the family of straight lines containing the rulings of $\mathcal{D}$ intersect.

11.1. Consequences of force balance on $\mathcal{S}$

In this development, we first consider the tangential component (9.21 b) of force balance on $\mathcal{S}$ and, as in Section 10, we use the basis $\{\boldsymbol{e}, \boldsymbol{n}, \boldsymbol{d}\}$ shown in Fig. 4 to write

$$\boldsymbol{T} = T_{dd}\boldsymbol{d}\otimes\boldsymbol{d} + T_{de}(\boldsymbol{d}\otimes\boldsymbol{e} + \boldsymbol{e}\otimes\boldsymbol{d}) + T_{ee}\,\boldsymbol{e}\otimes\boldsymbol{e}. \tag{11.2}$$

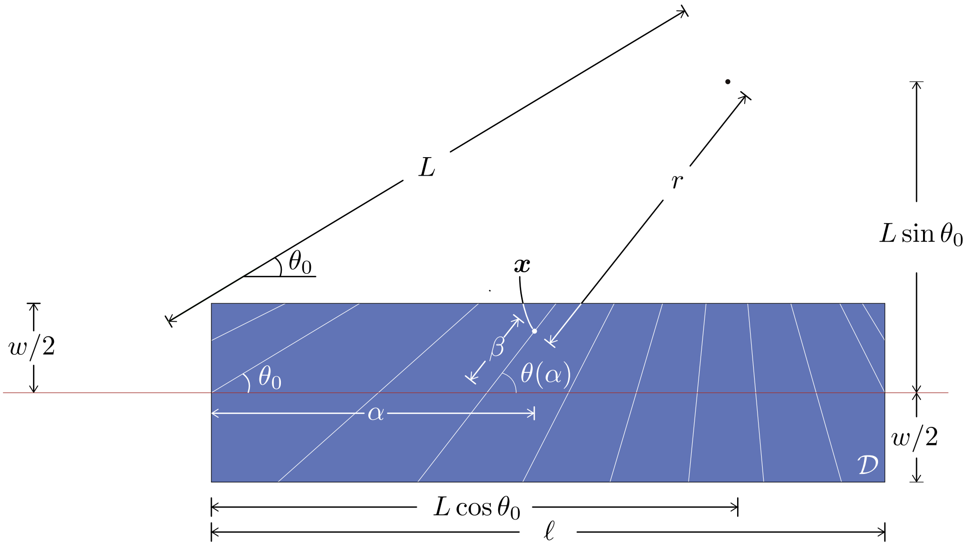

Fig. 3. A rectangular material strip $\mathcal{D}$ of length ℓ and width w showing some members of the family of intersecting straight lines which become the lines of zero principal curvature of the isometrically deformed material surface $\mathcal{S}$ which lies on a right circular conical surface $\mathcal{K}$. Here, L is the length of a generator of the portion of the right circular conical surface $\mathcal{K}$ that appears in Fig. 4.

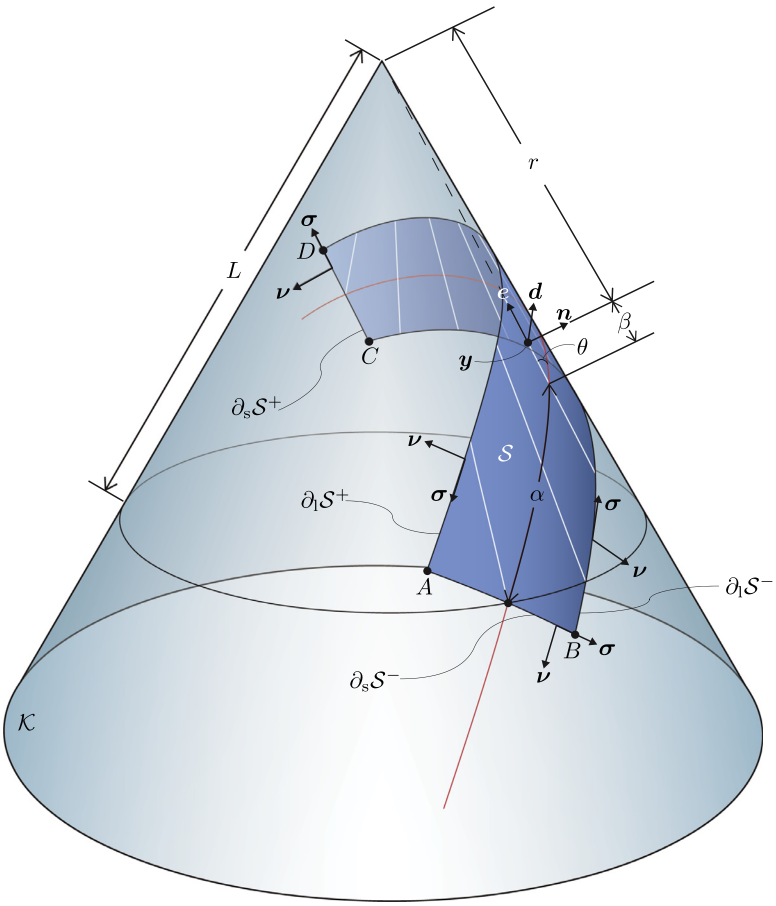

Fig. 4. Spatial configuration $\mathcal{S}$ of the deformed strip lying on the conical surface $\mathcal{K}$. The boundary $\partial\mathcal{S}$ is composed of long sides $\partial_l\mathcal{S} = \partial_l\mathcal{S}^+ \cup \partial_l\mathcal{S}^-$ and short sides $\partial_s\mathcal{S} = \partial_s\mathcal{S}^+ \cup \partial_s\mathcal{S}^-$ having oriented unit tanget $\boldsymbol{\sigma}$ and tangent-normal $\boldsymbol{\nu} := \boldsymbol{\sigma} \times \boldsymbol{n}$, where $\boldsymbol{n}$ is the unit normal to $\mathcal{S}$ oriented outward to $\mathcal{K}$. The generators of $\mathcal{S}$ are parallel to the unit vector $\boldsymbol{e}$ and $\boldsymbol{d} := \boldsymbol{e} \times \boldsymbol{n}$ is orthogonal to the generators so that $\{\boldsymbol{e}, \boldsymbol{n}, \boldsymbol{d}\}$ is a positively oriented basis for $\mathbb{V}^3$ at the generic spatial point $\boldsymbol{y}$ on $\mathcal{S}$. The angle θ, and distances α, β, L, and r shown here are the same as those in Fig. 3, and $\boldsymbol{y}$ is supposed to be the deformed position of the material particle $\boldsymbol{x}$ shown in Fig. 3. (Note: To increase legibility, this figure is not to scale relative to Fig. 3.).

Then, as shown in the Appendix A.2.1, we may write (9.21 b) in the form

$$\Big(\boldsymbol{d}\cdot\mathrm{grad}_{\mathcal{S}}T_{dd}+\boldsymbol{e}\cdot\mathrm{grad}_{\mathcal{S}}T_{de}-\frac{2T_{de}}{r}\Big)\boldsymbol{d}+\Big(\boldsymbol{d}\cdot\mathrm{grad}_{\mathcal{S}}T_{de}+\boldsymbol{e}\cdot\mathrm{grad}_{\mathcal{S}}T_{ee}+\frac{T_{dd}-T_{ee}}{r}\Big)\boldsymbol{e}=\boldsymbol{0}\quad\text{on }\mathcal{S}. \tag{11.3}$$

In the following, we shall assume, as seems reasonable, that $\{\boldsymbol{d},\boldsymbol{e}\}$ is a principal eigenvector basis for $\boldsymbol{T}$, so that $T_{de}=0$. This being so, (11.3) becomes the requirement that

$$\boldsymbol{d}\cdot\mathrm{grad}_{\mathcal{S}}T_{dd}=0\qquad\text{and}\qquad\boldsymbol{e}\cdot\mathrm{grad}_{\mathcal{S}}T_{ee}+\frac{T_{dd}-T_{ee}}{r}=0\quad\text{on }\mathcal{S}; \tag{11.4}$$

more explicitly, taking note from Fig. 4 that r is the distance measured from the tip of the conical surface $\mathcal{K}$ to a generic point $\boldsymbol{y}$ in $\mathcal{S}$, and that the base vector $\boldsymbol{e}$ is directed toward that tip (in the direction of decreasing r) of $\mathcal{K}$, we conclude that (9.21 b) can be expressed as two conditions,

$$T_{dd}\text{ depends at most on }r\qquad\text{and}\qquad\frac{\partial T_{ee}}{\partial r}-\frac{T_{dd}-T_{ee}}{r}=0\quad\text{on }\mathcal{S}, \tag{11.5}$$

to which we shall return later.[14]

Next, we reduce the normal component (9.21 a) of force balance on $\mathcal{S}$ to a manageable differential equation in terms of the single indpendent variable r. To begin, we see from (A.29) that

$$\begin{aligned}\mathrm{grad}_{\mathcal{S}}H&=-\frac{1}{2}\mathrm{grad}_{\mathcal{S}}\Big(\frac{\cot\varphi}{r}\Big)\\&=\frac{1}{2}\frac{d}{dr}\Big(\frac{\cot\varphi}{r}\Big)\boldsymbol{e}\\&=-\frac{\cot\varphi}{2r^2}\boldsymbol{e}.\end{aligned} \tag{11.6}$$

Thus, with the aid of (A.9) we have

$$\begin{aligned}\Delta_{\mathcal{S}}H&=-\frac{1}{2}\cot\varphi\,\mathrm{div}_{\mathcal{S}}\Big(\frac{1}{r^2}\boldsymbol{e}\Big)\\&=-\frac{1}{2}\cot\varphi\Big(\mathrm{grad}_{\mathcal{S}}\Big(\frac{1}{r^2}\Big)\boldsymbol{e}+\frac{1}{r^2}\mathrm{div}_{\mathcal{S}}\,\boldsymbol{e}\Big)\\&=-\frac{1}{2}\cot\varphi\Big(-\frac{d}{dr}\Big(\frac{1}{r^2}\Big)-\frac{1}{r^3}\Big)\\&=-\frac{\cot\varphi}{2r^3},\end{aligned} \tag{11.7}$$

and with these last two results we see that (9.21 a) becomes

$$\mu\Big(\frac{\cot\varphi}{r^3}+\frac{\cot^3\varphi}{r^3}\Big)+\frac{\cot\varphi}{r}T_{dd}-\frac{\cot\varphi}{r}\boldsymbol{e}\cdot(\mathrm{grad}_{\mathcal{S}}\,\mathrm{grad}_{\mathcal{S}}p)\boldsymbol{e}=0\quad\text{in }\mathcal{S}. \tag{11.8}$$

Moreover, since $\cot\varphi>0$ for $0<\varphi<\pi/2$ and since $\boldsymbol{e}\cdot(\mathrm{grad}_{\mathcal{S}}\,\mathrm{grad}_{\mathcal{S}}p)\boldsymbol{e}=\partial^2p/\partial r^2$, we find that (9.21 a) can be written in the reduced form

$$\mu\Big(1+\frac{\cot^2\varphi}{2}\Big)\frac{1}{r^3}+\frac{T_{dd}}{r}-\frac{1}{r}\frac{\partial^2p}{\partial r^2}=0\quad\text{in }\mathcal{S}. \tag{11.9}$$

11.2. Consequence of the edge conditions

Having reduced the field equations (9.21 a) and (9.21 b) expressing the normal and tangential components of force balance on $\mathcal{S}$ to their respective elementary forms (11.9) and (11.5), we turn to the remaining conditions (9.21 c), (9.21 d), and (9.21 e) that respectively express the normal component of force balance on $\partial\mathcal{S}$, the tangential component of force balance on $\partial\mathcal{S}$, and the tangent-normal component of couple balance on $\partial\mathcal{S}$. First, we use (A.29) to rewrite (9.21 e) as

$$\frac{\cot\varphi(p(\boldsymbol{d}\cdot\boldsymbol{\sigma})^2-\mu)}{r}+\boldsymbol{c}\cdot\boldsymbol{\nu}=0\quad\text{on }\partial\mathcal{S}. \tag{11.10}$$

[14] Alternatively, because of the uniformity of the bending along the curves of constant r on $\mathcal{S}$, it seems reasonable to assume that T_{dd}, T_{de}, and T_{ee} depend at most on r. This being so, we have $\boldsymbol{d}\cdot\mathrm{grad}_{\mathcal{S}}T_{dd}=\boldsymbol{d}\cdot\mathrm{grad}_{\mathcal{S}}T_{de}=0$ on $\mathcal{S}$, whereby (11.3) requires that $T_{de}=C/r^2$, where $C=$ constant, and we see that (11.5) holds as an ordinary differential equation. If this strategy is followed in the analysis of the remaining Euler–Lagrange equations in (9.21), we find that $C=0$ and the conclusions reached, below, in (11.9), (11.16), (11.23), (11.24), (11.26) and (11.30) stand.

For a clear understanding of (11.10), it is helpful to recall the decomposition of the boundary $\partial\mathcal{S}$ into $\partial\mathcal{S} = \partial_s\mathcal{S} \cup \partial_l\mathcal{S}$, where the subscripts 's' and 'l' denote, respectively, the 'short' and 'long' edges of $\mathcal{S}$. In addition, we identify the finer decomposition

$$\partial_s\mathcal{S} = \partial_s\mathcal{S}^- \cup \partial_s\mathcal{S}^+, \tag{11.11}$$

of the short edges $\partial_s\mathcal{S}$ into its two disjoint parts

$$\partial_s\mathcal{S}^- := \partial_s\mathcal{S}\big|_{x_1=0} \qquad \text{and} \qquad \partial_s\mathcal{S}^+ := \partial_s\mathcal{S}\big|_{x_1=\ell} \tag{11.12}$$

along with the finer decomposition

$$\partial_l\mathcal{S} = \partial_l\mathcal{S}^- \cup \partial_l\mathcal{S}^+ \tag{11.13}$$

of its long edges $\partial_l\mathcal{S}$ into its two disjoint parts

$$\partial_l\mathcal{S}^- := \partial_l\mathcal{S}\big|_{x_2=-w/2} \qquad \text{and} \qquad \partial_l\mathcal{S}^+ := \partial_l\mathcal{S}\big|_{x_2=w/2}, \tag{11.14}$$

as depicted in Fig. 4.[15] In addition, we observe from the geometry in Fig. 3 and/or Fig. 4 that

$$\boldsymbol{d}\cdot\boldsymbol{\sigma} = \begin{cases} \sin\theta & \text{on } \partial_l\mathcal{S}^-, \\ -\sin\theta & \text{on } \partial_l\mathcal{S}^+, \\ \cos\theta & \text{on } \partial_s\mathcal{S}^-, \\ -\cos\theta & \text{on } \partial_s\mathcal{S}^+. \end{cases} \tag{11.15}$$

Thus, recalling the notation $\boldsymbol{c} = c_l^\sigma\boldsymbol{\sigma} + c_l^\nu\boldsymbol{\nu}$ on $\partial_l\mathcal{S}$ and $\boldsymbol{c} = c_s^\sigma\boldsymbol{\sigma} + c_s^\nu\boldsymbol{\nu}$ on $\partial_s\mathcal{S}$, it follows that we may rewrite (11.10), which represents (9.21 e), as

$$c_l^\nu = \frac{\cot\varphi(\mu - p\sin^2\theta)}{r} \quad \text{on } \partial_l\mathcal{S} \tag{11.16a}$$

and

$$c_s^\nu = \frac{\cot\varphi(\mu - p\cos^2\theta)}{r} \quad \text{on } \partial_s\mathcal{S}, \tag{11.16b}$$

wherein to evaluate r and θ on $\partial_s\mathcal{S}$ we use (A.35) and (A.36), and to evaluate r and θ on $\partial_l\mathcal{S}$ we use (A.37) and (A.38).

Regarding (9.21 c), we recall that $T_{de} = 0$, so that $\boldsymbol{T} = T_{dd}\boldsymbol{d}\otimes\boldsymbol{d} + T_{ee}\boldsymbol{e}\otimes\boldsymbol{e}$, and use (A.29) to first express (9.21 d) in terms of its components relative to $\boldsymbol{d}$ and $\boldsymbol{e}$, namely

$$\left.\begin{aligned} \boldsymbol{d}\cdot\boldsymbol{T}\boldsymbol{\nu} - \frac{\mu\cot^2\varphi}{2r^2}\boldsymbol{d}\cdot\boldsymbol{\nu} + \frac{\cot\varphi}{r}\boldsymbol{c}\cdot\boldsymbol{d} = 0, \\ \boldsymbol{e}\cdot\boldsymbol{T}\boldsymbol{\nu} - \frac{\mu\cot^2\varphi}{2r^2}\boldsymbol{e}\cdot\boldsymbol{\nu} = 0, \end{aligned}\right\} \quad \text{on } \partial\mathcal{S}. \tag{11.17}$$

More explicitly, we express (11.17) as

$$\left.\begin{aligned} \Big(T_{dd} - \frac{\mu\cot^2\varphi}{2r^2}\Big)\boldsymbol{d}\cdot\boldsymbol{\nu} + \frac{\cot\varphi}{r}\boldsymbol{c}\cdot\boldsymbol{d} = 0, \\ \Big(T_{ee} - \frac{\mu\cot^2\varphi}{2r^2}\Big)\boldsymbol{e}\cdot\boldsymbol{\nu} = 0, \end{aligned}\right\} \quad \text{on } \partial\mathcal{S}. \tag{11.18}$$

Again, we observe from the geometry in Fig. 3 and/or Fig. 4 that

$$\boldsymbol{e}\cdot\boldsymbol{\sigma} = \boldsymbol{d}\cdot\boldsymbol{\nu} = \begin{cases} \cos\theta & \text{on } \partial_l\mathcal{S}^-, \\ -\cos\theta & \text{on } \partial_l\mathcal{S}^+, \\ -\sin\theta & \text{on } \partial_s\mathcal{S}^-, \\ \sin\theta & \text{on } \partial_s\mathcal{S}^+, \end{cases} \tag{11.19}$$

and that

$$\boldsymbol{e}\cdot\boldsymbol{\nu} = \begin{cases} -\sin\theta & \text{on } \partial_l\mathcal{S}^-, \\ \sin\theta & \text{on } \partial_l\mathcal{S}^+, \\ -\cos\theta & \text{on } \partial_s\mathcal{S}^-, \\ \cos\theta & \text{on } \partial_s\mathcal{S}^+. \end{cases} \tag{11.20}$$

[15] Because the deformation from $\mathcal{D}$ to $\mathcal{S}$ is isometric, the vectors $\boldsymbol{d}$, $\boldsymbol{e}$, $\boldsymbol{\nu}$, and $\boldsymbol{\sigma}$ on $\mathcal{S}$ have clear pre-images which respect the isometric deformation on $\mathcal{D}$.

Further, noting that $\boldsymbol{c}\cdot\boldsymbol{d} = c_{\mathrm{l}}^{\sigma}\boldsymbol{\sigma}\cdot\boldsymbol{d} + c_{\mathrm{l}}^{\nu}\boldsymbol{\nu}\cdot\boldsymbol{d}$ on $\partial_{\mathrm{l}}\mathcal{S}$ and $\boldsymbol{c}\cdot\boldsymbol{d} = c_{\mathrm{s}}^{\sigma}\boldsymbol{\sigma}\cdot\boldsymbol{d} + c_{\mathrm{s}}^{\nu}\boldsymbol{\nu}\cdot\boldsymbol{d}$ on $\partial_{\mathrm{s}}\mathcal{S}$, we have

$$\boldsymbol{c}\cdot\boldsymbol{d} = \begin{cases} c_{\mathrm{l}}^{\sigma}\sin\theta + c_{\mathrm{l}}^{\nu}\cos\theta & \text{on } \partial_{\mathrm{l}}\mathcal{S}^{-}, \\ -(c_{\mathrm{l}}^{\sigma}\sin\theta + c_{\mathrm{l}}^{\nu}\cos\theta) & \text{on } \partial_{\mathrm{l}}\mathcal{S}^{+}, \\ c_{\mathrm{s}}^{\sigma}\cos\theta - c_{\mathrm{s}}^{\nu}\sin\theta & \text{on } \partial_{\mathrm{s}}\mathcal{S}^{-}, \\ -(c_{\mathrm{s}}^{\sigma}\cos\theta - c_{\mathrm{s}}^{\nu}\sin\theta) & \text{on } \partial_{\mathrm{s}}\mathcal{S}^{+}. \end{cases} \tag{11.21}$$

By substituting (11.19), (11.20) and (11.21) into (11.18), we thus find that the form of the equations that correspond to the two components of $\partial_{\mathrm{l}}\mathcal{S} = \partial_{\mathrm{l}}\mathcal{S}^{-}\cup\partial_{\mathrm{l}}\mathcal{S}^{+}$ are the same and that the same is true of the forms of the two equations that correspond to the two components of $\partial_{\mathrm{s}}\mathcal{S} = \partial_{\mathrm{s}}\mathcal{S}^{-}\cup\partial_{\mathrm{s}}\mathcal{S}^{+}$. We see that (11.18), which represents (9.21 c), amounts to the conditions

$$\left.\begin{aligned} T_{dd}\cos\theta - \frac{\mu\cot^2\varphi\cos\theta}{2r^2} + \frac{\cot\varphi(c_{\mathrm{l}}^{\sigma}\sin\theta + c_{\mathrm{l}}^{\nu}\cos\theta)}{r} &= 0, \\ T_{ee}\sin\theta - \frac{\mu\cot^2\varphi\sin\theta}{2r^2} &= 0 \end{aligned}\right\} \quad \text{on } \partial_{\mathrm{l}}\mathcal{S}, \tag{11.22a}$$

and

$$\left.\begin{aligned} T_{dd}\sin\theta - \frac{\mu\cot^2\varphi\sin\theta}{2r^2} - \frac{\cot\varphi(c_{\mathrm{s}}^{\sigma}\cos\theta - c_{\mathrm{s}}^{\nu}\sin\theta)}{r} &= 0, \\ T_{ee}\cos\theta - \frac{\mu\cot^2\varphi\cos\theta}{2r^2} &= 0 \end{aligned}\right\} \quad \text{on } \partial_{\mathrm{s}}\mathcal{S}. \tag{11.22b}$$

From the second of (11.22 a) and the second of (11.22 b), we see that $T_{ee} = \mu\cot^2\varphi/2r^2$ on $\partial\mathcal{S}$. To satisfy this boundary condition and to support the observation, and its natural consequence, that each line of constant r within $\mathcal{S}$ is bent uniformly, we suppose, more generally,

$$T_{ee} = \frac{\mu\cot^2\varphi}{2r^2} \quad \text{on } \mathcal{S}. \tag{11.23}$$

As a result of (11.23), we see from $(11.5)_1$ that

$$T_{dd} = -\frac{\mu\cot^2\varphi}{2r^2} \quad \text{on } \mathcal{S}. \tag{11.24}$$

Consequently, we find from (11.9) that p must satisfy

$$\frac{\partial^2 p}{\partial r^2} = \frac{\mu}{r^2}, \tag{11.25}$$

and, thus, be of the general form

$$p = \mu\Big(a + \frac{b\,r}{L} - \log\frac{r}{L}\Big), \tag{11.26}$$

with $a = \hat{a}(\theta)$ and $b = \hat{b}(\theta)$ as yet unknown. With this, we have satisfied each of the Euler–Lagrange equations in (9.21).

Toward determining how a and b depend on θ, we substitute (11.16) and (11.24) into (11.22) to find that, for $0<\theta<\pi$,

$$c_{\mathrm{l}}^{\sigma} = \frac{p\cot\varphi\cos\theta\sin\theta}{r} \quad \text{on } \partial_{\mathrm{l}}\mathcal{S}, \tag{11.27a}$$

while, for $0<\theta<\pi/2$ and $\pi/2<\theta<\pi$,

$$c_{\mathrm{s}}^{\sigma} = -\frac{p\cot\varphi\cos\theta\sin\theta}{r} \quad \text{on } \partial_{\mathrm{s}}\mathcal{S}. \tag{11.27b}$$

By continuity, we may extend (11.27 b) to apply for $0<\theta<\pi$. Next, we consider (9.21 c) and first observe that the expression $\boldsymbol{\sigma}\cdot(p\boldsymbol{L}\boldsymbol{\nu}+\boldsymbol{c})$, which appears in that condition, vanishes; specifically, recalling that $\boldsymbol{L} = k\boldsymbol{d}\otimes\boldsymbol{d}$ with $k = -\cot\varphi/r$, we see from (11.15), (11.19), and (11.27) that

$$\boldsymbol{\sigma}\cdot(p\boldsymbol{L}\boldsymbol{\nu}+\boldsymbol{c}) = pk(\boldsymbol{d}\cdot\boldsymbol{\nu})(\boldsymbol{d}\cdot\boldsymbol{\sigma}) + \boldsymbol{c}\cdot\boldsymbol{\sigma} = 0 \quad \text{on } \partial\mathcal{S}. \tag{11.28}$$

We then note that, with $\boldsymbol{t} = \boldsymbol{0}$ on $\partial\mathcal{S}$, the relation $2H = k = -\cot\varphi/r$, the representation (11.26) for p, and the identity $\boldsymbol{e}\cdot\mathrm{grad}_{\mathcal{S}}p = \partial p/\partial r$, (9.21 c) yields

$$\begin{aligned} -2\mu\boldsymbol{\nu}\cdot\mathrm{grad}_{\mathcal{S}}H + (2H\boldsymbol{1}_{\mathcal{S}} - \boldsymbol{L})\mathrm{grad}_{\mathcal{S}}p\cdot\boldsymbol{\nu} &= \Big(\frac{\mu}{r} + \frac{\partial p}{\partial r}\Big)\frac{\cot\varphi}{r}\boldsymbol{\nu}\cdot\boldsymbol{e} \\ &= \frac{\mu b\cot\varphi}{Lr}\boldsymbol{e}\cdot\boldsymbol{\nu} \\ &= 0 \quad \text{on } \partial\mathcal{S}. \end{aligned} \tag{11.29}$$

Invoking (11.20), we hence see that, to ensure that (9.21 c) holds, b must satisfy $b=\hat{b}(\theta)=0$ for $0<\theta<\pi$ and, thus, that (11.26) simplifies to

$$p=\mu\Big(a-\log\frac{r}{L}\Big), \tag{11.30}$$

where $a=\hat{a}(\theta)$ is yet undetermined.

11.3. Global moment balance

Now, as in (10.24), the specific moment on $\partial\mathcal{S}$ is given by $\boldsymbol{m}:=\boldsymbol{n}\times\boldsymbol{c}$, and has the form

$$\boldsymbol{m}=\begin{cases}\boldsymbol{m}_{\mathrm{l}}:=-c_{\mathrm{l}}^{\sigma}\,\boldsymbol{\nu}+c_{\mathrm{l}}^{\nu}\boldsymbol{\sigma} & \text{on } \partial_{l}\mathcal{S},\\ \boldsymbol{m}_{\mathrm{s}}:=-c_{\mathrm{s}}^{\sigma}\,\boldsymbol{\nu}+c_{\mathrm{s}}^{\nu}\boldsymbol{\sigma} & \text{on } \partial_{s}\mathcal{S},\end{cases} \tag{11.31}$$

wherein, for specifics, we may substitute (11.16) and (11.27). Having done this, we lastly must consider the global moment balance condition $(4.8)_2$ in which we set $\boldsymbol{t}=\boldsymbol{0}$, that is

$$\int_{\partial\mathcal{S}}\boldsymbol{m}\,\mathrm{d}s=\int_{\partial_{\mathrm{l}}\mathcal{S}}\boldsymbol{m}_{\mathrm{l}}\,\mathrm{d}s+\int_{\partial_{\mathrm{s}}\mathcal{S}}\boldsymbol{m}_{\mathrm{s}}\,\mathrm{d}s=\boldsymbol{0}. \tag{11.32}$$

Toward this aim, with the aid of (11.15), (11.19), (11.20), and after substituting (11.16) and (11.27) into (11.31), we readily find that

$$\left.\begin{aligned}\boldsymbol{m}_{\mathrm{l}}\cdot\boldsymbol{d}&=\pm\frac{\cot\varphi\,(p-\mu)\sin\theta}{r},\\ \boldsymbol{m}_{\mathrm{l}}\cdot\boldsymbol{e}&=\mp\frac{\mu\cot\varphi\cos\theta}{r},\end{aligned}\right\}\quad\text{on } \partial_{\mathrm{l}}\mathcal{S}^{\pm}, \tag{11.33a}$$

and that

$$\left.\begin{aligned}\boldsymbol{m}_{\mathrm{s}}\cdot\boldsymbol{d}&=\pm\frac{\cot\varphi\,(p-\mu)\cos\theta}{r},\\ \boldsymbol{m}_{\mathrm{s}}\cdot\boldsymbol{e}&=\pm\frac{\mu\cot\varphi\sin\theta}{r},\end{aligned}\right\}\quad\text{on } \partial_{\mathrm{s}}\mathcal{S}^{\pm}. \tag{11.33b}$$

Using the expression (11.30) for p, we thus see that

$$\begin{aligned}\int_{\partial_{\mathrm{l}}\mathcal{S}}\boldsymbol{m}_{\mathrm{l}}\,\mathrm{d}s&=\mu\cot\varphi\int_{\partial_{\mathrm{l}}\mathcal{S}^{-}}\frac{(1-\hat{a}(\theta)+\log(r/L))\sin\theta\,\boldsymbol{d}+\cos\theta\,\boldsymbol{e})\,\mathrm{d}s}{r}\\ &-\mu\cot\varphi\int_{\partial_{\mathrm{l}}\mathcal{S}^{+}}\frac{((1-\hat{a}(\theta)+\log(r/L))\sin\theta\,\boldsymbol{d}+\cos\theta\,\boldsymbol{e})\,\mathrm{d}s}{r}\end{aligned} \tag{11.34a}$$

and

$$\begin{aligned}\int_{\partial_{\mathrm{s}}\mathcal{S}}\boldsymbol{m}_{\mathrm{s}}\,\mathrm{d}s&=\mu\cot\varphi\int_{\partial_{\mathrm{s}}\mathcal{S}^{-}}\frac{(1-\hat{a}(\theta)+\log(r/L))\cos\theta\,\boldsymbol{d}-\sin\theta\,\boldsymbol{e})\,\mathrm{d}s}{r}\\ &-\mu\cot\varphi\int_{\partial_{\mathrm{s}}\mathcal{S}^{+}}\frac{(1-\hat{a}(\theta)+\log(r/L))\cos\theta\,\boldsymbol{d}-\sin\theta\,\boldsymbol{e})\,\mathrm{d}s}{r}.\end{aligned} \tag{11.34b}$$

To simplify the integrals in (11.34), we refer to the notation for the corners A, B, C, and D of $\mathcal{S}$ in Fig. 4 and introduce readily established identities

$$\sin\theta\,\mathrm{d}s=r\,\mathrm{d}\theta\qquad\text{and}\qquad\cos\theta\,\mathrm{d}s=-\mathrm{d}r \tag{11.35}$$

for integrating from B to C on $\partial_{\mathrm{l}}\mathcal{S}^{-}$ and from A to D on $\partial_{\mathrm{l}}\mathcal{S}^{+}$, and

$$\cos\theta\,\mathrm{d}s=r\,\mathrm{d}\theta\qquad\text{and}\qquad\sin\theta\,\mathrm{d}s=\mathrm{d}r \tag{11.36}$$

for integrating from A to B on $\partial_{\mathrm{s}}\mathcal{S}^{-}$ and from D to C on $\partial_{\mathrm{s}}\mathcal{S}^{+}$. Observe, for example, that

$$\int_{\partial_{\mathrm{l}}\mathcal{S}^{-}}\frac{(1-\hat{a}(\theta)+\log(r/L))\sin\theta\,\boldsymbol{d}\,\mathrm{d}s}{r}=\int_{\theta_B}^{\theta_C}\Big(1-\hat{a}(\theta)+\log\frac{r}{L}\Big)\boldsymbol{d}\,\mathrm{d}\theta, \tag{11.37}$$

wherein it is understood from (A.37 a) that we may express r in terms of a function $\bar{r}$ of θ on $\partial_{\mathrm{l}}\mathcal{S}^{-}$, and that

$$\int_{\partial_{\mathrm{l}}\mathcal{S}^{-}}\frac{\cos\theta\,\boldsymbol{e}\,\mathrm{d}s}{r}=-\int_{\partial_{\mathrm{l}}\mathcal{S}^{-}}\frac{\boldsymbol{e}\,\mathrm{d}r}{r}$$

$$
\begin{aligned}
&= -\int_{\partial_l\mathcal{S}^-} \frac{L}{\bar{r}(\theta)}\frac{d\bar{r}(\theta)}{d\theta}\boldsymbol{e}\,\mathrm{d}\theta \\
&= -\int_{\partial_l\mathcal{S}^-} \left(\frac{d}{d\theta}\log\frac{\bar{r}(\theta)}{L}\right)\boldsymbol{e}\,\mathrm{d}\theta \\
&= -\int_{\partial_l\mathcal{S}^-} \left(\frac{d}{d\theta}\left(\left(\log\frac{\bar{r}(\theta)}{L}\right)\boldsymbol{e}\right) - \left(\log\frac{\bar{r}(\theta)}{L}\right)\frac{d\boldsymbol{e}}{d\theta}\right)\mathrm{d}\theta \\
&= -\left[\left(\log\frac{r}{L}\right)\boldsymbol{e}\right]_{\theta_B}^{\theta_C} - \int_{\theta_B}^{\theta_C}\left(\log\frac{r}{L}\right)\boldsymbol{d}\,\mathrm{d}\theta,
\end{aligned}
\tag{11.38}
$$

where we have used the readily established identity $d\boldsymbol{e}/d\theta = -\boldsymbol{d}$. Consequently, the first integral on the right hand side of (11.34 a) simplifies to

$$
\int_{\partial_l\mathcal{S}^-} \frac{(1-\hat{a}(\theta)+\log(r/L))\sin\theta\,\boldsymbol{d}+\cos\theta\,\boldsymbol{e})\,\mathrm{d}s}{r} = \int_{\theta_B}^{\theta_C}(1-\hat{a}(\theta))\boldsymbol{d}\,\mathrm{d}\theta - \left[\left(\log\frac{r}{L}\right)\boldsymbol{e}\right]_{\theta_B}^{\theta_C}.
\tag{11.39}
$$

Similarly, with the understanding that we may set $r=\bar{r}(\theta)$ from (A.38 a) on $\partial_l\mathcal{S}^+$, from (A.35 a) on $\partial_s\mathcal{S}^-$, and from (A.36 a) on $\partial_s\mathcal{S}^+$, we find that the remaining integral on the right-hand side of (11.34 a) simplifies to

$$
\int_{\partial_l\mathcal{S}^+} \frac{(1-\hat{a}(\theta)+\log(r/L))\sin\theta\,\boldsymbol{d}+\cos\theta\,\boldsymbol{e})\,\mathrm{d}s}{r} = \int_{\theta_A}^{\theta_D}(1-\hat{a}(\theta))\boldsymbol{d}\,\mathrm{d}\theta + \left[\left(\log\frac{r}{L}\right)\boldsymbol{e}\right]_{\theta_A}^{\theta_D},
\tag{11.40}
$$

while the integrals on the right-hand side of (11.34 b) simplify to

$$
\int_{\partial_s\mathcal{S}^-} \frac{(1-\hat{a}(\theta)+\log(r/L))\cos\theta\,\boldsymbol{d}-\sin\theta\,\boldsymbol{e})\,\mathrm{d}s}{r} = \int_{\theta_A}^{\theta_B}(1-\hat{a}(\theta))\boldsymbol{d}\,\mathrm{d}\theta - \left[\left(\log\frac{r}{L}\right)\boldsymbol{e}\right]_{\theta_A}^{\theta_B},
\tag{11.41}
$$

and

$$
\int_{\partial_s\mathcal{S}^+} \frac{((1-\hat{a}(\theta)+\log(r/L))\cos\theta\,\boldsymbol{d}-\mu\sin\theta\,\boldsymbol{e})\,\mathrm{d}s}{r} = \int_{\theta_D}^{\theta_C}(1-\hat{a}(\theta))\boldsymbol{d}\,\mathrm{d}\theta + \left[\left(\log\frac{r}{L}\right)\boldsymbol{e}\right]_{\theta_D}^{\theta_C}.
\tag{11.42}
$$

Thus, (11.34) takes the form

$$
\begin{aligned}
\int_{\partial\mathcal{S}}\boldsymbol{m}\,\mathrm{d}s = \mu\cot\varphi\Big(&\int_{\theta_B}^{\theta_C}(1-\hat{a}(\theta))\boldsymbol{d}\,\mathrm{d}\theta - \int_{\theta_A}^{\theta_D}(1-\hat{a}(\theta))\boldsymbol{d}\,\mathrm{d}\theta + \int_{\theta_A}^{\theta_B}(1-\hat{a}(\theta))\boldsymbol{d}\,\mathrm{d}\theta - \int_{\theta_D}^{\theta_C}(1-\hat{a}(\theta))\boldsymbol{d}\,\mathrm{d}\theta \\
&-\left[\left(\log\frac{r}{L}\right)\boldsymbol{e}\right]_{\theta_B}^{\theta_C} + \left[\left(\log\frac{r}{L}\right)\boldsymbol{e}\right]_{\theta_A}^{\theta_D} - \left[\left(\log\frac{r}{L}\right)\boldsymbol{e}\right]_{\theta_A}^{\theta_B} + \left[\left(\log\frac{r}{L}\right)\boldsymbol{e}\right]_{\theta_D}^{\theta_C}\Big) \\
= \mathbf{0},&
\end{aligned}
\tag{11.43}
$$

which shows that the moment balance condition (11.32) is identically satisfied for any choice of $a=\hat{a}(\theta)$ in (11.30).

11.4. Summary and interpretation of the solution

We have satisfied the Euler–Lagrange equations (9.21) and the global force and moment balance conditions (4.8), and we have found from (11.23) and (11.24) that the Lagrangian multiplier $\boldsymbol{T}$ is given by

$$
\boldsymbol{T} = -\frac{\mu\cot^2\varphi}{2r^2}(\boldsymbol{d}\otimes\boldsymbol{d}-\boldsymbol{e}\otimes\boldsymbol{e}),
\tag{11.44}
$$

and, according to (11.30), that the Lagrangian multiplier p is given by

$$
p = \mu\left(a-\log\frac{r}{L}\right),
\tag{11.45}
$$

where $a=\hat{a}(\theta)$ is indeterminate. We have specified that the applied specific traction $\boldsymbol{t}=\mathbf{0}$ on $\partial\mathcal{S}$ and we found that the applied specific moment $\boldsymbol{m}$ on $\partial\mathcal{S}$ is given by (11.31), which is parametrized by the indeterminate *arbitrary* quantity $a=\hat{a}(\theta)$. Thus, for any given isometric deformation of a rectangular strip $\mathcal{D}$ to a right cylindrical conical ribbon $\mathcal{S}$, the applied specific moment (11.31) is indeterminate up to the specification of $a=\hat{a}(\theta)$. As remarked earlier, a certain indeterminacy in the Lagrangian multiplier fields generally is to be expected because the ribbon is unstretchable and can only undergo an isometric deformation. Consequently, because the material ribbon is unstretchable, it is rendered unresponsive to certain loading systems. Interestingly, we observe from (11.33) that $a=\hat{a}(\theta)$ appears only in the $\boldsymbol{d}$-component of $\boldsymbol{m}$ on $\partial\mathcal{S}$, and not in the $\boldsymbol{e}$-component, which is the axis about which bending takes place. This also was the case for the problem, considered in Section 10, of the rectangular strip which is deformed isometrically to a circular helical band. A major difference, though, is that in the present problem, where $\mathcal{S}$ lies on a conical surface, the nonzero principal curvature of $\mathcal{S}$ varies along the generatricies $\boldsymbol{e}$ of $\mathcal{S}$, and the generatracies are not parallel. Thus, contrary to the situation in Section 10, it is not possible to make a natural choice of the function $a = \hat{a}(\theta)$ that eliminates the component $\boldsymbol{m} \cdot \boldsymbol{d}$ that is applied all around the edge $\partial\mathcal{S}$ of $\mathcal{S}$. The conical-form ribbon $\mathcal{S}$ is not uniformly bent about its generatricies; rather, it consists of a combination of nonuniform bending and twisting.

Aside from the indeterminacy issue, it also is interesting to recall that the ribbon $\mathcal{S}$ cannot intersect the apex of the conical surface $\mathcal{K}$ because of the geometric restriction $w < 2L\sin\theta_0$ as recorded in the Appendices A.2.2 and A.2.3. Thus, $r > 0$ for all points $\boldsymbol{y}$ on $\mathcal{S}$, and we see that for any configuration $\mathcal{S}$ whose edge $\partial_l\mathcal{S}^+$ lies on $\mathcal{K}$ close to the apex, both $\boldsymbol{T}$ and p are large at this point of closest approach. This is to be expected because the curvature of $\partial_l\mathcal{S}^+$ is large at that point and the constraint reaction in response to the material surface being unstretchable must be correspondingly large.

12. Discussion

A variational setting for studying the isometric deformation of unstretchable material surfaces can be posited in either the referential or spatial form. Here, we have followed a spatial variational development for an unstretchable material surface that in its undistorted referential state $\mathcal{D}$ in two-dimensional Euclidean point space $\mathbb{E}^2$ is flat, and in its distorted configuration $\mathcal{S}$ in three-dimensional Euclidean point space $\mathbb{E}^3$ may be subject to various edge conditions of either the loading type (specific traction and moment) or the kinematic type (placement and tangent-normal slope). Lagrangian multiplier fields — a symmetric tensor-valued field $\boldsymbol{T}$ that maps the translation space $\mathbb{V}^2$ of $\mathbb{E}^2$ to itself and a scalar-valued field p — appropriate for a spatial variational set-up, were introduced as constraint reactions in response to the requirement that the material be unstretchable or, equivalently, that the deformation be isometric, from a flat undistorted reference state $\mathcal{D}$. These reactions are conjugate to first and second order kinematic conditions that express the constraint of unstretchability and that restrict, respectively, the first and second fundamental forms of the deformed surface. The first order constraint, expressed in (2.4) relative to the reference $\mathcal{D}$, is a restriction on the surface gradient of the deformation, and therefore contains only first derivatives of the deformation. The second order constraint, expressed in (2.14) in the distorted state $\mathcal{S}$, ensures that the Gaussian curvature of $\mathcal{S}$ vanishes pointwise, and therefore places a restriction on the second derivatives of the deformation. For a spatial variational set-up, where variations are taken relative to the distorted state $\mathcal{S}$, the constraints which guarantee isometric deformation and the faithfulness to remain flat as $\mathcal{S}$ is varied need to be expressed in a spatial form that convects with the variation.

The Euler–Lagrange equations were derived in Section 9, using a classical Lagrangian multiplier spatial variational setup, for the general problem concerning the isometric deformation of an undistorted, planar region $\mathcal{D}$. The variational problem, being posed in a spatial form where the variation is defined relative to the assumed equilibrium configuration $\mathcal{S}$, does not explicitly mention the reference configuration $\mathcal{D}$ nor its flatness. As a consequence, the constraint of unstretchability relative to the configuration $\mathcal{S}$ does not carry with it an expression of the flatness of $\mathcal{D}$, and a spatial awareness of this flatness must be made explicit. In addition to enforcing the relative unstretchability of $\mathcal{S}$ in any variation of $\mathcal{S}$, it is natural to constrain the Gaussian curvature in any variation of $\mathcal{S}$ so that it remains zero, and a presence of the referential flatness of $\mathcal{D}$ is included.[16] Thus, we apply the constraints (9.2) and (9.3) in the derivation of Section 9.

In this derivation, the stored energy density function W was assumed to depend only on the curvature tensor $\boldsymbol{L}$ of $\mathcal{S}$. Two specific problems were solved for the special case $W(\boldsymbol{L}) = \frac{1}{2}\mu(\operatorname{tr}\boldsymbol{L})^2$, wherein the Euler–Lagrange equations were shown to have the specific form given in (9.21): In Section 10, we studied the deformation of a rectangular strip to a circular helical band and in Section 11, we studied the deformation of a rectangular strip to a conical band. In both of these problems, the applied specific edge traction $\boldsymbol{t} = \boldsymbol{0}$, and the applied specific edge moment $\boldsymbol{m}$ was presumed to be given on the edge $\partial\mathcal{S}$ of the distorted equilibrium configuration $\mathcal{S}$ such that the resultant force and resultant moment on $\mathcal{S}$ vanish, in accord with (4.8).

From a mechanics perspective, the Lagrangian multiplier fields $\boldsymbol{T}$ and p are necessary reactions to the constraint that the material surface is unstretchable and can only deform isometrically. Moreover, it is common when considering kinematic constraints that the corresponding reaction fields may not be uniquely determined for certain equilibrium (relative minimum potential energy) configurations of a body. For the two problems of Sections 10 and 11, we determined $\boldsymbol{T}$ but p remained partially indeterminate after the governing Euler–Lagrange equations were satisfied and the distorted ribbon $\mathcal{S}$ was force and moment balanced. We found that the degree of indeterminacy in p reflected itself in the form of the specific edge moment $\boldsymbol{m}$ that could be applied on $\partial\mathcal{S}$.

For example, in Section 10 we found, in particular, that the bending of a rectangular, unstretchable material strip $\mathcal{D}$ into a given isometrically deformed right circular ring $\mathcal{S}$ can be accomplished with many different applied specific moment systems. All of these systems, whose general form was shown in (10.31), have an equal and oppositely directed constant specific bending moment on the ends of width w of the ring $\mathcal{S}$, but, because of the indeterminacy in p, they allow an *arbitrary* equal and oppositely directed specific bending moment distribution to be applied to the two circularly bent edges of the ring. Thus, the unstretchable property of the material strip made it impossible for the circularly bent ring $\mathcal{S}$ to isometrically deform further in response to this wide class of specific moment edge loading. This circumstance may be made more intuitively clear by envisioning the arc-like shape of the bent configuration $\mathcal{S}$ and, because of its unstretchable nature, contemplating its rigid-like resistance to any opposing specific bending moment distribution that is applied on each of its

[16] See Remark 9.1 for more discussion of this matter.

circularly bent edges. Of course, because we found that for this deformation the applied specific bending moment distribution on the circularly bent edges is arbitrary, it may, in particular, be taken to be zero with the commonplace conclusion that the rectangular strip can be bent to a circular form solely by applying equal and oppositely directed constant end specific bending moments.

More generally, we found in Section 10 that the indeterminacy in the specific edge moment $\boldsymbol{m}$ that is applied on the boundary $\partial\mathcal{S}$ of the circular helical form of $\mathcal{S}$, consistent with a circular helical isometric deformation from $\mathcal{D}$, has no effect on the component $\boldsymbol{m}\cdot\boldsymbol{e}$ that is applied around the axis $\boldsymbol{e}$ on $\partial\mathcal{S}$ (namely, around the generatracies of $\mathcal{S}$). However, this indeterminacy was shown to allow for an *arbitrary* choice of the distribution of the applied component $\boldsymbol{m}\cdot\boldsymbol{d}$ on $\partial\mathcal{S}$, where $\boldsymbol{d} := \boldsymbol{e}\times\boldsymbol{n}$, with $\boldsymbol{n}$ the outer directed unit normal to the cylindrical surface on which $\mathcal{S}$ lies. These conclusions were made specific in (10.24), (10.25), and (10.26). Based upon the observation that the distorted circular helical form of $\mathcal{S}$ corresponds to a shape which is uniformly bent around its generatracies $\boldsymbol{e}$, as was shown in Fig. 2, we suggested that a natural choice is to set $\boldsymbol{m}\cdot\boldsymbol{d}=0$ on $\partial\mathcal{S}$. In that case, we concluded that this bending into the form of $\mathcal{S}$ could be sustained solely by a corresponding applied specific *bending moment* about $\boldsymbol{e}$ on its edge $\partial\mathcal{S}$. We recorded this conclusion in (10.36), and discussed its consequences in the last two paragraphs of Section 10.

The constrained variational problem of an unstretchable rectangular flat material strip that is isometrically deformed into a right circular conical ribbon was solved in Section 11 for the special case of a quadratic stored energy density function. The indeterminacy in the Lagrangian multiplier p due to the internal constraint of material unstretchability, and its effect on the degree of arbitrariness of the applied edge moment $\boldsymbol{m}$ was described. Also, we characterized the interesting near singular behavior of the Lagrangian multiplier functions $\boldsymbol{T}$ and p and, consequently, the applied edge moment $\boldsymbol{m}$ for a deformed ribbon $\mathcal{S}$ whose edge passes near the apex of the conical surface on which $\mathcal{S}$ lies. As we have included a short summary and interpretation of this solution and, in particular, these issues in Section 11.4, any further discussion at this point is unnecessary.

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Appendix A

A1. Appendix for Section 10

Here, we justify (10.2), starting with the representation (10.1), namely

$$\boldsymbol{T} = T_{dd}\boldsymbol{d}\otimes\boldsymbol{d} + T_{de}(\boldsymbol{d}\otimes\boldsymbol{e}+\boldsymbol{e}\otimes\boldsymbol{d}) + T_{ee}\boldsymbol{e}\otimes\boldsymbol{e}, \tag{A.1}$$

and the following observation:

$$\begin{aligned}\mathrm{div}_{\mathcal{S}}\boldsymbol{T} = {} & (\boldsymbol{d}\cdot\mathrm{grad}_{\mathcal{S}}T_{dd} + \boldsymbol{e}\cdot\mathrm{grad}_{\mathcal{S}}T_{de})\boldsymbol{d} + (\boldsymbol{d}\cdot\mathrm{grad}_{\mathcal{S}}T_{de} + \boldsymbol{e}\cdot\mathrm{grad}_{\mathcal{S}}T_{ee})\boldsymbol{e} \\ & + T_{dd}\mathrm{div}_{\mathcal{S}}(\boldsymbol{d}\otimes\boldsymbol{d}) + T_{de}(\mathrm{div}_{\mathcal{S}}(\boldsymbol{d}\otimes\boldsymbol{e}) + \mathrm{div}_{\mathcal{S}}(\boldsymbol{e}\otimes\boldsymbol{d})) + T_{ee}\mathrm{div}_{\mathcal{S}}(\boldsymbol{e}\otimes\boldsymbol{e}).\end{aligned} \tag{A.2}$$

Because $\boldsymbol{e}=$ constant, we know that $\mathrm{div}_{\mathcal{S}}(\boldsymbol{e}\otimes\boldsymbol{e})=\boldsymbol{0}$ and that we may write

$$\mathrm{div}_{\mathcal{S}}(\boldsymbol{d}\otimes\boldsymbol{d}) = \mathrm{div}_{\mathcal{S}}(\boldsymbol{1}_{\mathcal{S}}-\boldsymbol{e}\otimes\boldsymbol{e}) = \mathrm{div}_{\mathcal{S}}\boldsymbol{1}_{\mathcal{S}} = \mathrm{div}_{\mathcal{S}}(\boldsymbol{1}-\boldsymbol{n}\otimes\boldsymbol{n}) = -(\mathrm{div}_{\mathcal{S}}\boldsymbol{n})\boldsymbol{n} = 2H\boldsymbol{n}. \tag{A.3}$$

Also, because $\boldsymbol{d}=\boldsymbol{e}\times\boldsymbol{n}$ and $-\mathrm{grad}_{\mathcal{S}}\boldsymbol{n}=\boldsymbol{L}=k\boldsymbol{d}\otimes\boldsymbol{d}$, so $(\mathrm{grad}_{\mathcal{S}}\boldsymbol{n})\boldsymbol{e}=\boldsymbol{0}$, we see that $\boldsymbol{d}$ is constant in the direction of $\boldsymbol{e}$ and consequently $(\mathrm{grad}_{\mathcal{S}}\boldsymbol{d})\boldsymbol{e}=\boldsymbol{0}$. Thus, we obtain

$$\mathrm{div}_{\mathcal{S}}(\boldsymbol{d}\otimes\boldsymbol{e}) = (\mathrm{grad}_{\mathcal{S}}\boldsymbol{d})\boldsymbol{e} + (\mathrm{div}_{\mathcal{S}}\boldsymbol{e})\boldsymbol{d} = \boldsymbol{0}. \tag{A.4}$$

Finally, since $\boldsymbol{d}\cdot\boldsymbol{d}=1 \Rightarrow (\mathrm{grad}_{\mathcal{S}}\boldsymbol{d})^{\top}\boldsymbol{d}=\boldsymbol{0}$, we may conclude that

$$\mathrm{div}_{\mathcal{S}}\boldsymbol{d} = \boldsymbol{d}\cdot(\mathrm{grad}_{\mathcal{S}}\boldsymbol{d})\boldsymbol{d} + \boldsymbol{e}\cdot(\mathrm{grad}_{\mathcal{S}}\boldsymbol{d})\boldsymbol{e} = 0, \tag{A.5}$$

which, with $\boldsymbol{e}=$ constant, then yields

$$\mathrm{div}_{\mathcal{S}}(\boldsymbol{e}\otimes\boldsymbol{d}) = (\mathrm{grad}_{\mathcal{S}}\boldsymbol{e})\boldsymbol{d} + (\mathrm{div}_{\mathcal{S}}\boldsymbol{d})\boldsymbol{e} = \boldsymbol{0}. \tag{A.6}$$

Finally, (10.2) follows from (A.2) and these past few calculations.

A2. Appendix for Section 11

A2.1. Derivation of the tangential force balance on $\mathcal{S}$

In the first part of this Appendix we wish to show that (11.3) follows from (11.2), and, as in Appendix A.1, we start with the observation (A.2) and concentrate on determining $\mathrm{div}_{\mathcal{S}}(\boldsymbol{d}\otimes\boldsymbol{d})$, $\mathrm{div}_{\mathcal{S}}(\boldsymbol{d}\otimes\boldsymbol{e})$, $\mathrm{div}_{\mathcal{S}}(\boldsymbol{e}\otimes\boldsymbol{d})$, and $\mathrm{div}_{\mathcal{S}}(\boldsymbol{e}\otimes\boldsymbol{e})$. We shall first show that

$$\mathrm{div}_{\mathcal{S}}\,(\boldsymbol{e}\otimes\boldsymbol{e}) = (\boldsymbol{d}\cdot(\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{e})\boldsymbol{d})\boldsymbol{e}, \qquad \mathrm{div}_{\mathcal{S}}\,(\boldsymbol{d}\otimes\boldsymbol{d}) = k\boldsymbol{n} - (\boldsymbol{d}\cdot(\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{e})\boldsymbol{d})\boldsymbol{e}, \tag{A.7a}$$

where k is the nonzero principal curvature of $\mathcal{S}\subset\mathcal{K}$, and that

$$\mathrm{div}_{\mathcal{S}}\,(\boldsymbol{d}\otimes\boldsymbol{e}) = \mathrm{div}_{\mathcal{S}}\,(\boldsymbol{e}\otimes\boldsymbol{d}) = (\boldsymbol{d}\cdot(\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{e})\boldsymbol{d})\boldsymbol{d}. \tag{A.7b}$$

Then, we show that

$$\boldsymbol{d}\cdot(\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{e})\boldsymbol{d} = -\frac{1}{r} \quad \text{in } \mathcal{S}\subset\mathcal{K}, \tag{A.8}$$

where r is the distance from the tip of the conical surface $\mathcal{K}$ to $\boldsymbol{y}\in\mathcal{S}$. Thus, (11.3) readily follows from (A.7) and (A.8).

Let us start by noting that the unit vector $\boldsymbol{e}$ is constant along a generator of $\mathcal{K}$. Thus, $\mathrm{div}_{\mathcal{S}}\,(\boldsymbol{e}\otimes\boldsymbol{e}) = (\mathrm{div}_{\mathcal{S}}\,\boldsymbol{e})\boldsymbol{e}$ and $\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{e} = (\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{e})(\boldsymbol{d}\otimes\boldsymbol{d}+\boldsymbol{e}\otimes\boldsymbol{e}) = (\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{e})\boldsymbol{d}\otimes\boldsymbol{d}$ so we have

$$\mathrm{div}_{\mathcal{S}}\,(\boldsymbol{e}\otimes\boldsymbol{e}) = (\mathrm{div}_{\mathcal{S}}\,\boldsymbol{e})\boldsymbol{e} = (\boldsymbol{d}\cdot(\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{e})\boldsymbol{d})\boldsymbol{e}. \tag{A.9}$$

Next, observe that

$$\begin{aligned}\mathrm{div}_{\mathcal{S}}\,(\boldsymbol{d}\otimes\boldsymbol{d}) &= \mathrm{div}_{\mathcal{S}}\,(\boldsymbol{1}-\boldsymbol{n}\otimes\boldsymbol{n}-\boldsymbol{e}\otimes\boldsymbol{e}) = -\boldsymbol{n}\,\mathrm{div}_{\mathcal{S}}\,\boldsymbol{n} - \mathrm{div}_{\mathcal{S}}\,(\boldsymbol{e}\otimes\boldsymbol{e})\\ &= 2H\boldsymbol{n} - (\boldsymbol{d}\cdot(\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{e})\boldsymbol{d})\boldsymbol{e}\\ &= k\boldsymbol{n} - (\boldsymbol{d}\cdot(\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{e})\boldsymbol{d})\boldsymbol{e}.\end{aligned} \tag{A.10}$$

Further, note that, because the unit vector $\boldsymbol{d}$ is constant in the direction of $\boldsymbol{e}$, we have $(\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{d})\boldsymbol{e} = \boldsymbol{0}$ and, consequently,

$$\mathrm{div}_{\mathcal{S}}\,(\boldsymbol{d}\otimes\boldsymbol{e}) = (\mathrm{div}_{\mathcal{S}}\,\boldsymbol{e})\boldsymbol{d} = (\boldsymbol{d}\cdot(\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{e})\boldsymbol{d})\boldsymbol{d}. \tag{A.11}$$

Again, noting that $\boldsymbol{d}$ is constant in the direction of $\boldsymbol{e}$, we may write

$$\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{d} = (\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{d})(\boldsymbol{d}\otimes\boldsymbol{d}+\boldsymbol{e}\otimes\boldsymbol{e}) = (\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{d})\boldsymbol{d}\otimes\boldsymbol{d}, \tag{A.12}$$

which may be combined with the consequence $(\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{d})^{\top}\boldsymbol{d} = \boldsymbol{0}$ of differentiating the constraint $\boldsymbol{d}\cdot\boldsymbol{d} = 1$ to reach

$$\mathrm{div}_{\mathcal{S}}\,\boldsymbol{d} = \mathrm{tr}(\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{d}) = 0. \tag{A.13}$$

Thus, we see that $\mathrm{div}_{\mathcal{S}}\,(\boldsymbol{e}\otimes\boldsymbol{d}) = (\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{e})\boldsymbol{d}$ and, since $\boldsymbol{e}\cdot\boldsymbol{e} = 1 \Rightarrow (\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{e})^{\top}\boldsymbol{e} = \boldsymbol{0}$, we arrive at

$$\mathrm{div}_{\mathcal{S}}\,(\boldsymbol{e}\otimes\boldsymbol{d}) = (\boldsymbol{d}\cdot(\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{e})\boldsymbol{d})\boldsymbol{d}. \tag{A.14}$$

With (A.9)–(A.14), we conclude that (A.7) holds and, with (A.8), we arrive at (11.3).

To complete this verification we need to show that (A.8) holds. To do this it is convenient to draw upon the general development in Chen et al. (2018, Section 4) of an isometric deformation of a flat domain $\mathcal{D}\subset\mathbb{E}^2$ to a surface $\mathcal{S}\subset\mathbb{E}^3$. The notation in Chen et al. (2018) relates to the present notation in the following way: $\boldsymbol{a}_2\mapsto\boldsymbol{e}$; $\eta^1\mapsto\alpha$; $\eta^2\mapsto\beta$; $\boldsymbol{a}^1/|\boldsymbol{a}^1|\mapsto\boldsymbol{d}$, the latter because the dual base vector $\boldsymbol{a}^1$ is orthogonal to the base vector $\boldsymbol{a}_2$, and $\boldsymbol{d}$ is orthogonal to $\boldsymbol{e}$ in the same way. Because of (A.8), we are interested in the form of $\boldsymbol{d}\cdot(\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{e})\boldsymbol{d}$ and so it is useful to record the relation

$$\frac{\boldsymbol{a}^1\cdot(\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{a}_2)\boldsymbol{a}^1}{|\boldsymbol{a}^1|^2} \mapsto \boldsymbol{d}\cdot(\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{e})\boldsymbol{d}. \tag{A.15}$$

To go further, note that because we know from Chen et al. (2018, Section 4) that $\boldsymbol{a}_2 = \hat{\boldsymbol{a}}_2(\eta^1)$, then

$$\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{a}_2 = \frac{\partial\boldsymbol{a}_2}{\partial\eta^1}\otimes\boldsymbol{a}^1 = \dot{\boldsymbol{a}}_2\otimes\boldsymbol{a}^1, \tag{A.16}$$

so that

$$\frac{\boldsymbol{a}^1\cdot(\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{a}_2)\boldsymbol{a}^1}{|\boldsymbol{a}^1|^2} = \frac{\boldsymbol{a}^1\cdot(\dot{\boldsymbol{a}}_2\otimes\boldsymbol{a}^1)\boldsymbol{a}^1}{|\boldsymbol{a}^1|^2} = \dot{\boldsymbol{a}}_2\cdot\boldsymbol{a}^1. \tag{A.17}$$

Now, according to (4.9) in Chen et al. (2018, Section 4), we showed that $\boldsymbol{a}_2 = \boldsymbol{Q}\boldsymbol{b}_2$, where $\boldsymbol{Q}\in\mathrm{Orth}^+$ and, according to (4.2b) in that work, that $\boldsymbol{b}_2 = \cos\theta\boldsymbol{\imath}_1 + \sin\theta\boldsymbol{\imath}_2$, where $\theta = \hat{\theta}(\eta^1)\in(0,\pi)$. Thus, $\dot{\boldsymbol{a}}_2 = \dot{\boldsymbol{Q}}\boldsymbol{b}_2 + \boldsymbol{Q}\dot{\boldsymbol{b}}_2$. We also showed, according to (4.13) of that work, that for an isometric deformation we must have $\dot{\boldsymbol{Q}}\boldsymbol{b}_2 = \boldsymbol{0}$. Thus, it follows that

$$\dot{\boldsymbol{a}}_2\cdot\boldsymbol{a}^1 = \boldsymbol{a}^1\cdot\boldsymbol{Q}\dot{\boldsymbol{b}}_2 = \dot{\boldsymbol{b}}_2\cdot\boldsymbol{Q}^{\top}\boldsymbol{a}^1 = \dot{\boldsymbol{b}}_2\cdot\boldsymbol{b}^1, \tag{A.18}$$

where, according to (4.16) of Chen et al. (2018, Section 4),

$$\boldsymbol{b}^1 = \frac{\sin\theta\boldsymbol{\imath}_1 - \cos\theta\boldsymbol{\imath}_2}{\sin\theta - \eta^2\dot{\theta}}. \tag{A.19}$$

Reverting to the notation of the present paper, wherein $\alpha = \eta^1$ and $\beta = \eta^2$, we may conclude, using the identification (A.15) together with (A.17) and (A.18), that

$$\boldsymbol{d}\cdot(\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{e})\boldsymbol{d} = -\frac{\dot{\theta}}{\sin\theta - \beta\dot{\theta}}, \tag{A.20}$$

where $\theta = \hat{\theta}(\alpha)$ and the superposed 'dot' denotes differentiation with respect to α.

Now, to determine $\hat{\theta}(\alpha)$ we note, from Fig. 3, that the 'sine law' for triangles yields

$$\frac{\alpha}{\sin(\theta-\theta_0)}=\frac{L}{\sin(\pi-\theta)}, \tag{A.21}$$

which, for any $\alpha\in(-\infty,+\infty)$, may be reduced to

$$\tan\theta=\frac{\sin\theta_0}{\cos\theta_0-\alpha/L}, \tag{A.22}$$

where $\theta\in(0,\pi)$. Recall, we have assumed in Section 11 that $\theta_0\in(0,\pi/2)$. Thus, (A.22) yields a unique $\theta=\hat{\theta}(\alpha)\in(0,\pi)$ for each $\alpha\in(-\infty,+\infty)$, and gives

$$\dot{\theta}=\frac{\sin^2\theta}{L\sin\theta_0}>0, \tag{A.23}$$

so that, with (A.20), we may reach

$$\boldsymbol{d}\cdot(\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{e})\boldsymbol{d}=-\frac{\sin\theta}{L\sin\theta_0-\beta\sin\theta}. \tag{A.24}$$

Again, referring to Fig. 3, it readily follows that

$$L\sin\theta_0=(r+\beta)\sin(\pi-\theta), \tag{A.25}$$

which yields

$$r=\frac{L\sin\theta_0-\beta\sin\theta}{\sin\theta}, \tag{A.26}$$

and, with (A.24) we conclude that

$$\boldsymbol{d}\cdot(\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{e})\boldsymbol{d}=-\frac{1}{r}. \tag{A.27}$$

Thus, we see that (A.8) holds and we see from (A.2), (A.7), and (A.8) that (11.3) holds.

A2.2. Representations for the nonvanishing principal curvature of $\mathcal{S}$

First, it is helpful to recall, from (9.58) of Chen et al. (2018), that the nonzero principal curvature $k=\tilde{k}(\alpha,\beta)$ of $\mathcal{S}\subset\mathcal{K}$ may be written as

$$k=-\frac{\cot\varphi\sin\theta}{L\sin\theta_0-\beta\sin\theta}. \tag{A.28}$$

Thus, using (A.26), we see that the curvature tensor associated with $\mathcal{S}$ has the explicit form

$$\boldsymbol{L}=-\mathrm{grad}_{\mathcal{S}}\,\boldsymbol{n}=k\boldsymbol{d}\otimes\boldsymbol{d},\qquad k=-\frac{\cot\varphi}{r}. \tag{A.29}$$

In addition, in (9.60) of Chen et al. (2018) we also showed that the principal curvature k may be represented in terms of the reference coordinates (x_1, x_2) for $\boldsymbol{x}\in\mathcal{D}$ as

$$\tilde{k}(x_1,x_2)=-\frac{\cot\varphi}{L\sqrt{(\cos\theta_0-x_1/L)^2+(\sin\theta_0-x_2/L)^2}}, \tag{A.30}$$

from which it is clear that the restriction $w<2L\sin\theta_0$ keeps k bounded and negative for all $\boldsymbol{x}\in\mathcal{D}$. Consequently, we see that r may be written in terms of (x_1, x_2) for $\boldsymbol{x}\in\mathcal{D}$ as

$$\tilde{r}(x_1,x_2)=L\sqrt{(\cos\theta_0-x_1/L)^2+(\sin\theta_0-x_2/L)^2}, \tag{A.31}$$

which is useful in Section 11 to determine the relationship between r and θ when r is evaluated on the boundary $\partial\mathcal{S}$ of the isometrically deformed surface $\mathcal{S}$. Clearly, the boundary points $\boldsymbol{y}\in\partial\mathcal{S}$ correspond to the boundary points $\boldsymbol{x}\in\partial\mathcal{D}$ which, for $\partial_s\mathcal{S}^{\mp}$, are characterized respectively as $x_1=0$ or $x_1=\ell$ with $x_2\in(-w/2,w/2)$ and, for $\partial_l\mathcal{S}^{\pm}$, are characterized respectively as $x_2=\pm w/2$ with $x_1\in(0,\ell)$.

A2.3. Relationship between r *and θ on $\partial\mathcal{S}$*

To arrive explicitly at the relationship between r and θ when r is evaluated on $\partial\mathcal{S}$, we shall first eliminate α in (A.22) in favor of (x_1, x_2). From Fig. 3, we see that

$$\alpha=x_1-\frac{x_2}{\tan\theta}. \tag{A.32}$$

On the other hand, from (A.22) it is easy to see that

$$\alpha = L\left(\cos\theta_0 - \frac{\sin\theta_0}{\tan\theta}\right), \tag{A.33}$$

and eliminating α we find

$$\tan\theta = \frac{\sin\theta_0 - x_2/L}{\cos\theta_0 - x_1/L}, \tag{A.34}$$

where $\theta \in (0, \pi)$.

To obtain the relationship between r and θ for r evaluated on the subset $\partial_s\mathcal{S} \subset \partial\mathcal{S}$, we concentrate on $x_1 = 0$ or $x_1 = \ell$ with $x_2 \in (-w/2, w/2)$ and we use (A.34) to first write (A.31) as

$$r = L|\cos\theta_0 - x_1/L|\sqrt{1 + \tan^2\theta} = L\left|\frac{\cos\theta_0 - x_1/L}{\cos\theta}\right| \quad \text{on } \partial_s\mathcal{S}.$$

Then, recalling that $\theta_0 \in (0, \pi/2)$ and noting the decomposition $\partial_s\mathcal{S} = \partial_s\mathcal{S}^- \cup \partial_s\mathcal{S}^+$, as shown in Fig. 4, we explicitly have

$$r = \frac{L\cos\theta_0}{|\cos\theta|} \quad \text{on } \partial_s\mathcal{S}^-, \tag{A.35a}$$

wherein for determining $\theta \in (0, \pi)$ we must use, from (A.34),

$$\tan\theta = \frac{\sin\theta_0 - x_2/L}{\cos\theta_0}, \qquad -\frac{w}{2} < x_2 < \frac{w}{2}. \tag{A.35b}$$

Similarly, we explicitly have

$$r = L\left|\frac{\cos\theta_0 - \ell/L}{\cos\theta}\right| \quad \text{on } \partial_s\mathcal{S}^+, \tag{A.36a}$$

wherein for determining $\theta \in (0, \pi)$ we must use, from (A.34),

$$\tan\theta = \frac{\sin\theta_0 - x_2/L}{\cos\theta_0 - \ell/L}, \qquad -\frac{w}{2} < x_2 < \frac{w}{2}. \tag{A.36b}$$

To obtain the relationship between r and θ for r evaluated on the subset $\partial_l\mathcal{S} \subset \partial\mathcal{S}$, we concentrate on $x_2 = \pm w/2$ with $x_1 \in (0, \ell)$ and note that, because $w < 2L\sin\theta_0$ and $\theta \in (0, \pi)$, we may use (A.34) to first write (A.31) as

$$r = L(\sin\theta_0 - x_2/L)\sqrt{\cot^2\theta + 1} = \frac{L\sin\theta_0 - x_2}{\sin\theta} \quad \text{on } \partial_l\mathcal{S}.$$

Thus, with the decomposition $\partial_l\mathcal{S} = \partial_l\mathcal{S}^- \cup \partial_l\mathcal{S}^+$, as shown in Fig. 4, we explicitly obtain

$$r = \frac{L\sin\theta_0 + w/2}{\sin\theta} \quad \text{on } \partial_l\mathcal{S}^-, \tag{A.37a}$$

wherein for determining $\theta \in (0, \pi)$ we must use, from (A.34),

$$\tan\theta = \frac{\sin\theta_0 + w/2L}{\cos\theta_0 - x_1/L}, \qquad 0 < x_1 < \ell, \tag{A.37b}$$

and

$$r = \frac{L\sin\theta_0 - w/2}{\sin\theta} \quad \text{on } \partial_l\mathcal{S}^+, \tag{A.38a}$$

wherein for determining $\theta \in (0, \pi)$ we must use, from (A.34),

$$\tan\theta = \frac{\sin\theta_0 - w/2L}{\cos\theta_0 - x_1/L}, \qquad 0 < x_1 < \ell. \tag{A.38b}$$

Note that $r > 0$ on $\partial_l\mathcal{S}^+$ since $w < 2L\sin\theta_0$. Thus, according to Fig. 4 the ribbon $\mathcal{S}$ cannot touch the apex of the conical surface $\mathcal{K}$.

Of course, the domain over which θ varies is different for $x_1 = 0$ than it is for $x_1 = \ell$, but that is of secondary importance at this juncture.

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